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NECESSARY AND SUFFICIENT CONDITIONS FOR THE WELL-POSEDNESS OF A BOUNDARY VALUE PROBLEM FOR A LINEAR LOADED HYPERBOLIC EQUATION

Problems for loaded hyperbolic equations have acquired particular relevance in connection with the study of the stability of vibrations of the wings of an aircraft loaded with masses, and in the calculation of the natural vibrations of antennas loaded with lumped capacities and self-inductions. Loaded differential equations have a number of features that must be taken into account when setting problems for these equations and creating methods for their solution. One of the features of loaded differential equations is that such equations can be undecidable without additional conditions. The main idea of the research work is to expand the class of solvable boundary value problems and develop methods that provide a numerical-analytical solution. The paper considers a boundary value problem for a linear hyperbolic equation with a mixed derivative, where the load points are set in terms of the spatial variable. By introducing unknown functions, the problem is reduced to an equivalent boundary value problem for a linear loaded hyperbolic equation of the first order. With the help of the well posed of the equivalent boundary value problem, the well posed of the original problem is established. The paper presents the necessary and sufficient conditions for the well-posedness of a periodic boundary value problem for a linear loaded hyperbolic equation with two independent variables.

Key words: well-posedness solvability, necessary and sufficient conditions, loaded hyperbolic equation, linear hyperbolic equation, semi-periodic boundary value problem.

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Сызықтық жүктелген гиперболалық теңдеу үшін периодты шеттік есептің корректілі шешілімділігінің қажетті және жеткілікті шарттары

Жүктелген гиперболалық теңдеулер үшін есептер массалармен жүктелген ұшақ қанаттарының тербелістерінің тұрақтылығын зерттеуге және шоғырланған сыйымдылықтар мен өзін-өзі индукциялармен жүктелген антенналардың өзіндік тербелістерін есептеуге байланысты ерекше өзекті болып табылды. Жүктелген дифференциалдық теңдеулердің бірқатар ерекшеліктері бар, оларды осы теңдеулер үшін есептер шығару және оларды шешу әдістерін құру кезінде ескеру қажет. Жүктелген дифференциалдық теңдеулердің бір ерекшелігі мұндай теңдеулер қосымша шарттарсыз шешілмеуі мүмкін.

Бұл зерттеудің негізгі мақсаты шешілімді шекаралық есептер класын кеңейту және аналитикалық шешім беретін әдістерді құрастыру болып табылады. Жұмыста жүктеме нүктелері кеңістіктік айнымалыға қойылған аралас туындылы сызықтық гиперболалық теңдеу үшін шекттік есеп қарастырылады. Белгісіз функцияларды енгізу арқылы есеп бірінші ретті сызықтық жүктелген гиперболалық теңдеу үшін эквивалентті шеттік есепке келтіріледі. Эквивалентті шеттік есептің дұрыс шешілімділігінің көмегімен бастапқы есептің дұрыс шешілімділігі алынады. Жұмыста аралас туындысы бар сызықтық жүктелген гиперболалық теңдеу үшін жартылай периодты шекаралық есептің дұрыс шешілімділігінің қажетті және жеткілікті шарттары алынған.

Кілт сөздер: корректілі шешілімділік, қажетті және жеткілікті шарттар, жүктелген гиперболалық теңдеу, сызықтық гиперболалық теңдеу, жартылай периодты шеттік есеп.

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Необходимые и достаточные условия корректной разрешимости краевой задачи для линейного нагруженного гиперболического уравнения

Задачи для нагруженных гиперболических уравнений приобрели особую актуальность в связи с изучением устойчивости вибраций крыльев самолета, нагруженного массами, и при расчете собственных колебаний антенн, нагруженных сосредоточенными емкостями и самоиндукциями. Нагруженные дифференциальные уравнения имеют ряд особенностей, которые должны быть учтены при постановке задач для этих уравнений и создании методов их решений. Одним из особенностей нагруженных дифференциальных уравнений является то, что такие уравнения могут быть неразрешимыми без дополнительных условий. Основной целью данного исследования заключается в том, чтобы расширить класс разрешимых краевых задач и разработать методы которые дают аналитический вид решения задачи. В работе рассматривается краевая задача для линейного гиперболического уравнения со смешанной производной, где точки нагрузки ставятся по пространственной переменной. Задача путем введения неизвестных функции сводиться к эквивалентной краевой задаче для линейного нагруженного гиперболического уравнения первого порядка. С помощью корректной разрешимости эквивалентной краевой задачи устанавливается корректная разрешимость исходной задачи. В работе получены необходимые и достаточные условия корректной разрешимости полупериодической краевой задачи для линейного нагруженного гиперболического уравнения со смешанной производной.
Ключевые слова: корректная разрешимость, необходимые и достаточные условия, нагруженное гиперболическое уравнение, линейное гиперболическое уравнение, полупериодическая краевая задача.

1 Introduction

1.1 Problem statement

In the domain $\bar{\Omega} = [0, T] \times [0, \omega]$, we consider the semi-periodic boundary value problem for the linear loaded hyperbolic equation

$$\frac{\partial^2 u}{\partial t \partial x} = A(x,t)\frac{\partial u}{\partial x} + B(x,t)\frac{\partial u}{\partial t} + C(x,t)u + f(x,t) + A_0(x,t)\frac{\partial u(x_0,t)}{\partial x}, \tag{1}$$

$$u(0,t) = \psi(t), \quad t \in [0,T], \tag{2}$$

$$u(x,0) = u(x,T), \quad x \in [0,\omega], \tag{3}$$

where $A(x,t)$, $B(x,t)$, $C(x,t)$, and $f(x,t)$ are continuous on $\bar{\Omega}$, $\psi(t)$ is continuously differentiable on $[0,T]$ and satisfies the condition $\psi(0) = \psi(T)$, and x_0 is the load point. Let $C(\overline{\Omega})$ be the space of continuous functions $u : \overline{\Omega} \to R$ on $\overline{\Omega}$ with the norm $\|u\|_C = \max_{\overline{\Omega}} |u(x,t)|$. By $C^{1,1}_{x,t}(\overline{\Omega})$ we denote the space of continuous and continuously differentiable functions $u(x,t)$ on $\bar{\Omega}$ with the norm $\|u\|_0 = \max(\|u\|_C, \|u_x\|_C, \|u_t\|_C)$. $C^1\big([0,T]\big)$ denotes the space of continuous and differentiable function $\psi(t)$ on $[0,T]$ with the norm $\|\psi\|_1 = \max\big(\max_{t\in[0,T]} |\psi(t)|, \max_{t\in[0,T]} |\dot{\psi}(t)|\big)$.

A function $u(x,t) \in C(\overline{\Omega})$, that has partial derivatives

$$\frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t},$$

$$\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega}),$$

is called a solution of problem (1)-(3), if it satisfies equation (1) for all $(x,t) \in \overline{\Omega}$, take the value $\psi(t)$, $t \in [0,T]$ on the characteristic $x = 0$ and has equal values on the characteristics $t = 0$, $t = T$ for all $x \in [0,\omega]$.

Loaded hyperbolic partial differential equations with non-local boundary conditions arise in many fields of science and technology. The general definition of the loaded equation was given by Nakhushev. Loaded differential equations have a number of features that should be taken into account when setting problems for these equations and creating methods for their solution. One of the features of loaded differential equations is that such equations can be unsolvable unless additional conditions are imposed. There are some examples of linear loaded ordinary differential equations and loaded hyperbolic equations with mixed derivatives that have no solution.Boundary value problems for loaded differential equations have been studied by many authors [1-9].

Definition 1. *The boundary value problem (1)-(3) is called well-posed if for any $f(x,t) \in C(\overline{\Omega})$, continuous and continuously-differentiable on $[0,T]$ functions $\psi(t)$, it has a unique solution $u(x,t)$ and the inequality*

$$\|u\|_0 \le K \max\big\{\|\psi\|_1, \|f\|_C\big\},$$

is valid, where K is a constant, independent of $f(x,t)$ and $\psi(t)$.

1.2 Equivalent problem

Let us introduce new unknown functions $v(x,t) = \dfrac{\partial u(x,t)}{\partial x}$ and $w(x,t) = \dfrac{\partial u(x,t)}{\partial t}$.
The problem (1)-(3) is reduced it to the equivalent problem:

$$\frac{\partial v}{\partial t} = A(x,t)v + B(x,t)w + C(x,t)u + f(x,t) + A_0(x,t)v(x_0,t), \tag{4}$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \tag{5}$$

$$u(x,t)=\psi(t)+\int_0^x v(\xi,t)d\xi,\ w(x,t)=\dot{\psi}(t)+\int_0^x v_t(\xi,t)d\xi \tag{6}$$

The triple of continuous functions $\{u(x,t),w(x,t),v(x,t)\}$ on $\bar{\Omega}$ is called solution of problem (4)-(6), if $v(x,t)\in C(\bar{\Omega})$ has the continuous derivative with respect to t on $\bar{\Omega}$ and satisfies the family of periodic boundary value problems (4), (5), where the functions $u(x,t)$ and $w(x,t)$ are related to $v(x,t)$ and $\dfrac{\partial v(x,t)}{\partial t}$ by the functional relations (6).

The problems (1)-(3) and (4)-(6) are equivalent in the sense that if $u(x,t)$ is solution of problem (1)-(3), then the triple of functions $\{u(x,t),w(x,t),v(x,t)\}$ will be solution of problem (4)-(6). Vice versa, if a triple of functions $\{u^*(x,t),w^*(x,t),v^*(x,t)\}$ is a solution of problem (4)-(6), then $u^*(x,t)$ will be a solution of problem (1)-(3).

We consider the family of periodic boundary value problems for loaded ordinary differential equations

$$\frac{dv}{dt}=A(x,t)v+A_0(x,t)v(x_0,t)+F(x,t),\ (x,t)\in\overline{\Omega} \tag{7}$$

$$v(x,0)=v(x,T),\quad x\in[0,\omega], \tag{8}$$

where the functions $A(x,t),A_0(x,t)$ and $F(x,t)$ are continuous on $\overline{\Omega}$.

A function $v(x,t)\in C(\overline{\Omega})$ having the continuous derivative with respect to t is called a solution to the boundary value problem (7), (8) if it satisfies the system of equations (7) for all $(x,t)\in\overline{\Omega}$ and the boundary condition (8) for $x\in[0,\omega]$.

Definition 2. *The boundary value problem (7),(8) is called well posed if for any function $F(x,t)$ it has a unique solution $v(x,t)$ and the estimate*

$$\max_{t\in[0,T]}|v(x,t)|\le K\max_{t\in[0,T]}|F(x,t)|,$$

is valid, where K is a const, independent of $F(x,t)$.

Similarity Theorem 3 [10, C.23], we can show that the semi-periodic boundary value problem (1)-(3) is well-posed if and only if the periodic boundary value problem (7), (8) is wellposed.

2 Materials and methods

2.1 The well-posedness of the equivalent problem

The following statement establishes the necessary and sufficient conditions for the well-posedness of problem (7), (8).

Theorem 1. *The problem (7), (8) is well posed if and only if, for some $\delta_1>0,\delta_2>0$ the following inequalities holds:*
1) $\left|\int_0^T A(x,\tau)d\tau\right|\ge\delta_1$ *for all* $x\in[0,\omega]$,
2) $\left|\int_0^T\left[A(x_0,\tau)+A_0(x_0,\tau)\right]d\tau\right|\ge\delta_2, x_0\in[0,\omega]$.

Proof. Sufficiency. We consider the periodic boundary value problems (7), (8). For a fixed $x\in[0,\omega]$, we solve the differential equation (7). Its general solution is written as

$$v(x,t)=\exp\Big(\int_0^t A(x,\tau)d\tau\Big)\times$$

$$\times\Big[C(x)+\int_0^t F(x,\tau)\exp\Big(-\int_0^\tau A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\int_0^t A_0(x,\tau)v(x_0,\tau)\exp\Big(-\int_0^\tau A(x,\tau_1)d\tau_1\Big)d\tau\Big],\ t\in[0,T] \tag{9}$$

where $C(x)$ is a function continuous on $[0,\omega]$. Given the condition 2) of the theorem and the boundary condition (8), we find $C(x)$. Substituting it into (9), we find the solution of problem (7), (8) in the following form

$$v(x,t)=\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T\Big[F(x,\tau)+A_0(x,\tau)v(x_0,\tau)\Big]\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$

$$+\int_0^t\Big[F(x,\tau)+A_0(x,\tau)v(x_0,\tau)\Big]\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau,\ t\in[0,T] \tag{10}$$

Setting $x=x_0$, in problem (7), (8), we obtain the problem

$$\frac{dv(x_0,t)}{dt}=\big[A(x_0,t)+A_0(x_0,t)\big]v(x_0,t)+F(x_0,t),\ (x_0,t)\in\overline{\Omega}, \tag{11}$$

$$v(x_0,0)=v(x_0,T),\quad x_0\in[0,\omega]. \tag{12}$$

We then find the solution of problem (11), (12):

$$v(x_0,t)=\exp\Big(\int_0^t\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)\Big[C(x_0)+$$

$$+\int_0^t F(x_0,\tau)\exp\Big(-\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau,t\in[0,T]. \tag{13}$$

Substituting (13) into condition (12), we get

$$C(x_0)=C(x_0)\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)+$$

$$+\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau$$

When the conditions (2) $\Big|\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big|\ge\delta_2$ of the theorem 1 is fulfilled, we get that $\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)\ne 1$. Then the function $C(x_0)$ is defined as follows:

$$C(x_0)=\frac{1}{1-\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}\times$$

$$\times\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau.$$

Thus, the solution of the boundary value problem (11), (12) has the representation

$$v(x_0,t)=\frac{\exp\Big(\int_0^t\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big)}\times$$
$$\times\int_0^T F(x_0,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$
$$+\int_0^t F(x_0,\tau)\exp\Big(\int_\tau^t\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau. \tag{14}$$

Substituting (14) into the right-hand of (10), we get

$$v(x,t)=\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T F(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$
$$+\int_0^t F(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)d\tau+$$
$$+\frac{\exp\Big(\int_0^t A(x,\tau)d\tau\Big)}{1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)}\int_0^T A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\times$$
$$\times\Bigg\{\frac{\exp\Big(\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$
$$\times\int_0^T F(x,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$
$$+\int_0^\tau F(x_0,\tau_1)\exp\Big(\int_{\tau_1}^\tau\big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\}+$$
$$+\int_0^t A_0(x,\tau)\exp\Big(\int_\tau^t A(x,\tau_1)d\tau_1\Big)\Bigg\{\frac{\exp\Big(\int_0^\tau\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}{1-\exp\Big(\int_0^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)}\times$$
$$\times\int_0^T F(x,\tau)\exp\Big(\int_\tau^T\big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big])d\tau_1\Big)d\tau+$$
$$+\int_0^\tau F(x_0,\tau_1)\exp\Big(\int_{\tau_1}^\tau\big[A(x_0,\tau_2)+A_0(x_0,\tau_2)\big]d\tau_2\Big)d\tau_1 d\tau\Bigg\},\ t\in[0,T] \tag{15}$$

Uniqueness. Assume the opposite, let $v^*(x,t)$ and $\overline{v}(x,t)$ be two solutions of the periodic boundary value problem (7), (8). Then their difference $\Delta v(x,t)=v^*(x,t)-\overline{v}(x,t)$ satisfies the periodic boundary value problem for ordinary differential equation

$$\frac{d\Delta v}{dt}=A(x,t)\Delta v+A_0(x_0,t)\Delta v(x_0,t), \tag{16}$$

$$\Delta v(x,0)=\Delta v(x,T). \tag{17}$$

The general solution of equation (16) is of the form

$$\Delta v(x,t)=C(x)\cdot\exp\Big(\int_0^t A(x,\tau)d\tau\Big)+\int_0^T A(x_0,t)\exp\Big(\int_t^\tau A(x,\tau_1)d\tau_1\Big)\times$$

$$\times\exp\Big(\int_0^t[A(x_0,\tau)+A_0(x_0,\tau)]d\tau\Big)dt.$$

From the boundary condition (17), we get

$$C(x)\Bigg[1-\exp\Big(\int_0^T A(x,t)dt\Big)+C(x_0)\Big(\int_0^T A(x_0,t)\exp\Big(\int_t^\tau A(x,\tau_1)d\tau_1\Big)\Big)\times$$

$$\times\exp\Big(\int_0^\tau[A(x_0,\tau)+A_0(x_0,\tau)]d\tau\Big)\Bigg]=0.$$

For all $x\in[0,\omega]$, we have

$$\Big|\int_0^T A(x,t)dt\Big|\geq\delta_1>0\ \text{ and }\ \Big|\int_0^T\big[A(x_0,t)+A_0(x_0,t)\big]dt\Big|\geq\delta_2>0.$$

Then

$$\Big|1-\exp\Big(\int_0^T A(x,t)dt\Big)\Big|\neq 0\quad\text{and}\quad\Big|\int_0^T\big[A(x_0,t)+A_0(x_0,t)\big]dt\Big|\neq 0.$$

This implies that the function $C(x)$ is equal to zero for all $x\in[0,\omega]$. Then $\Delta v(x,t)\equiv 0$, i.e. problem (16),(17) has only the trivial solution. Therefore, $v^*(x,t)=\overline{v}(x,t)$ for all $x\in[0,\omega]$. Let us show that the inequality

$$\frac{1}{\Big|1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\Big|}\leq\frac{e^{\delta_1}}{e^{\delta_1}-1} \tag{18}$$

is correct.
Let us consider two cases: (a) $\int_0^T A(x,\tau)d\tau\leq-\delta_1$, and (b) $\int_0^T A(x,\tau)d\tau\geq\delta_1>0$.

$$\text{In the case (a), }\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\leq e^{-\delta_1},\ \ 1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\geq 1-e^{-\delta_1},$$

$$\text{and }\Big[1-\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\Big]^{-1}\leq\frac{1}{1-e^{-\delta_1}}=\frac{e^{\delta_1}}{e^{\delta_1}-1}.$$

$$\text{In the case (b) }\exp\Big(\int_0^T A(x,\tau)d\tau\Big)\geq e^{\delta_1},\ \exp\Big(\int_0^T A(x,\tau)d\tau\Big)-1\geq e^{\delta_1}-1,$$

$$\text{and }\Big[\exp\Big(\int_0^T A(x,\tau)d\tau\Big)-1\Big]^{-1}\leq\frac{1}{e^{\delta_1}-1}.$$

From these inequalities, we obtain (18).

Let us show the inequality is correct

$$\frac{1}{\left|1-\int_0^T\big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\right|}\le\frac{e^{\delta_2}}{e^{\delta_2}-1}. \tag{19}$$

In view of (18) and (19), we get the following estimate for $v(x,t)$:

$$\|v(x,\cdot)\|_1\le\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}\cdot\frac{e^{\alpha T}-1}{\alpha}\|F(x,\cdot)\|_1+\frac{e^{\alpha T}-1}{\alpha}\|F(x,\cdot)\|_1+$$

$$+\frac{e^{\delta_1}}{e^{\delta_1}-1}\cdot\frac{\alpha_0\cdot(e^{\alpha T}-1)}{\alpha}33\left[\frac{e^{\delta_2}}{e^{\delta_2}-1}\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1+\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1\right]+$$

$$+\frac{\alpha_0(e^{\alpha T}-1)}{\alpha}\cdot\left[\frac{e^{\delta_2}}{e^{\delta_2}-1}\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1+\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\|F(x,\cdot)\|_1\right]\le$$

$$\le\left\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_2}}{e^{\delta_2}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\times$$

$$\times\frac{e^{\alpha T}-1}{\alpha}\cdot\|F(x,\cdot)\|_1=K_1(\alpha,\alpha_0,\delta_1,\delta_2,T)\|F(x,\cdot)\|_1, \tag{20}$$

where $\alpha=\max\limits_{(x,t)\in\overline{\Omega}}|A(x,t)|$, $\alpha_0=\max\limits_{(x,t)\in\overline{\Omega}}|A_0(x,t)|$, and

$$K_1(\alpha,\alpha_0,\delta_1,\delta_2,T)=$$

$$=\left\{1+\frac{e^{\alpha T}\cdot e^{\delta_1}}{e^{\delta_1}-1}+\alpha_0\cdot\Big(1+\frac{e^{\delta_1}}{e^{\delta_1}-1}\Big)\cdot\Big(1+\frac{e^{\delta_2}}{e^{\delta_2}-1}\Big)\cdot\frac{e^{(\alpha+\alpha_0)T}-1}{\alpha+\alpha_0}\right\}\cdot\frac{e^{\alpha T}-1}{\alpha}.$$

Thus, by definition, problem (7), (8) is well posed.

Necessity. Let problem (7), (8) be well posed and let K be a constant satisfying inequality (20).

Since the family of periodic boundary value problems (7), (8) is well posed, we take $F(x,t)=1$ and consider the boundary value problem for the ordinary differential equation

$$\frac{dv}{dt}=A(x,t)v+A_0(x,t)v(x_0,t)+1, \tag{21}$$

$$v(x,0)=v(x,T) \tag{22}$$

Let us assume that there is $\widetilde{x}\in[0,\omega]$ for which $\int_0^T A(\widetilde{x},t)dt=0$ and $\int_0^T[A(\widetilde{x}_0,t)+A_0(\widetilde{x}_0,t)]dt=0$. The well-posedness of problem (7), (8) implies the existence of a unique $v_1(x,t)$ problem (21), (22). The function $v_1(x,t)$ satisfies the differential equation (21) for all $x,x_0\in[0,\omega]$, then for $x=\widetilde{x},x_0=\widetilde{x}_0$ we have

$$v_1(\widetilde{x},t)=\exp\Big(\int_0^t A(\widetilde{x},\tau)d\tau\Big)\Big[v_1(\widetilde{x},0)+\int_0^t\exp\Big(-\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)d\tau\Big]+$$

$$+\exp\Big(\int_0^t A(\widetilde{x},\tau)d\tau\Big)\int_0^t A(\widetilde{x}_0,\tau)\exp\Big(\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)\times$$

$$\times\exp\int_0^t \big[A(\widetilde{x}_0,\tau_1)+A_0(\widetilde{x}_0,\tau_1)\big]d\tau_1\times$$

$$\times\Big[\widetilde{v}_1(\widetilde{x}_0,0)+\int_0^t\exp\Big(-\int_0^\tau \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau\Big]d\tau. \quad (23)$$

From (23), setting $t=T$, we get

$$v(\widetilde{x},T)=v(\widetilde{x},0)+\int_0^T\exp\Big(-\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)d\tau+\int_0^T A(\widetilde{x}_0,\tau)\exp\Big(\int_0^\tau A(\widetilde{x},\tau_1)d\tau_1\Big)\times$$

$$\times\Big[\widetilde{v}_1(\widetilde{x}_0,0)+\int_0^T\exp\Big(-\int_0^\tau \big[A(x_0,\tau_1)+A_0(x_0,\tau_1)\big]d\tau_1\Big)d\tau\Big]d\tau.$$

It follows from our assumption, that $v(\widetilde{x},0)\neq v(\widetilde{x},T)$. The boundary condition (8) does not hold, hence we get that problem (7), (8) has no solution. Thus, we have a contradiction. Assuming that there is $\widetilde{x}\in[0,\omega]$ such that $\int_0^x A(\widetilde{x},t)dt=0$ and $\int_0^T[A(\widetilde{x}_0,t)+A_0(\widetilde{x}_0,t)]dt=0$ are not valid. It follows that if problem (7), (8) is well posed, then we have $\big|\int_0^T A(x,\tau)dt\big|\neq 0$ and $\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\neq 0$ for all $x,x_0\in[0,\omega]$. Since the functions $A(x,t)$ and $A_0(x,t)$ are continuous on $\overline{\Omega}$, then $\widetilde{A}(x)=\int_0^T A(x,t)dt$ and $\widetilde{A}(x_0)=\int_0^T[A(x_0,t)+A_0(x_0,t)]dt$ are also continuous functions on $[0,\omega]$. Hence $\big|\int_0^T A(x,t)dt\big|\neq 0$ and $\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\neq 0$ for all $x\in[0,\omega]$. From the well-known theorem [11, p. 175 - 176], it follows that there are $\delta_1>0,\delta_2>0$ such that the inequalities $\big|\int_0^T A(x,t)dt\big|\geq\delta_1$ and $\big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\big|\geq\delta_2$ hold for all $x,x_0\in[0,\omega]$. Theorem is proved.

2.2 Well-posedness of the main problem

Theorem 2. *Problem (1)-(3) is well posed if and only if for some $\delta>0$ following inequality hold:*

1. $\big|\int_0^T A(x,\tau)d\tau\big|\geq\delta_1$ *for all* $x\in[0,\omega]$.
2. $\Big|\int_0^T \big[A(x_0,\tau)+A_0(x_0,\tau)\big]d\tau\Big|\geq\delta_2, x_0\in[0,\omega]$.

Proof. Necessity. Let problem (1)-(3) be well posed. By Theorem 3 from [10, p.23], we obtain that problem (7), (8) is well posed. Then Theorem 1 implies the existence of $\delta_1>0,\delta_2>0$ such that the inequalities $\Big|\int_0^T A(x,\tau)d\tau\Big|\geq\delta_1$ and $\Big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\Big|\geq\delta_2$ hold for all $x\in[0,\omega]$.

Sufficiency. Let there exist $\delta_1>0$ and $\delta_2>0$ such that $\Big|\int_0^T A(x,\tau)d\tau\Big|\geq\delta_1$ and $\Big|\int_0^T[A(x_0,t)+A_0(x_0,t)]dt\Big|\geq\delta_2$ for all $x\in[0,\omega]$. Then, by Theorem 1, we obtain the well-posedness of problem (7), (8). The equivalence of boundary value problems (7), (8) and (1)-(3), and Theorem 3 from [10, p.23] imply the well-posedness of the boundary value problem (1)-(3). Theorem 2 is proved.

3 Conclusions

In this research, we considered a boundary value problem for a linear loaded hyperbolic equation with a mixed derivative, where the load points are set in terms of the spatial variable. An explicit form of the solution of an equivalent boundary value problem for a linear loaded equation is constructed. With its help, necessary and sufficient conditions of well-posedness solvability are obtained for a linear loaded hyperbolic equation.

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References

[1] Nakhushev A.M. Loaded equations and their applications, //Differential equations, –1983.– V. 19, No 1. –P. 86-94.

[2] Dzhumabaev D.S. Computational methods of solving the boundary value problems for the loaded differential and Fredholm integro-differential equations, //Mathematical Methods in the Applied Sciences, –2008.– V.41, No 4. – P. 1439-1462.

[3] Dzhumabaev D.S. Well-posedness of nonlocal boundary value problem for a system of loaded hyperbolic equations and an algorithm for finding its solution, //Journal of Mathematical Analysis and Applications, –2018– V.461, No 1. –P.817-836.

[4] Nakhushev A.M. Loaded equations and their applications, Science, Moscow.– 2012.–[in Russian]

[5] Aida-zade K.R., Abdullaev V.M. WOn a numerical solution of loaded differential equations, //Journal of computational mathematics and mathematical physics,–2004– V. 44, No 9. –P.1585-1595.

[6] Assanova A. T., Imanchiyev A. E., Kadirbayeva Zh. M. Numerical solution of systems of loaded ordinary differential equations with multipoint conditions, //Computational Mathematics and Mathematical Physics,–2009.– Vol 58, No 4. –P.508-516.

[7] Faramarz Tahamtani Blow-Up Results for a Nonlinear Hyperbolic Equation with Lewis Function, //Boundary Value Problems, –2009.– 9 pages. **doi:10.1155/2009/691496**.

[8] Kabdrakhova S. S. Criterion for the correct solvability of a semi-periodic boundary value problem for a linear hyperbolic equation,//Mathematical Journal,–2010.– V. 10 No 4(20). –P. 33-37. [in Russian]

[9] Genaliev M. T., Ramazanov M. I. Blow-Up Results for a Nonlinear Hyperbolic Equation with Lewis Function, //Boundary Value Problems,–2009.– 9 pages.
doi:10.1155/2009/691496.

[10] Asanova A.T., Dzhumabaev D. S. Criterion for the well-posedness solvability of a boundary value problem for a system of hyperbolic equations//Izv.NAS RK. Series of Physics and Mathematics, –2002–. No. 3. – P.20-26.

[11] Fichtenholz G.M. Course of differential and integral calculus, Moscow. The science. – 1969.– Volume 1. –P.608.

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ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE INTEGRAL BOUNDARY VALUE PROBLEM FOR SINGULARLY PERTURBED INTEGRO-DIFFERENTIAL EQUATIONS

The work is devoted to clarifying asymptotic with respect to a small parameter behavior of the solution of the integral boundary value problem for singularly perturbed linear integro-differential equation. We study the boundary value problem for singularly perturbed integro-differential equations with the phenomena of the so-called boundary jumps, when the fast solution variable becomes unbounded at both boundaries. The exceptions of the qualitative influence of integral terms on the asymptotic behavior of the solutions for singularly perturbed integro-differential equations are shown. The presence of integral terms will significantly change the degenerate equation: the solution of the assumed singularly perturbed integro-differential equation does not tend to the solution of the usual degenerate equation, obtained from the supposed equation with the zero value of a small parameter and will tend to solve a specially modified degenerate integro-differential equation with an additional term called the jump of the integral term. Boundary and initial functions are defined; their existence and uniqueness are proved. On the basis of the constructed boundary and initial functions are obtained analytical formula and asymptotic estimates of the solution for the integral boundary value problem. It is established that the solution of the considered boundary value problem at the ends of a given segment has the phenomena of boundary jumps of the same orders. A modified degenerate boundary value problem is constructed, to the solution of which approaches the solution of assumed singularly perturbed integral boundary value problem. The value of the jump of integral terms is found. An example was made based on the initial results.
Key words: singular perturbation, small parameter, the initial jump, asymptotics.

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Сингулярлы ауытқыған интегралды-дифференциалдық теңдеуге арналған шекаралық есеп шешімінің асимптотикалық сипаты

Жұмыс сингулярлы ауытқыған интегро-дифференциалдық теңдеу үшін шекаралық есеп шешімінің кіші параметр бойынша асимптотикалық құбылысын анықтауға арналған. Біз сингулярлы ауытқыған интегро-дифференциалдық теңдеуді, кіші параметр нөлге ұмтылғанда шешім кесіндінің екі жақ шетінде шексіздікке ұмтылатын шекаралық секіріс құбылысымен қарастырамыз. Жұмыста интегралдық мүшенің сингулярлы ауытқыған интегро-дифференциалдық теңдеу үшін шекаралық есеп шешімінің асимптотикалық сипатына әсері көрсетілген. Интегралдық мүше ауытқымаған шекаралық есепті айтарлықтай өзгертеді. Кіші параметр нөлге ұмтылған кезде сингулярлы ауытқыған шекаралық есеп шешімі қарапайым ауытқымаған шекаралық есеп шешіміне ұмтылмайды. Сингулярлық ауытқыған шекаралық есеп шешімі интегралдық мүшенің секірісі бар өзгертілген ауытқымаған есеп шешіміне ұмтылады.

Жұмыста интегралдық шекаралық есеп шешімінің аналитикалық формуласы мен асимптотикалық бағалаулары алынды. Берілген кесіндінің екі жақ шетінде де қарастырылып отырған шекаралық есеп шешімінің бастапқы секіріс құбылысының реті бірдей екендігі анықталды. Осы сингулярлы ауытқыған шекаралық есебіне сәйкес өзгертілген ауытқымаған шекаралық есеп құрылған. Берілген сингулярлы ауытқыған шекаралық есеп шешімінің өзгертілген ауытқымаған шекаралық есебіне ұмтылатыны дәлелденген. Интегралдық мүшенің бастапқы секірісінің шамасы анықталды. Мысал бастапқы нәтижелер негізінде жасалды.
Түйін сөздер: сингулярлық ауытқу, кіші параметр, бастапқы секіріс, асимптотика.

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Асимптотическое поведение решения интегральной краевых задач для сингулярно возмущенных интегро-дифференциальных уравнений

Работа посвящена выяснению асимптотического по малому параметру поведения решения интегральной краевой задачи для сингулярного возмущенного линейного интегро-дифференциального уравнения. Мы изучаем краевую задачу для сингулярно возмущенных интегро-дифференциальных уравнений с явлениями так называемых граничных скачков, когда переменная быстрого решения становится неограниченной на обеих границах. Показаны исключения качественного влияния интегральных членов на асимптотическое поведение решений сингулярно возмущенных интегро-дифференциальных уравнений. Наличие интегральных членов существенно изменит вырожденное уравнение. Решение предполагаемого сингулярно возмущенного интегро-дифференциального уравнение не стремится к решению обычного вырожденного уравнения, полученного из предполагаемого уравнения с нулевым значением малого параметра, и будет стремиться специально модифицированное вырожденное интегро-дифференциальное уравнение с дополнительным членом, называемым скачком интегрального члена. В работе получены аналитическая формула и асимптотические оценки решения интегральной краевой задачи. Установлено, что решение рассматриваемой краевой задачи на концах заданного отрезка имеет явления граничных скачков тех же порядков. Построена модифицированная вырожденная краевая задача, к решению которой стремится решение исходной сингулярно возмущенной интегральной краевой задачи. Определена величина начального скачка интегрального члена. Пример был сделан на основе начальных результатов.

Ключевые слова: малый параметр, сингулярное возмущение, асимптотика, начальный скачок.

1 Introduction

Singularly perturbed equations is called equations that contain a small parameter with higher derivatives. Many applied problems in physics, mechanics, technology, etc. are modeled using this type of equations in mathematics. In the following work of the authors L. Schlesinger [1], G.D. Birkhoff [2], P. Noaillon [3], W. Wasow [4], A.H. Nayfeh [5], A. N. Tikhonov [6,7], M.I. Vishik, L.A. Lusternik [8,9], N.N. Bogolyubov, U.A Mitropolsky [10], A.B. Vasilieva and V.F. Butuzov [11,12], R.E. O'Malley [13,14], D.R. Smith [15], W. Eckhaus [16], K. W. Chang and F. A. Howes [17], J. Kevorkian and J.D. Cole [18], Sanders and F. Verhulst [19], E.F. Mischenko, N. Rozov [20], S.A. Lomov [21], M.I. Imanaliev [22], K.A.Kassymov [23-25] was appeared and devoloped the theory of such type equations.

In the study of certain singularly perturbed problems, it can be found that the fast variable of the solution near the boundary of the set takes on an infinitely large value, since the small parameter tends to zero. The study of the Cauchy problem with an initial jump for a single nonlinear ODE was began by authors' work L.A. Lusternik and M.I. Vishik [26] and K.A.Kassymov [27]. The most common cases of the Cauchy problem with initial jump were studied by K.A. Kassymov. A characteristic exception of such problems is that the solution of this problem approaches to solution of degenerate equation with modified initial conditions when the small parameter goes to zero. In this instance, it is said that the phenomenon of the initial jump for the solution is valid.

In [28-31] was investigated boundary value problems (BVP) for singularly perturbed ODE and integro-differential equations with initial jumps.

In [32, 33] was studied BVP for integro-differential equations third-order with a small parameter at two higher derivatives, when hold the so-called phenomena of boundary jumps, i.e. when some derivatives of the solution for sufficiently small values of the parameter become infinitely large at both ends of the interval. But as this takes place at the ends of the considered interval of solving these problems, there were jumps of different orders. Singularly perturbed differential equations with a piecewise constant argument considered in [34].

In the present work, we investigate integral BVP for singularly perturbed linear integro-differential equations third-order, solution that at the ends of a given segment has jumps of the same order. The main goal of this paper is to establish the asymptotic behavior of the solution on a small parameter and the construction of the modified degenerate problem. A similar work but without integral conditions considered in [35].

2 Statement of the problem and auxiliary materials

Consider the following singularly perturbed linear integro-differential equation

$$L_\mu y \equiv \mu^2 y''' + \mu A_0(t)y'' + A_1(t)y' + A_2(t)y = F(t) + \int_0^1 \sum_{i=0}^{2} H_i(t,x)y^{(i)}(x,\mu)dx \tag{1}$$

with integral boundary conditions

$$h_1 y \equiv y(0,\mu) = \alpha, \quad h_2 y \equiv y'(0,\mu) = \beta, \quad h_3 y \equiv y'(1,\mu) - \int_0^1 \sum_{i=0}^{2} a_i(x)y^{(i)}(x,\mu)dx = \gamma, \tag{2}$$

where $\mu > 0$ is a small parameter, α, β, γ are known constants independent of μ.

Assume that following conditions hold:

C1) Functions $A_i(t), i = \overline{0,2}$, $F(t)$, $a_i(t), i = \overline{0,2}$ are defined on the interval $0 \leq t \leq 1$, $H_0(t,x), H_1(t,x), H_2(t,x)$ are defined in the domain $D = \{0 \leq t \leq 1, 0 \leq x \leq 1\}$ and sufficiently smooth.

C2) The roots $\kappa_i(t),\ i = 1,2$ of "additional characteristic equation" $\kappa^2 + A_0(t)\kappa + A_1(t) = 0$ satisfy the following inequalities $\kappa_1(t) \leqslant -\gamma_1 < 0$, $\kappa_2(t) \geqslant \gamma_2 > 0$.

C3) $a_2(1) \neq 1$.

We consider homogeneous singularly perturbed equation associated with (1):

$$L_\varepsilon y \equiv \mu^2 y''' + \mu A_0(t)y'' + A_1(t)y' + A_2(t)y = 0. \tag{3}$$

The fundamental set of solutions of equation (3) takes the form [35]

$$\begin{aligned} y_1^{(q)}(t,\mu) &= \frac{1}{\mu^q} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} \cdot \left(\kappa_1^q(t) y_{10}(t) + O(\mu)\right), \quad q=\overline{0,2}, \\ y_2^{(q)}(t,\mu) &= \frac{1}{\mu^q} e^{-\frac{1}{\mu}\int\limits_t^1 \kappa_2(x)dx} \cdot \left(\kappa_2^q(t) y_{20}(t) + O(\mu)\right), \quad q=\overline{0,2}, \\ y_3^{(q)}(t,\mu) &= y_{30}^{(q)}(t) + O(\mu), \quad q=\overline{0,2}, \end{aligned} \tag{4}$$

where $y_{30}(t) = e^{-\int\limits_0^t \frac{A_2(x)}{A_1(x)}dx}$, functions $y_{i0}(t), i=1,2$ are solutions of the problem:

$$p_i(t)\cdot y_{i0}'(t) + q_i(t)\cdot y_{i0}(t) = 0, \;\; y_{i0}(0)=1, \quad i=1,2,$$

where

$$p_i(t) = (A_0(t) + 2\kappa_i(t))\kappa_i(t), \quad q_i(t) = 3\kappa_i(t)\kappa_i'(t) + A_0(t)\kappa_i'(t) + A_2(t).$$

Wronskian, composed of the fundamental system of solutions of equation (3) has the form:

$$\begin{aligned} W(t,\mu) = \frac{1}{\mu^3} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx - \frac{1}{\mu}\int\limits_t^1 \kappa_2(x)dx} & (y_{10}(t)y_{20}(t)y_{30}(t)\kappa_1(t)\kappa_2(t)\times \\ & \times (\kappa_2(t)-\kappa_1(t)) + O(\mu)). \end{aligned} \tag{5}$$

We introduce the functions [35]

$$K_0(t,s,\mu) = \frac{P_0(t,s,\mu)}{W(s,\mu)}; \quad K_1(t,s,\mu) = \frac{P_1(t,s,\mu)}{W(s,\mu)}, \tag{6}$$

where $P_0(t,s,\mu)$, $P_1(t,s,\mu)$ are the third order determinant derived from the Wronskian $W(s,\mu)$ by replacing the third row with $y_1(t,\mu), 0, y_3(t,\mu)$ and $0, y_2(t,\mu), 0$ respectively. Sum of $K_0(t,s,\mu)$ and $K_1(t,s,\mu)$ are the Cauchy function. Therefore, these functions have the following properties:

1. With respect to the variable t satisfy equation (3), i.e.

$$L_\mu K_0(t,s,\mu) = 0, \;\; L_\mu K_1(t,s,\mu) = 0, \;\; t\in[0,1], \;\; t\neq s.$$

2. When $t=s$ satisfy the conditions:

$$K_0(s,s,\mu) + K_1(s,s,\mu) = 0, \quad K_0'(s,s,\mu) + K_1'(s,s,\mu) = 0, \quad K_0''(s,s,\mu) + K_1''(s,s,\mu) = 1.$$

For the functions $K_0(t,s,\mu)$, $K_1(t,s,\mu)$ in view (4), (5), (6) the following asymptotic representation hold as $\mu\to 0$ [35]:

$$\begin{aligned} K_0^{(q)}(t,s,\mu) = \mu^2 \Bigg(& \frac{y_{30}^{(q)}(t)}{y_{30}(s)\kappa_1(s)\kappa_2(s)} - \frac{\kappa_1^q(t) y_{10}(t)}{\mu^q y_{10}(s)\kappa_1(s)\left(\kappa_2(s)-\kappa_1(s)\right)} e^{\frac{1}{\mu}\int\limits_s^t \kappa_1(x)dx} + \\ & + O(\mu)\Bigg), \quad t\geq s, \quad q=\overline{0,2}, \end{aligned} \tag{7}$$

$$K_1^{(q)}(t,s,\mu)=\mu^2\left(\frac{\kappa_2^q(t)y_{20}(t)}{\mu^q y_{20}(s)\kappa_2(s)\left(\kappa_2(s)-\kappa_1(s)\right)}e^{-\frac{1}{\mu}\int\limits_t^s\kappa_2(x)dx}+O(\mu)\right),t\le s, q=\overline{0,2}.$$

Let functions $\Phi_i(t,\mu), i=1,2,3$ are solutions for the following problem:

$$L_\mu\Phi_i(t,\mu)=0,\quad h_k\Phi_i(t,\mu)=\delta_{ki},\qquad i,k=1,2,3,$$

where δ_{ki} is Kronecker symbol. Functions $\Phi_i(t,\mu),\ i=1,2,3$ are called boundary functions. The boundary functions are determined by the formula

$$\Phi_i(t,\mu)=\frac{J_i(t,\mu)}{J(\mu)}, \tag{8}$$

where $J(\mu)$ is the determinant consisting of the fundamental solution system of equation (3):

$$J(\mu)=\begin{vmatrix} h_1y_1(t,\mu) & h_1y_2(t,\mu) & h_1y_3(t,\mu)\\ h_2y_1(t,\mu) & h_2y_2(t,\mu) & h_2y_3(t,\mu)\\ h_3y_1(t,\mu) & h_3y_2(t,\mu) & h_3y_3(t,\mu)\end{vmatrix}, \tag{9}$$

$J_i(t,\mu)$ is the determinant derived from $J(\mu)$ by replacing the i-th row by the fundamental set of solutions $y_1(t,\mu),y_2(t,\mu),y_3(t,\mu)$ of the equation (3). From (9), by virtue of (2), (3), we obtain for $J(\mu)$ the asymptotic representation:

$$J(\mu)=\frac{1}{\mu^2}\left(\kappa_2(1)y_{20}(1)\kappa_1(0)(1-a_2(1))+O(\mu)\right)\ne 0. \tag{10}$$

Then from (8) in view (4), (10) we get asymptotic representation as $\mu\to 0$ for boundary functions $\Phi_i(t,\mu),\ i=1,2,3$:

$$\Phi_1^{(q)}(t,\mu)=y_{30}^{(q)}(t)-\frac{y'_{30}(0)\kappa_1^q(t)y_{10}(t)}{\mu^{q-1}\kappa_1(0)}e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}-$$

$$-\frac{\kappa_2^q(t)y_{20}(t)\left(h_3y_{30}(t)-a_2(0)y'_{30}(0)\right)}{\mu^{q-1}\kappa_2(1)y_{20}(1)(1-a_2(1))}e^{-\frac{1}{\mu}\int\limits_t^1\kappa_2(x)dx}+$$

$$+O\left(\mu+\frac{1}{\mu^{q-2}}e^{-\frac{t}{\mu}\gamma_1}+\frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right),\ q=\overline{0,2}, \tag{11}$$

$$\Phi_2^{(q)}(t,\mu)=-\mu\frac{y_{30}^{(q)}(t)}{\kappa_1(0)}+\frac{\kappa_1^q(t)y_{10}(t)}{\mu^{q-1}\kappa_1(0)}e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}-\frac{\kappa_2^q(t)y_{20}(t)a_2(0)}{\mu^{q-1}\kappa_2(1)y_2(1)(1-a_2(1))}\times$$

$$\times e^{-\frac{1}{\mu}\int\limits_t^1\kappa_2(x)dx}+O\left(\mu^2+\frac{1}{\mu^{q-2}}e^{-\frac{t}{\mu}\gamma_1}+\frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right),\ q=\overline{0,2},$$

$$\Phi_3^{(q)}(t,\mu)=\frac{\kappa_2^q(t)y_{20}(t)}{\mu^{q-1}\kappa_2(1)y_{20}(1)(1-a_2(1))}e^{-\frac{1}{\mu}\int\limits_t^1\kappa_2(x)dx}+O\left(\frac{1}{\mu^{q-2}}e^{-\frac{1-t}{\mu}\gamma_2}\right).$$

3 Main results

We are looking for a solution of the boundary value problem (1),(2) in the form:

$$y(t,\mu) = C_1\Phi_1(t,\mu) + C_2\Phi_2(t,\mu) + C_3\Phi_3(t,\mu) + \\ + \frac{1}{\mu^2}\int_0^t K_0(t,s,\mu)v(s,\mu)ds + \frac{1}{\mu^2}\int_1^t K_1(t,s,\mu)v(s,\mu)ds, \tag{12}$$

where $\Phi_i(t,\mu), i = 1,2,3$ are boundary functions and expressed by the formula (8), $K_0(t,s,\mu), K_1(t,s,\mu)$ are auxiliary functions, determined by the formula (6) and having an asymptotic representation (7), $C_i, i = 1,2,3$ are unknown constants, $v(t,\mu)$ is an unknown function.

Replacing (12) into equation (1) we get that $v(t,\mu)$ satisfies the following Fredholm integral equation of the second kind:

$$v(t,\mu) = p(t,\mu) + \int_0^1 H(t,s,\mu)v(s,\mu)ds, \tag{13}$$

where

$$p(t,\mu) = F(t) + C_1\int_0^1 \sum_{i=0}^{2} H_i(t,x)\Phi_1^{(i)}(x,\mu)dx + C_2\int_0^1 \sum_{i=0}^{2} H_i(t,x)\Phi_2^{(i)}(x,\mu)dx +$$

$$+C_3\int_0^1 \sum_{i=0}^{2} H_i(t,x)\Phi_3^{(i)}(x,\mu)dx, \tag{14}$$

$$H(t,s,\mu) = \frac{1}{\mu^2}\int_s^1 \sum_{i=0}^{2} H_i(t,x)K_0^{(i)}(x,s,\mu)dx - \frac{1}{\mu^2}\int_0^s \sum_{i=0}^{2} H_i(t,x)K_1^{(i)}(x,s,\mu)dx.$$

C4) 1 is not an eigenvalue of the kernel $H(t,s,\mu)$.

Taking into account the condition (C4) integral equation (13) has an unique solution, that can be represented in the form

$$v(t,\mu) = p(t,\mu) + \int_0^1 R_\mu(t,s,1)p(s,\mu)ds, \tag{15}$$

where $R_\mu(t,s,1)$ is a resolvent of the kernel $H(t,s,\mu)$, representable by (14) as an asymptotic formula $R_\mu(t,s,1) = R_0(t,s,1) + O(\mu)$. Substituting (15) taking account of (14) into (12), we obtain solution of the boundary value problem (1),(2) in the form:

$$y(t,\mu) = \sum_{i=1}^{3} C_iT_i(t,\mu) + Q(t,\mu), \tag{16}$$

where

$$T_i(t,\mu)=\Phi_i(t,\mu)+\frac{1}{\mu^2}\int\limits_0^t K_0(t,s,\mu)\overline{\phi_i}(s,\mu)ds+\frac{1}{\mu^2}\int\limits_1^t K_1(t,s,\mu)\overline{\phi_i}(s,\mu)ds,$$

$$Q(t,\mu)=\frac{1}{\mu^2}\int\limits_0^t K_0(t,s,\mu)\overline{f}(s,\mu)ds+\frac{1}{\mu^2}\int\limits_1^t K_1(t,s,\mu)\overline{f}(s,\mu)ds, \tag{17}$$

$$\overline{\phi}_i(s,\mu)=\int\limits_0^1\sum_{j=0}^2\overline{H}_j(s,x,\mu)\Phi_i^{(j)}(x,\mu)dx,\ \ \overline{f}(s,\mu)=F(s)+\int\limits_0^1 R_\mu(s,p,1)F(p)dp,$$

$$\overline{H}_j(s,x,\mu)=H_j(s,x)+\int\limits_0^1 R_\mu(s,p,1)H_j(p,x)dp.$$

For the functions $\overline{H}_j(s,x,\mu)$, $\overline{f}(s,\mu)$ are valid asymptotic representations:

$$\begin{aligned}&\overline{H}j(s,x,\mu)=Hj(s,x)+\int\limits_0^1 R_0(s,p,1)H_j(p,x)dp+O(\mu)\equiv\overline{H}_j(s,x)+O(\mu),\\&\overline{f}(s,\mu)=F(s)+\int\limits_0^1 R_0(s,p,1)F(p)dp+O(\mu)\equiv\overline{f}(s)+O(\mu).\end{aligned} \tag{18}$$

For the functions $\overline{\phi}_i(s,\mu)$ applying (11), (18), we can obtain the asymptotic formula:

$$\overline{\phi}_i(s,\mu)=\overline{\phi}_i(s)+O(\mu),\quad i=\overline{1,3}, \tag{19}$$

where

$$\begin{aligned}&\overline{\phi}_1(s)=\int\limits_0^1\sum_{j=0}^2\overline{H}_j(s,x)y_{30}^{(j)}(x)dx+\overline{H}_2(s,0)y'_{30}(0)-\frac{(h_3y_{30}(t)-a_2(0)y'_{30}(0))}{1-a_2(1)}\overline{H}_2(s,1),\\&\overline{\phi}_2(s)=-\overline{H}_2(s,0)-\overline{H}_2(s,1)\frac{a_2(0)}{1-a_2(1)},\quad \overline{\phi}_3(s)=\overline{H}_2(s,1)\frac{1}{1-a_2(1)}.\end{aligned} \tag{20}$$

From (17) applying (7),(11), (18)-(20), for the functions $T_i(t,\mu)$, $i=1,2,3$, $Q(t,\mu)$, we get asymptotic representations:

$$T_1^{(q)}(t,\mu)=y_{30}^{(q)}(t)+\int\limits_0^t\frac{y_{30}^{(q)}(t)\overline{\phi}_1(s)}{y_{30}(s)A_1(s)}ds-\frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)}\left(y'_{30}(0)+\frac{\overline{\phi}_1(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))}\right)\times$$

$$\times e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}-\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{h_3y_{30}(t)-a_2(0)y'_{30}(0)}{1-a_2(1)}-\frac{\overline{\phi}_1(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))}\right)\times$$

$$\times e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_1(t) +$$

$$+O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx}\right);$$

$$T_2^{(q)}(t,\mu) = \int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)} ds + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_2(t) -$$

$$-\frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)} \left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0) - \kappa_1(0))} - 1\right) e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} +$$

$$+\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} - \frac{a_2(0)}{1 - a_2(1)}\right) e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} +$$

$$+O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx}\right);$$

$$T_3^{(q)}(t,\mu) = \int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_1^q(t)y_{10}(t)\overline{\phi}_3(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \tag{21}$$

$$+\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)} \left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1) - \kappa_1(1))} + \frac{1}{1 - a_2(1)}\right) e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} +$$

$$+\frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{\phi}_3(t) + O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx}\right);$$

$$Q^{(q)}(t,\mu) = \int\limits_0^t \frac{y_{30}^{(q)}(t)\overline{f}(s)}{y_{30}(s)A_1(s)} ds - \frac{\kappa_1^q(t)y_{10}(t)\overline{f}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0) - \kappa_1(0))} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} +$$

$$+\frac{\kappa_2^q(t)y_{20}(t)\overline{f}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1) - \kappa_1(1))} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx} + \frac{\kappa_1^{q-2}(t) - \kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t) - \kappa_1(t))} \cdot \overline{f}(t) +$$

$$+O\left(\mu + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_0^t \kappa_1(x)dx} + \frac{1}{\mu^{q-2}} e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx}\right).$$

To find the unknown constants $C_i,\ i = 1,2,3$ from (16), taking account of the boundary conditions (2), we get a system of algebraic equations:

$$\begin{cases} C_1h_1T_1(t,\mu) + C_2h_1T_2(t,\mu) + C_3h_1T_3(t,\mu) = \alpha - h_1Q(t,\mu), \\ C_1h_2T_1(t,\mu) + C_2h_2T_2(t,\mu) + C_3h_2T_3(t,\mu) = \beta - h_2Q(t,\mu). \\ C_1h_3T_1(t,\mu) + C_2h_3T_2(t,\mu) + C_3h_3T_3(t,\mu) = \gamma - h_3Q(t,\mu) \end{cases} \tag{22}$$

For the elements of this system are valid asymptotic representations:

$$h_1T_1(t,\mu)=1+O(\mu),\ h_1T_i(t,\mu)=O(\mu),\ i=2,3,\ \ h_1Q(t,\mu)=O(\mu),$$

$$h_2T_i(t,\mu)=-\frac{\overline{\phi}_i(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}+O(\mu),\ \ i=1,3,$$

$$h_2T_2(t,\mu)=1-\frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}+O(\mu),$$

$$h_2Q(t,\mu)=-\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}+O(\mu),$$

$$h_3T_i(t,\mu)=[\overline{\phi}_i]+O(\mu),\ i=1,2,\ h_3T_3(t,\mu)=1+[\overline{\phi}_3]+O(\mu),\ h_3Q(t,\mu)=[\overline{f}]+O(\mu),$$

where

$$[\overline{\phi}_i]=\int\limits_0^1\frac{\overline{\phi}_i(s)}{y_{30}(s)A_1(s)}\left(y'_{30}(1)-\int\limits_s^1\sum_{j=0}^{2}a_j(x)y_{30}^{(j)}(x)-a_1(s)y_{30}(s)\right)ds-$$

$$-\frac{a_2(0)\overline{\phi}_i(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))}+\frac{\overline{\phi}_i(1)}{\kappa_1(1)(\kappa_2(1)-\kappa_1(1))}-\frac{a_2(1)\overline{\phi}_i(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))},\ \ i=\overline{1,3},$$

$$[\overline{f}]=\int\limits_0^1\frac{\overline{f}(s)}{y_{30}(s)A_1(s)}\left(y'_{30}(1)-\int\limits_s^1\sum_{j=0}^{2}a_j(x)y_{30}^{(j)}(x)-a_1(s)y_{30}(s)\right)ds- \tag{23}$$

$$-\frac{a_2(0)\overline{f}(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))}+\frac{\overline{f}(1)}{\kappa_1(1)(\kappa_2(1)-\kappa_1(1))}-\frac{a_2(1)\overline{f}(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))}.$$

Then for the main determinant of the system (22):

$$I(\mu)=\begin{vmatrix}h_1T_1(t,\mu) & h_1T_2(t,\mu) & h_1T_3(t,\mu)\\ h_2T_1(t,\mu) & h_2T_2(t,\mu) & h_2T_3(t,\mu)\\ h_3T_1(t,\mu) & h_3T_2(t,\mu) & h_3T_3(t,\mu)\end{vmatrix},$$

we get the following asymptotic representation:

$$I(\mu)=(1+[\overline{\phi}_3])\left(1-\frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)+\frac{\overline{\phi}_3(0)[\overline{\phi}_2]}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\equiv\overline{I}+O(\mu).$$

C5) $\overline{I}\neq 0$.

Then from the system (22), we determine the unknown constants $C_i,\ i=1,2,3$. Thus, the following theorem will be true.

Theorem 1 *If conditions (C1)-(C5) are satisfied, then the boundary value problem (1),(2) on the interval* $[0,1]$ *has a unique solution, expressed by formula (16), where* $T_i(t,\mu),\ i=1,2,3,\ Q(t,\mu)$ *are expressed by formula (17) and the constants* $C_i,\ i=1,2,3$ *are determined from system (22).*

For $C_i,\ i=1,2,3$ are valid asymptotic representations:

$$C_1=\alpha+O(\mu);\ C_2=\frac{1}{\overline{\overline{I}}}\left[\alpha\cdot\frac{\overline{\phi}_1(0)(1+[\overline{\phi}_3])-\overline{\phi}_3(0)[\overline{\phi}_1]}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}+\right.$$

$$\left.+\left(\beta+\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)(1+[\overline{\phi}_3])+(\gamma-[\overline{f}])\ \frac{\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right]+O(\mu);\quad(24)$$

$$C_3=\frac{1}{\overline{\overline{I}}}\left[\alpha\left(\frac{\overline{\phi}_2(0)[\overline{\phi}_1]-\overline{\phi}_1(0)[\overline{\phi}_2]}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}-[\overline{\phi}_1]\right)-\left(\beta+\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)[\overline{\phi}_2]+\right.$$

$$\left.+(\gamma-[\overline{f}])\left(1-\frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)\right]+O(\mu).$$

Theorem 2 *If conditions (C1)-(C5) are satisfied, then for solutions of problem (1),(2) are valid the following asymptotic representations:*

$$y^{(q)}(t,\mu)=\alpha\left[y_{30}^{(q)}(t)+\int\limits_0^t\frac{y_{30}^{(q)}(t)\overline{\phi}_1(s)}{y_{30}(s)A_1(s)}ds-\frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\cdot\overline{\phi}_1(t)-\right.$$

$$-\frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)}\left(y_{30}'(0)+\frac{\overline{\phi}_1(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))}\right)e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}-\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\times$$

$$\left.\times\left(\frac{h_3y_{30}(t)-a_2(0)y_{30}'(0)}{1-a_2(1)}-\frac{\overline{\phi}_1(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))}\right)e^{\frac{1}{\mu}\int\limits_1^t\kappa_2(x)dx}\right]+$$

$$+\frac{1}{\overline{\overline{I}}}\left[\alpha\frac{\overline{\phi}_1(0)(1+[\overline{\phi}_3])-[\overline{\phi}_1]\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}+\right.$$

$$\left.+\left(\beta+\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)(1+[\overline{\phi}_3])+\frac{(\gamma-[\overline{f}])\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right]\times$$

$$\times\left[\int\limits_0^t\frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)}ds-\frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\varepsilon^{q-1}(\kappa_2(t)-\kappa_1(t))}\cdot\overline{\phi}_2(t)-\right.$$

$$-\frac{y_{10}(t)\kappa_1^q(t)}{\mu^{q-1}\kappa_1(0)}\left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))}-1\right)e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}+$$

$$\left.+\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))}-\frac{a_2(0)}{1-a_2(1)}\right)e^{\frac{1}{\mu}\int\limits_1^t\kappa_2(x)dx}\right]+$$

$$+\frac{1}{\overline{\overline{I}}}\left[\alpha\left(\frac{\overline{\phi}_2(0)[\overline{\phi}_1]-[\overline{\phi}_2]\overline{\phi}_1(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}-[\overline{\phi}_1]\right)-\left(\beta+\frac{\overline{f}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)[\overline{\phi}_2]+\right.$$

$$+(\gamma-[\overline{f}])\left(1-\frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)\Bigg]\cdot\Bigg[\int\limits_0^t\frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)}ds- \tag{25}$$

$$-\frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\overline{\phi}_3(t)-\frac{\kappa_1^q(t)y_{10}(t)\overline{\phi}_3(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0)-\kappa_1(0))}e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}+$$

$$+\frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))}+\frac{1}{1-a_2(1)}\right)e^{\frac{1}{\mu}\int\limits_1^t\kappa_2(x)dx}\Bigg]+$$

$$+\int\limits_0^t\frac{y_{30}^{(q)}(t)\overline{f}(s)}{y_{30}(s)A_1(s)}ds-\frac{\kappa_1^q(t)y_{10}(t)\overline{f}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0)-\kappa_1(0))}e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}+$$

$$+\frac{\kappa_2^q(t)y_{20}(t)\overline{f}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1)-\kappa_1(1))}e^{\frac{1}{\mu}\int\limits_1^t\kappa_2(x)dx}+\frac{\kappa_1^{q-2}(t)-\kappa_2^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\cdot\overline{f}(t)+$$

$$+O\left(\mu+\frac{1}{\mu^{q-2}}e^{\frac{1}{\mu}\int\limits_0^t\kappa_1(x)dx}+\frac{1}{\mu^{q-2}}e^{\frac{1}{\mu}\int\limits_1^t\kappa_2(x)dx})\right),\quad q=\overline{0,2}.$$

The proof of the theorem follows from (16) taking account of (21), (24).

From formula (25) for solutions of problem (1), (2), we obtain the following order of growth:

$$y''(0,\mu)=O\left(\frac{1}{\mu}\right),\quad y''(1,\mu)=O\left(\frac{1}{\mu}\right),\quad \mu\to 0.$$

It implies that at points $t=0$ and $t=1$ the solution of the considered problem (1), (2) have the first order boundary jumps.

4 Modified degenerate problem

Singularly perturbed boundary value problem (1), (2), we put in accordance with the following modified degenerate problem:

$$L_0\overline{y}\equiv A_1(t)\overline{y}'(t)+A_2(t)\overline{y}(t)=F(t)+\int\limits_0^1\sum_{i=0}^2 H_i(t,x)\overline{y}^{(i)}(x)dx+\Theta(t), \tag{26}$$

$$h_1\overline{y}\equiv\overline{y}(0)=\alpha,\quad h_2\overline{y}\equiv\overline{y}'(0)=\beta+\Theta_0,$$

$$h_3\overline{y}\equiv\overline{y}'(1)-\int\limits_0^1\sum_{i=0}^2 a_i(x)\overline{y}^{(i)}(x)dx=\gamma+a_2(0)\Theta_0+(1-a_2(1))\Theta_1, \tag{27}$$

where $\Theta(t)$ and $\Theta_0,\ \Theta_1$ are called jumps of the integral term and solution respectively.

For the difference $u(t,\mu)$ between the solution $y(t,\mu)$ of the singularly perturbed problem (1), (2) and the solution $\overline{y}(t)$ of the degenerate problem, we get the problem:

$$\begin{aligned} L_\mu u &\equiv \mu^2 u''' + \mu A_0(t)u'' + A_1(t)u' + A_2(t)u = \\ &= \int_0^1 \sum_{i=0}^2 H_i(t,x)u^{(i)}(x,\mu)dx - \Theta(t) - \mu^2 y''' - \mu A_0(t)y'', \end{aligned} \tag{28}$$

$$h_1 u \equiv u(0,\mu) = 0, \quad h_2 u \equiv u'(0,\mu) = -\Theta_0,$$

$$h_3 u \equiv u'(1,\mu) - \int_0^1 \sum_{i=0}^2 a_i(x)u^{(i)}(x,\mu)dx = -a_2(0)\Theta_0 - (1-a_2(1))\Theta_1. \tag{29}$$

The problem (28),(29) is of the same type as the problem (1),(2), by applying asymptotic formula (25). As a result, we obtain asymptotic representation for the solution of the problem (28), (29):

$$\begin{aligned} u^{(q)}(t,\mu) = \frac{1}{\overline{\overline{I}}}\Bigg[-\left(\Theta_0 + \frac{\overline{\Theta}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)(1+[\overline{\phi}_3]) + \\ -\frac{\big(a_2(0)\Theta_0 + (1-a_2(1))\Theta_1 - [\overline{\Theta}]\big)\overline{\phi}_3(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\Bigg] \cdot \Bigg[\int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_2(s)}{y_{30}(s)A_1(s)}ds - \frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\overline{\phi}_2(t) - \\ -\frac{\kappa_1^q(t)y_{10}(t)}{\mu^{q-1}\kappa_1(0)}\left(\frac{\overline{\phi}_2(0)}{\kappa_1(0)(\kappa_2(0)-\kappa_1(0))} - 1\right)e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \\ + \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{\overline{\phi}_2(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))} - \frac{a_2(0)}{1-a_2(1)}\right)e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx}\Bigg] + \\ +\frac{1}{\overline{\overline{I}}}\Bigg[\left(\Theta_0 + \frac{\overline{\Theta}(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)[\overline{\phi}_2] + \\ + \big(a_2(0)\Theta_0 + (1-a_2(1))\Theta_1 - [\overline{\Theta}]\big)\left(1 - \frac{\overline{\phi}_2(0)}{\kappa_2(0)(\kappa_2(0)-\kappa_1(0))}\right)\Bigg] \cdot \Bigg[\int_0^t \frac{y_{30}^{(q)}(t)\overline{\phi}_3(s)}{y_{30}(s)A_1(s)}ds - \\ -\frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\overline{\phi}_3(t) - \frac{\kappa_1^q(t)y_{10}(t)\overline{\phi}_3(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0)-\kappa_1(0))}e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \\ + \frac{y_{20}(t)\kappa_2^q(t)}{\mu^{q-1}y_{20}(1)\kappa_2(1)}\left(\frac{\overline{\phi}_3(1)}{\kappa_2(1)(\kappa_2(1)-\kappa_1(1))} + \frac{1}{1-a_2(1)}\right)e^{\frac{1}{\mu}\int_1^t \kappa_2(x)dx}\Bigg] - \\ -\int_0^t \frac{y_{30}^{(q)}(t)\overline{\Theta}(s)}{y_{30}(s)A_1(s)}ds + \frac{\kappa_1^q(t)y_{10}(t)\overline{\Theta}(0)}{\mu^{q-1}\kappa_1^2(0)(\kappa_2(0)-\kappa_1(0))}e^{\frac{1}{\mu}\int_0^t \kappa_1(x)dx} + \end{aligned} \tag{30}$$

$$+\frac{\kappa_2^q(t)y_{20}(t)\overline{\Theta}(1)}{\mu^{q-1}\kappa_2^2(1)y_{20}(1)(\kappa_2(1)-\kappa_1(1))}e^{\frac{1}{\mu}\int\limits_1^t \kappa_2(x)dx}+\frac{\kappa_2^{q-2}(t)-\kappa_1^{q-2}(t)}{\mu^{q-1}(\kappa_2(t)-\kappa_1(t))}\overline{\Theta}(t)+$$
$$+O\left(\mu+\frac{1}{\mu^{q-2}}e^{-\gamma_1\frac{t}{\mu}}+\frac{1}{\mu^{q-2}}e^{-\gamma_2\frac{1-t}{\mu}}\right),\quad q=\overline{0,2},$$

where

$$\overline{\Theta}(t)\equiv\Theta(t)+\int\limits_0^1 R_0(t,s,1)\Theta(s)ds. \tag{31}$$

If the function $\overline{\Theta}(t)$ is selected in the form

$$\overline{\Theta}(t)=-\Theta_0\left(\overline{\phi}_2(t)+a_2(0)\overline{\phi}_3(t)\right)-\Theta_1(1-a_2(1))\overline{\phi}_3(t), \tag{32}$$

it can be presented that $u(t,\mu)\to 0,\ \mu\to 0$. From (32) in view of (18),(20), (31) to determine the jump of integral terms $\Theta(t)$, we obtain the following formula:

$$\Theta(t)=\Theta_0 H_2(t,0)-\Theta_1 H_2(t,1). \tag{33}$$

Theorem 3 *If conditions (C1)-(C5) are satisfied, then for the solution $y(t,\mu)$ of singularly perturbed integral boundary value problem (1),(2) are valid the limiting equalities:*

$$\lim_{\mu\to 0} y(t,\mu)=\overline{y}(t),\quad 0\le t\le 1,$$

$$\lim_{\mu\to 0} y^{(i)}(t,\mu)=\overline{y}^{(i)}(t),\ \ i=1,2,\ \ 0<t<1,$$

where $\overline{y}(t)$ is a solution of modified degenerate boundary value problem (26),(27), the jump of the integral terms $\Theta(t)$ is determined by formula (33).

Example. To illustrate the results, we give an example. For simplicity, we consider the integral boundary value problem for a singularly perturbed integro-differential equation with constant coefficients:

$$\begin{aligned}&L_\mu y\equiv\mu^2 y'''-\mu y''-2y'=1+\int\limits_0^1 \delta y''(x,\mu)dx,\ \ \delta\neq 0,\\&h_1 y\equiv y(0,\mu)=\alpha,\ \ h_2 y\equiv y'(0,\mu)=\beta,\\&h_3 y\equiv y'(1,\mu)-\int\limits_0^1 \nu y''(x,\mu)dx=\gamma,\ \ \nu\neq 0,\ \nu\neq 1,\end{aligned} \tag{34}$$

where

$$A_0(t)=-1,\ A_1(t)=-2,\ A_2(t)=0,\ F(t)=1,\ H_i(t,x)=0,\ i=0,1,$$
$$H_2(t,x)=\delta,\ a_i(x)=0,\ i=0,1,\ a_2(x)=\nu.$$

The exact solution to this problem is:

$$y(t,\mu)=\mu\cdot\frac{\delta(\gamma-\beta)+(2\beta+1)(1-\nu)+\left[\beta(\delta+2\nu)-\gamma(\delta+2)+\nu-1\right]e^{-\frac{2}{\mu}}}{2(1-\nu)\left(e^{-\frac{3}{\mu}}-1\right)}\cdot e^{-\frac{t}{\mu}}+$$

$$+\mu\cdot\frac{\beta(\delta+2\nu)-\gamma(\delta+2)+\nu-1+\left[\delta(\gamma-\beta)+(2\beta+1)(1-\nu)\right]e^{-\frac{1}{\mu}}}{4(1-\nu)\left(e^{-\frac{3}{\mu}}-1\right)}\cdot e^{-\frac{2(t-1)}{\mu}}+\alpha-$$

$$-\mu\cdot\frac{2\left[\delta(\gamma-\beta)+(2\beta+1)(1-\nu)\right]+3\left[\beta(\delta+2\nu)-\gamma(\delta+2)+\nu-1\right]e^{-\frac{2}{\mu}}}{4(1-\nu)\left(e^{-\frac{3}{\mu}}-1\right)}-$$

$$-\mu\cdot\frac{\left[\delta(\gamma-\beta)+(2\beta+1)(1-\nu)\right]e^{-\frac{3}{\mu}}}{4(1-\nu)\left(e^{-\frac{3}{\mu}}-1\right)}+\frac{\delta(\beta-\gamma)}{2(1-\nu)}\cdot t-\frac{t}{2}.$$

Hence, passing to the limit at $\mu\to 0$, we get

$$\lim_{\mu\to 0}y(t,\mu)=\alpha+\frac{\delta(\beta-\gamma)+\nu-1}{2(1-\nu)}t\equiv\overline{y}(t),\quad 0\le t\le 1, \tag{35}$$

where $\overline{y}(t)$ is not a solution to the usual degenerate equation, obtained from (1) at $\mu=0$, and is a solution of modified degenerate boundary value problem:

$$-2\overline{y}'=1+\int_0^1\delta\overline{y}''(x)dx+\Theta(t), \tag{36}$$

$$\begin{aligned}&h_1\overline{y}\equiv\overline{y}(0)=\alpha,\quad h_2\overline{y}\equiv\overline{y}'(0)=\beta+\Theta_0,\\&h_3\overline{y}\equiv\overline{y}'(1)-\int_0^1\nu\overline{y}''(x)dx=\gamma+\nu\Theta_0+(1-\nu)\Theta_1,\end{aligned} \tag{37}$$

where $\Theta_0,\ \Theta_1$ are jumps of the solution at points $t=0$ and $t=1$ respectively, determined from problem (4.11), (4.12) and the value of the jump of the integral terms $\Delta(t)$ in accordance with formula (4.8) should be determined in the form:

$$\Theta(t)=\delta(\Theta_0-\Theta_1). \tag{38}$$

The exact solution of the problem (36), (37) according to the condition (38) has the form

$$\overline{y}(t)=\alpha+\frac{\delta(\beta-\gamma)+\nu-1}{2(1-\nu)}t.$$

Therefore, the passage to the limit (35) will be valid. The limit transitions are similarly valid.

$$\lim_{\mu\to 0}y^{(i)}(t,\mu)=\overline{y}^{(i)}(t),\quad i=1,2,\quad 0<t<1.$$

5 Conclusion

In this paper, we study the boundary value problem for singularly perturbed integro-differential equations with the phenomena of the so-called boundary jumps, when the fast solution variable becomes unbounded at both boundaries. The exceptions of the qualitative influence of integral terms on the asymptotic behavior of the solutions for singularly perturbed integro-differential equations are shown. The presence of integral terms will significantly change the degenerate equation: the solution of the assumed singularly perturbed integro-differential equation does not tend to the solution of the usual degenerate equation, obtained from the supposed equation with the zero value of a small parameter and will tend to solve a specially modified degenerate integro-differential equation with an additional term called the jump of the integral term. Moreover, the value of the jump of the integral term is determined differently than in [33] because of occurence of jumps in solutions of the same order. In addition, in the boundary conditions of the modified degenerate problem, there is also a change. The so-called jumps of the first derivative of the solution appear $\Theta_0 = \overline{y}'(0) - y'(0,\mu) \neq 0$ and $\Theta_1 = \overline{y}'(1) - y'(1,\mu) \neq 0$ at the ends of considered interval.

References

[1] Schlesinger L. "Uber asymptotische darstellungen der losungen linearer differential systeme als funktionen eines parameters" , *Mathematische Annalen,* (1907); 63(3): 277–300.

[2] Birkhoff GD. "On the asymptotic character of the solutions of certain linear differential equations containing a parameter" , *Transactions of the American Mathematical Society,* (1908); 9(2): 219–231.

[3] Noaillon P. "Developpements asymptotiques dans les equations differentielles lineaires a parametre variable" , *Mem. Soc. Sci. Liege,* (1912); 3(11): 197.

[4] Wasow W. "Singular perturbations of boundary value problems for nonlinear differential equations of the second order" , *Comm. On Pure and Appl. Math.,* (1956); 9 : 93–113.

[5] Nayfeh A. "Perturbation Methods" , *New York, USA: John Wiley,* (1973).

[6] Tihonov AN. "O zavisimosti reshenij differencial'nyh uravnenij ot malogo parametra" , *Matematicheskij sbornik* (1948); 22(2): 193–204 (in Russian).

[7] Tihonov AN. "O sistemah differencial'nyh uravnenij soderzhashhih parametry" , *Matematicheskij sbornik,* (1950); 27(69): 147–156 (in Russian).

[8] Vishik MI, Lyusternik LA. "Regular degeneration and boundary layer for linear differential equations with small parameter multiplying the highest derivatives" , *Usp.Mat. Nauk,* (1957); 12: 3–122 (in Russian), Amer. Math. Soc. Transl. 1962; 20(2): 239–364.

[9] Vishik MI, Lyusternik LA. "On the initial jump for non-linear differential equations containing a small parameter" ,*Doklady Akademii Nauk SSSR,* (1960); 132(6): 1242–1245 (in Russian).

[10] Bogoliubov N, Mitropolskii YA. "Asymptotic methods in the theory of nonlinear oscillations" ,*Delhi: Hindustan Publ. Corp.,* (1961).

[11] Vasil'eva A, Butuzov V. "Singularly perturbed differential equations of parabolic type in Asymptotic Analysis II" , *Lecture Notes in Math. Berlin: Springer-Verlag,* (1983).

[12] Vasil'eva A, Butuzov V, Kalachev L. "The boundary function method for singular perturbation problems" ,*Philadeplhia: SIAM Studiesin Applied Mathematics,* (1995).

[13] O'Malley R. "Introduction to singular perturbations" ,*New York, USA: Academic Press,* (1974).

[14] O'Malley R. "Singular perturbations methods for ordinary differential equations" , *Berlin: Springer-Verlag,* (1991).

[15] Smith DR. "Singular-perturbation theory" ,*Cambridge: Cambridge University Press,* (1985).

[16] Eckhaus W. "Matched asymptotic expansions and singular perturbations" , *London: North-Holland,* (1973).

[17] Chang K, Howes F. "Nonlinear Singular Perturbation Phenomena: Theory and Application" ,*New York, USA: Springer Verlag,* (1984).

[18] Kevorkian J, Cole JD. "Singular perturbation methods in applied mathematics" , *Berlin: Springer-Verlag,* (1981).

[19] Sanders, Verhulst F. "Averaging methods in nonlinear dynamical systems" , *Berlin: Springer-Verlag,* (1985).

[20] Mishchenko E, Rozov N. "Differential equations with small parameter and relaxation oscillations" , *New York, USA: Plenum Press,* (1980).

[21] Lomov S. "Introduction to the general theory of singular perturbations" , *American Mathematical Society, Providence,* (1992).

[22] Imanaliev MI. "Asimptoticheskie metody v teorii singulyarno vozmushchennyh integro-differencial'nyh sistem" , *Frunze: Ilim,* (1972) (in Russian).

[23] Kassymov KA. Ob asimptotike reshenija zadachi Koshi s bol'shimi nachal'nymi uslovijami dlja nelinejnogo obyknovennogo differencial'nogo uravnenija, soderzhashhego malyj parametr , *Uspehi matematicheskih nauk,* (1962); 17(5): 187–188 (in Russian).

[24] Kassymov KA. "O zadache s nachal'nym skachkom dlja nelinejnyh sistem differencial'nyh uravnenij soderzhashhih malyj parametr" , *DAN SSSR* (1968); 179(2): 275–278 (in Russian).

[25] Kassymov KA. "Asimptotika reshenija zadachi s nachal'nymi skachkami dlja sistemy differencial'nyh uravnenij giperbolicheskogo tipa s malym parametrom pri proizvodnoj" , *DAN SSSR,* (1971); 196(2): 274–277 (in Russian).

[26] Vishik MI, Lyusternik LA. "O nachal'nom skachke dlya nelinejnyh differencial'nyh uravnenij, soderzhashchih malyj parametr" , *DAN SSSR,* (1960); 132(6): 1242–1245 (in Russian).

[27] Kasymov KA. "Ob asimptotike resheniya zadachi Koshi s bol'shimi nachal'nymi usloviyami dlya nelinejnogo obyknovennogo differencial'nogo uravneniya, soderzhashchego malyj parametr" , *UMN,* (1962); 17(5): 187–188 (in Russian).

[28] Kassymov KA, Nurgabyl D. "Asymptotic estimates of solution of a singularly perturbed boundary value problem with an initial jump for linear differential equations" , *Differential Equations* (2004); 40(5): 641–651.

[29] Dauylbayev MK, Atakhan N. "The initial jumps of solutions and integral terms in singular BVP of linear higher order integro-differential equations" , *Miskolc Math. Notes* (2015); 16(2): 747–761.

[30] Kassymov KA, Dauylbaev MK. "Estimation of solutions to problems associated with linear singularly perturbed integro-differential equations" , *Differential Equations,* (1999); 35(6): 822–830.

[31] Dauylbaev MK. "The asymptotic behavior of solutions to singularly perturbed nonlinear integro-differential equations" ,*Siberian Mathematical Journal,* (2000); 41(1): 49–60.

[32] Dauylbaev MK, Mirzakulova AE. "Boundary value problems with initial jumps for singularly perturbed integrodifferential equations" , *Journal of Mathematical Sciences,* (2017); 222(3): 214–225. doi: 10.1007/s10958-017-3294-7

[33] Dauylbaev MK, Mirzakulova AE. "Asymptotic behavior of solutions of singular integro-differential equations" , *Journal of Discontinuity, Nonlinearity, and Complexity,* (2016); 5(2): 147–154.

[34] Akhmet M, Dauylbayev M, Mirzakulova A. "A singularly perturbed differential equation with piecewise constant argument of generalized type" ,*Turkish Journal of Mathematics,* (2018); 42(4): 1680–1685.

[35] Dauylbayev, M.K., Uaissov, A.B. "Asymptotic Behavior of the Solutions of Boundary-Value Problems for Singularly Perturbed Integrodifferential Equations" , *Ukrainian Mathematical Journal,* (2020); 71(11): 1677-1691

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ON ROOT FUNCTIONS OF NONLOCAL DIFFERENTIAL SECOND-ORDER OPERATOR WITH BOUNDARY CONDITIONS OF PERIODIC TYPE

In this paper we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. Such problems arise when solving by the method of separation of variables for a nonlocal heat equation. We investigate spectral properties of the problem for the nonlocal ordinary differential equation $\mathcal{L}y\left(x\right) \equiv -y''\left(x\right) + \varepsilon y''\left(-x\right) = \lambda y\left(x\right)$, $-1 < x < 1$. Here λ is a spectral parameter, $|\varepsilon| < 1$. Such equations are called nonlocal because they have a term $y''\left(-x\right)$ with involutional argument deviation. Boundary conditions are nonlocal $y'\left(-1\right) + ay'\left(1\right) = 0$, $y\left(-1\right) - y\left(1\right) = 0$. Earlier this problem has been investigated for the special case $a = -1$. We consider the case $a \neq -1$. A criterion for simplicity of eigenvalues of the problem is proved: the eigenvalues will be simple if and only if the number $r = \sqrt{\left(1-\varepsilon\right)/\left(1+\varepsilon\right)}$ is irrational. We show that if the number r is irrational, then all the eigenvalues of the problem are simple, and the system of eigenfunctions of the problem is complete and minimal but does not form an unconditional basis in $L_2(-1,1)$. For the case of rational numbers r, it is proved that a (chosen in a special way) system of eigen- and associated functions forms an unconditional basis in $L_2(-1,1)$.
Key words: Nonlocal differential operator, spectrum, eigenvalue, multiplicity of eigenvalues, eigenfunction, associated function, unconditional basis.

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Шекаралық шарттары периодты типті екінші ретті локалді емес дифференциалдық операторлдың түбірлік функциялары туралы

Бұл жұмыста локальды емес периодты шекаралық шартты қарапайым локальды емес дифференциалды оператор (басты бөлігінде инволюция бар) үшін спектрлік есептердің бір класы қарастырылды. Мұндай есептер локальды емес жылу өткізгіштік теңдеу үшін қойылған есептерді айнымалыларды ажырату әдісімен шешкенде пайда болады. Біз $\mathcal{L}y\left(x\right) \equiv -y''\left(x\right) + \varepsilon y''\left(-x\right) = \lambda y\left(x\right)$, $-1 < x < 1$ түріндегі локалды емес қарапайым дифференциалдық теңдеуге қойылған есептердің спектрлік қасиеттерін зерттейміз. Мұндағы λ - спектрлік параметр, $|\varepsilon| < 1$. Мұндай теңдеулер локальды емес, себебі оның $y''\left(-x\right)$ түріндегі аргументтің инволютиционды қ ауытқуы бар мүшесі болады. $y'\left(-1\right) + ay'\left(1\right) = 0$, $y\left(-1\right) - y\left(1\right) = 0$-локальды емес шекаралық шарт болып табылады. Бұрын бұл есептің $a = -1$ кезіндегі дербес жағдайы қарастырылды. Біз бұл есептің $a \neq -1$ жағдайын қарастырамыз.

Біз бұл есептің меншікті мәндерінің қарапайымдылық критериин дәлелдедік: меншікті мәндері қарапайым болады сонда тек сонда ғана, егер $r = \sqrt{\left(1-\varepsilon\right)/\left(1+\varepsilon\right)}$ ирроционал болса. Егер r ирроционал болса, онда есептің меншікті мәндерінің барлығы қарапайым болатынын, бірақ $L_2(-1,1)$-де сөзсіз базис болмайтынын көрсеттік. r рационал сан кезінде, меншікті және косылған функциялар $L_2(-1,1)$-де сөзсіз базис болатыны (арнайы таңдап алынған) дәлелденген.

Түйін сөздер: Локальды емес дифференциалдық оператор, спектр, меншікті мәндер, меншікті мәндердің еселігі, меншікті функция, сөзсіз базис.

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О корневых функциях нелокального дифференциального оператора второго порядка с краевыми условиями периодического типа

В настоящей работе рассматривается один класс спектральных задач для нелокального обыкновенного дифференциального оператора (с инволюцией в главной части) с нелокальными краевыми условиями. периодического типа. Такие задачи возникают при решении методом разделения переменных задач для нелокального уравнения теплопроводности. Мы исследуем спектральные свойства задачи для нелокального обыкновенного дифференциального уравнения $\mathcal{L}y\left(x\right) \equiv -y''\left(x\right) + \varepsilon y''\left(-x\right) = \lambda y\left(x\right)$, $-1 < x < 1$. Здесь λ - спектральный параметр, $|\varepsilon| < 1$. Такие уравнения называются нелокальными, так как они содержат член $y''\left(-x\right)$ с инволютиционным отклонением аргумента. Краевые условия являются нелокальными $y'\left(-1\right) + ay'\left(1\right) = 0$, $y\left(-1\right) - y\left(1\right) = 0$. Ранее эта задача была исследована для частного случая $a = -1$. Нами рассматривается случай $a \neq -1$. Нами доказан критерий простоты собственных значений задачи: собственные значения будут простыми если и только если число $r = \sqrt{\left(1-\varepsilon\right)/\left(1+\varepsilon\right)}$ является иррациональным. Мы показали, что если число r иррациональное, то собственные значения задачи - все простые, а система собственных функций задачи является полной и минимальной, но не образует безусловного базиса в $L_2(-1,1)$. Для случая рациональных r доказано, что (специальным образом выбранная) система собственных и присоединенных функций образует безусловный базис в $L_2(-1,1)$.

Ключевые слова: Нелокальный дифференциальный оператор, спектр, собственное значение, кратность собственных значений, собственная функция, присоединенная функция, безусловный базис.

1 Introduction

It is well known that many spectral problems for ordinary differential operators arise in using the method of separation of variables (Fourier method) to solve initial-boundary value problems for evolution equations. Due to the fact that spectral properties of self-adjoint problems are well studied and the system of eigenvectors of self-adjoint operators forms an orthonormal basis, researchers use self-adjoint boundary conditions to model various processes of natural science. And in order to use more complex differential equations and/or more complex boundary conditions in modeling, it is necessary to develop the spectral theory of nonlocal operators. Such operators, as a rule, are non-self-adjoint. Therefore, nothing is known about their spectral properties and additional research is required in each particular case.

In this paper we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. Such problems arise when solving by the method of separation of variables for a nonlocal heat equation.

For example, one can consider a problem of modeling thermal diffusion process which is close to one described in the article of Cabada and Tojo [1], where the example is given that describes a specific physical situation. We consider a closed metal wire (length 2) that is wrapped around a thin sheet of insulation material. Assume that the position $x = 0$ is the most low in the wire, and the wire goes around the insulation up to the left to the point $x = -1$ and to the right to the point $x = 1$. Since the wire is closed, then the points $x = -1$ and $x = 1$ physically coincide. It is assumed that the insulating layer is slightly permeable. Hence, the value of the temperature $u(x, t)$ (at point x of the wire at time t) on one side of the insulation affects the diffusion process on the other side of the insulation, at point $(-x, t)$. For this reason, the standard heat equation is modified by adding an additional term $\varepsilon u_{xx}(-x, t)$ to the "classical" term $u_{xx}(x, t)$. Thus, this process is described by the nonlocal heat equation

$$u_t(x, t) - u_{xx}(x, t) + \varepsilon u_{xx}(-x, t) = f(x, t), \tag{1}$$

in $\Omega = \{(x, t) : -1 < x < 1, 0 < t < T\}$. Here $f(x, t)$ is a function of influence of an external source; $|\varepsilon| < 1$ is a coefficient depending on the permeability of the insulating layer $t = 0$ is an initial moment of time; $t = T$ is a final moment of time.

The initial temperature distribution in the wire is considered known:

$$u(x, 0) = \tau(x), \quad -1 \leq x \leq 1. \tag{2}$$

Since the wire is closed, it is natural to assume that the temperature at the ends of wire is the same:

$$u(-1, t) = u(1, t), \quad 0 \leq t \leq T. \tag{3}$$

If we consider the case when an additional external thermal effect occurs at the junction of the ends of the wire then boundary conditions of periodic type but non-self-adjoint, arise. Consider the process, where the temperature flux at one end at each time t is proportional to the rate of change of the average temperature over the entire wire. After non-singular transformations such a boundary condition can be reduced to the form

$$u_x(-1, t) + a u_x(1, t) = 0, \quad 0 \leq t \leq T. \tag{4}$$

Here a is a certain coefficient characterizing the proportionality of the temperature flux at one end and the rate of change of the average temperature over the entire wire.

Such a mathematical model can serve as a direct justification of the need to consider the nonlocal differential equations and the nonlocal boundary conditions for them. Our paper is devoted to the investigation of spectral properties of the problem arising when solving the formulated problem (1)–(4) by the method of separation of variables.

2 Materials and methods

3 Spectral problem

The use of the Fourier method for solving problem (1)–(4) leads to a spectral problem for the operator $\mathcal{L}$ given by the differential expression

$$\mathcal{L}y\,(x) \equiv -y''\,(x) + \varepsilon y''\,(-x) = \lambda y\,(x)\,, \;-1 < x < 1, \tag{5}$$

and the boundary conditions of periodic type

$$\begin{cases} U_1(y) \equiv y'(-1) + ay'(1) = 0, \\ U_2(y) \equiv y(-1) - y(1) = 0, \end{cases} \tag{6}$$

where λ is a spectral parameter.

Spectral problems for Eq. (5) were first considered, apparently, in [2], [3]. Cases of the Dirichlet and Neumann boundary$a = -1$ were considered. Cases of the boundary conditions when the system of root vectors forms a Riesz basis in L_2 were singled out. Here we consider a case $a \neq -1$. This case has not yet been investigated before.

Close spectral problems were considered in the works [4]- [9]. In [4] for Eq. (5) a problem with the nonlocal conditions

$$y(-1) = 0, \quad y'(-1) = y'(1)$$

was studied. It was proved that if $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is irrational, then the system of eigenfunctions is complete and minimal in $L_2(-1,1)$ but is not an unconditional basis. For rational r, a method for choosing associated functions for which the system of root functions of the problem is the unconditional basis in $L_2(-1,1)$ was indicated. A similar result was proved in [5] for the case of the space $L_p(-1,1)$.

A problem for Eq. (5) with the nonlocal boundary conditions

$$y(-1) = \beta y(1), \quad y'(-1) = y'(1)$$

was investigated in [8] for the case of the space $L_2(-1,1)$ and in [9] for the space $L_p(-1,1)$. In these papers it was also shown that the multiplicity of eigenvalues depends on the rationality or irrationality of the number r.

Since for Eq. (5) the spectral theory of boundary value problems is not yet fully formed, then each separate case of boundary conditions must be considered separately. The spectral problems with the nonlocal conditions (6) have not been previously considered. In this connection, we note the works [10]– [18] in which close problems related with spectral properties of nonlocal problems were considered.

4 General solution of equation (5)

To construct a general solution of equation (5), consider the Cauchy problem with data at the interior point

$$\mathcal{L}y(x) \equiv -y''(x) + \varepsilon y''(-x) - \lambda y(x) = f(x), \ -1 < x < 1, \tag{7}$$

$$y(0) = A, \ y'(0) = B, \tag{8}$$

with arbitrary constants A and B. Here $f(x) \in C[-1,1]$.

By direct calculation it is easy to show that this problem (7) to (8) is equivalent to the integral equation

$$y(x) + \lambda \int_{-x}^{x} k(x,t) y(t) dt = A + Bx - \int_{-x}^{x} k(x,t) f(t) dt, \tag{9}$$

with the integral operator

$$\int_{-x}^{x} k(x,t)\varphi(t)dt \equiv \frac{1}{1-\alpha^2}\left\{\alpha\int_{-x}^{0}(x+t)\varphi(t)dt + \int_{0}^{x}(x-t)\varphi(t)dt.\right\}$$

Let us show that the integral equation (9) has a unique solution. For this, we introduce a new function

$$Y(x) = y(x)e^{-\mu|x|},$$

where $\mu > 0$ is a positive parameter which we will choose below.

Then for $Y(x)$ we obtain the integral equation

$$Y(x) + \lambda\int_{-x}^{x} k_1(x,t)Y(t)dt = \psi(x), \tag{10}$$

where it is indicated

$$k_1(x,t) = k(x,t)e^{-\mu[|x|-|t|]},$$

$$\psi(x) = (A+Bx)\,e^{-\mu|x|} - \int_{-x}^{x} k_1(x,t)f_1(t)dt,$$

$$f_1(x) = f(x)e^{-\mu|x|}.$$

By I_λ denote the integral operator in the left-hand side of (10). Estimating its norm in $L_2(-1,1)$, we have

$$\|I_\lambda\| \le \frac{|\lambda|}{1-\alpha^2}\frac{\sqrt{2\mu-1+e^{-2\mu}}}{\mu}.$$

Hence it is easy to see that for any λ we always can choose a positive number $\mu > 0$ such that the operator norm will be less than one: $\|I_\lambda\| \le \delta < 1$. Therefore, with this choice of μ, equation (10) has the unique solution $Y(x) \in L_2(-1,1)$.

That is why equation (9) has the unique solution $y(x) \in L_2(-1,1)$. Further, it is easy to justify by the classical formula that $y(x) \in C^2[-1,1]$ for $f(x) \in C[-1,1]$. Thus, it is proved

Lemma 1. *For any values of the parameter λ, of the constants A and B and for any function $f(x) \in C[-1,1]$ the Cauchy problem* (7) *to* (8) *has the unique solution $y(x) \in C^2[-1,1]$.*

As follows from this lemma, the general solution of equation (5) is two-parameter. As fundamental solutions we choose two functions $c(x,\lambda) \in C^2[-1,1]$ and $s(x,\lambda) \in C^2[-1,1]$ which are solutions of equation (5) and satisfy the Cauchy conditions:

$$c(0,\lambda) = s'(0,\lambda) = 1,\ c'(0,\lambda) = s(0,\lambda) = 0.$$

The existence of such solutions is ensured by Lemma 1.

By direct calculation it is easy to obtain these solutions explicitly:

$$c(x,\lambda) = \cos(\mu_1 x),\ s(x,\lambda) = \frac{1}{\mu_2}\sin(\mu_2 x),\quad \mu_1 = \sqrt{\frac{\lambda}{1-\varepsilon}},\quad \mu_2 = \sqrt{\frac{\lambda}{1+\varepsilon}}.$$

It is also easy to verify that the chosen solutions have the following symmetry properties:

$$c(-x,\lambda)=c(x,\lambda),\quad s(-x,\lambda)=-s(x,\lambda),\quad -1\le x\le 1. \tag{11}$$

Thus, the general solution of equation (5) has the form:

$$y\left(x,\lambda\right)=C_1c(x,\lambda)+C_2s(x,\lambda) \tag{12}$$

with arbitrary constants C_1 and C_2.

5 Eigenvalues of problem (5) to (6)

First of all, it is easy to see that $\lambda=0$ is an eigenvalue of problem (5) to (6). The corresponding eigenfunction has the form:

$$y_0(x)=1.$$

Consider a case $\lambda\neq 0$. Satisfying the general solution (12) of equation (5) to the boundary conditions (6), we get the linear system

$$\begin{cases} C_1U_1\left(c(x,\lambda)\right)+C_2U_1\left(s(x,\lambda)\right)=0,\\ C_1U_2\left(c(x,\lambda)\right)+C_2U_2\left(s(x,\lambda)\right)=0. \end{cases} \tag{13}$$

Its determinant will be the characteristic determinant of the spectral problem (5) to (6):

$$\triangle(\lambda)\equiv\begin{vmatrix} U_1\left(c(x,\lambda)\right) & U_1\left(s(x,\lambda)\right)\\ U_2\left(c(x,\lambda)\right) & U_2\left(s(x,\lambda)\right)\end{vmatrix}=0.$$

Therefore, taking into account the symmetry conditions (11), we calculate

$$\triangle(\lambda)\equiv 2(1-a)c'(1,\lambda)s(1,\lambda)=0. \tag{14}$$

First of all from (14) we get that for $a=1$ each number λ is the eigenvalue of problem (5) to (6), regardless of the value of ε. In this case system (13) has the form

$$\begin{cases} C_2s'(1,\lambda)=0,\\ C_2s(1,\lambda)=0. \end{cases}$$

Since $|s'(1,\lambda)|+|s(1,\lambda)|>0$, then it follows that $C_2=0$. Thus, it is proved

Lemma 2. *For $a=1$ each number λ is the eigenvalue of problem* (5) *to* (6). *Corresponding eigenfunctions have the form*

$$y\left(x,\lambda\right)=\cos\left(\mu_1x\right),\ \mu_1=\sqrt{\frac{\lambda}{1-\varepsilon}}. \tag{15}$$

Now consider the case when $a\neq 1$. Then from (14) we obtain $c'(1,\lambda)s(1,\lambda)=0$. Therefore, taking into account the explicit form of fundamental solutions, we have

$$\sin\left(\mu_1\right)\sin\left(\mu_2\right)=0.$$

Thus, problem (5) to (6) has two series of the eigenvalues

$$\begin{aligned} &\lambda_k^{(1)} = (1-\varepsilon)(k\pi)^2, \quad k = 0,1,2,..., \\ &\lambda_n^{(2)} = (1+\varepsilon)(n\pi)^2, \quad n = 1,2,.... \end{aligned} \tag{16}$$

Lemma 3. *Problem* (5) *to* (6) *has multiple eigenvalues if and only if the number* $r = \sqrt{(1-\varepsilon)\,/\,(1+\varepsilon)}$ *is rational.*

Proof. Indeed, suppose that any two eigenvalues from different series coincide:

$$\lambda_k^{(1)} = \lambda_n^{(2)}.$$

This is equivalent to the equality

$$(1-\varepsilon)(k\pi)^2 = (1+\varepsilon)(n\pi)^2.$$

That is, the coincidence of eigenvalues is possible if and only if for some $k_0, n_0 \in N$ $r = n_0/k_0$ holds. That is, only if the value r is rational.

6 Spectral problem for irrational numbers r

Let r be an irrational number. Then, by virtue of Lemma 3, all eigenvalues of problem (5) to (6) are simple and are given by the formulas (16). By direct calculation from (13) we get that

$$\begin{aligned} &y_k^{(1)}(x) = \cos(k\pi x), \\ &y_n^{(2)}(x) = (1+a)r\cos(n\pi)\cos\left(\tfrac{n\pi x}{r}\right) + (a-1)\sin\left(\tfrac{n\pi}{r}\right)\sin(n\pi x), \end{aligned} \tag{17}$$

where $k = 0,1,2,...$ and $n = 1,2,...$, correspond to these eigenvalues.

Lemma 4. *The system of functions* (17) *is complete and minimal in* $L_2(-1,1)$.

Proof Consider an arbitrary function $f(x)$ orthogonal to system (17). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0,1,2,...$, then we have

$$0 = \int_{-1}^{1} f(x)\cos(k\pi x)dx = \int_0^1 \Big\{f(x) + f(-x)\Big\}\cos(k\pi x)dx.$$

But the system $\{\cos(k\pi x), k = 0,1,2,...\}$ forms a basis in $L_2(0,1)$. Therefore, $f(x)+f(-x) = 0$ holds almost everywhere on the interval $(-1,1)$. That is, this function is odd.

Therefore, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$, $n = 1,2,...$ we obtain

$$0 = \int_{-1}^{1} f(x)y_n^{(2)}(x)dx = (a-1)\sin\left(\frac{n\pi}{r}\right)\int_{-1}^{1} f(x)\sin(n\pi x)dx.$$

Since $a \neq 1$ and the number r is irrational, then from this we have

$$0 = \int_{-1}^{1} f(x)\sin(n\pi x)dx = \int_0^1 \Big\{f(x) - f(-x)\Big\}\sin(n\pi x)dx.$$

But the system $\{\sin(n\pi x), n = 1, 2, ...\}$ forms a basis in $L_2(0, 1)$. Therefore, $f(x) - f(-x) = 0$ holds almost everywhere on the interval $(-1, 1)$. That is, this function is even.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1, 1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1, 1)$. This proves the completeness of the system of functions (17) in $L_2(-1, 1)$.

Since the system under consideration (17) is a system of eigenfunctions of an linear operator, then it has a biorthogonal system consisting of eigenfunctions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. But from the existence of the biorthogonal system follows the minimality of the system of functions (17) in $L_2(-1, 1)$. Lemma is proved.

Now let us show that despite the fact that the system of functions (17) is complete and minimal in $L_2(-1, 1)$, it does not form an unconditional basis. For this, we use the necessary condition for the basis property from [19].

Lemma 5. ([19], Th. 3.135, s. 219) *Let $\{u_j\}$ be a closed and minimal system in a Hilbert space H. If the system $\{u_j\}$ is an unconditional basis in H, then the strict inequality holds*

$$\limsup_{j\to\infty} \left| \left\langle \frac{u_j}{\|u_j\|}, \frac{u_{j+1}}{\|u_{j+1}\|} \right\rangle \right| < 1, \tag{18}$$

where $\langle \cdot, \cdot \rangle$ is the inner products in H.

By virtue of this lemma, for the unconditional basis property in $L_2(-1, 1)$ of the system of functions (17), it is necessary to satisfy the strict inequality

$$\limsup_{j\to\infty} \left| \left\langle \frac{y_{k_j}^{(1)}}{\left\|y_{k_j}^{(1)}\right\|}, \frac{y_{n_j}^{(2)}}{\left\|y_{n_j}^{(2)}\right\|} \right\rangle \right| < 1 \tag{19}$$

for all possible infinitely increasing subsequences k_j and n_j.

Calculating the norms of the eigenfunctions, we obtain

$$\left\|y_k^{(1)}\right\| = 1,$$

$$\left\|y_n^{(2)}\right\|^2 = (1+a)^2 r^2 \left\{1 + \frac{r}{2n\pi} \sin\left(\frac{2n\pi}{r}\right)\right\} + (1-a)^2 \sin^2\left(\frac{n\pi}{r}\right).$$

Therefore,

$$\left\|y_n^{(2)}\right\|^2 = (1+a)^2 r^2 + (1-a)^2 \sin^2\left(\frac{n\pi}{r}\right) + O\left(\frac{1}{n}\right) \tag{20}$$

for $n \to \infty$.

Calculate the inner products in $L_2(-1, 1)$:

$$\left|\left\langle y_k^{(1)}, y_n^{(2)} \right\rangle\right| = |1+a| r \left| \int_{-1}^{1} \cos(k\pi x) \cos\left(\frac{n\pi x}{r}\right) dx \right| =$$

$$= |1+a| r \left| \frac{\sin\left(k - \frac{n}{r}\right)\pi}{\left(k - \frac{n}{r}\right)\pi} + O\left(\frac{1}{k+n}\right) \right|$$

for $k, n \to \infty$.

According to the Dirichlet's approximation theorem (see, example, [20], Th. 1A, p. 34), for any irrational number α there exists an infinite set of irreducible fractions $\frac{p}{q}$ (where p and q are integers) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}.$$

Choosing here $\alpha = \frac{1}{r}$, we get that there exist infinite subsequences of the natural numbers k_j and n_j such that

$$\left|\frac{1}{r} - \frac{k_j}{n_j}\right| < \frac{1}{n_j^2}.$$

For these subsequences we will have

$$\left|k_j - \frac{n_j}{r}\right| < \frac{1}{n_j}.$$

Therefore, there exists the limit

$$\lim_{j\to\infty} \frac{\sin\left(k_j - \frac{n_j}{r}\right)\pi}{\left(k_j - \frac{n_j}{r}\right)\pi} = 1.$$

From this we have that the limit exists

$$\lim_{j\to\infty} \left|\left\langle y_{k_j}^{(1)}, y_{n_j}^{(2)}\right\rangle\right| = |1+a|r.$$

From (20) it is easily seen that the limit exists

$$\lim_{j\to\infty} \left\|y_{n_j}^{(2)}\right\| = |1+a|r.$$

Finally, substituting everything obtained in (21), we get

$$\lim_{j\to\infty} \left|\left\langle \frac{y_{k_j}^{(1)}}{\left\|y_{k_j}^{(1)}\right\|}, \frac{y_{n_j}^{(2)}}{\left\|y_{n_j}^{(2)}\right\|}\right\rangle\right| = 1 \tag{21}$$

for our chosen (according to the Dirichlet's approximation theorem) infinitely increasing subsequences k_j and n_j. That is, the necessary condition of the unconditional basis property (3.3) is not satisfied. Thus, the following lemma is proved.

Lemma 6. *Let r be irrational. Then the system of eigenfunctions (3.2) of the spectral problem (1.8), (1.9) is complete and minimal but does not form an infinite basis in $L_2(-1, 1)$.*

7 Spectral problem for rational numbers r

Now consider the case when r is a rational number. Then there exist natural numbers n_0 and k_0 such that $r = \frac{n_0}{k_0}$. In this case, as follows from (16), problem (5) to (6) has an infinite number of double eigenvalues

$$\lambda_{k_0 j}^{(1)} = \lambda_{n_0 j}^{(2)}, \quad j \in \mathbb{N}. \tag{22}$$

As mentioned above, the spectral problems for Eq. (5) with periodic boundary conditions ($a = -1$) were considered in [2], [3]. For periodic problems it was shown that root subspaces, consisting of two eigenfunctions, correspond to the double eigenvalues. Here we consider the case $a \neq -1$.

For $a \neq -1$, one eigenfunction and one associated function correspond to the double eigenvalues (22).

By direct calculation it is easily shown that for the cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (5) to (6) has the eigenfunctions

$$\begin{aligned} &y_k^{(1)}(x) = \cos(k\pi x), \\ &y_n^{(2)}(x) = (1+a)r\cos(n\pi)\cos\left(\tfrac{n\pi x}{r}\right) + (a-1)\sin\left(\tfrac{n\pi}{r}\right)\sin(n\pi x), \end{aligned} \tag{23}$$

where $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for those cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (5) to (6) has the eigenfunctions $y_{k_0 j}^{(1)}(x)$ and the corresponding associated functions $y_{n_0 j,1}(x)$:

$$\begin{aligned} &y_{k_0 j}^{(1)}(x) = \cos\left(k_0 j\pi x\right), \\ &y_{n_0 j,1}(x) = -\tfrac{1}{2k_0 j\pi(1-\varepsilon)}\left\{x\sin(k_0 j\pi x) + \tfrac{1-a}{1+a}\tfrac{1}{r}(-1)^{(n_0+k_0)j}\sin\left(n_0 j\pi x\right)\right\}. \end{aligned} \tag{24}$$

Here we mean by the associated functions (according to M.V. Keldysh) solutions of the inhomogeneous equation

$$\mathcal{L}y_{k,1}(x) \equiv -y_{k,1}''(x) + \varepsilon y_{k,1}''(-x) = \lambda_k^{(1)} y_{k,1}(x) + y_k^{(1)}(x), \ -1 < x < 1, \tag{25}$$

satisfying the boundary conditions (6).

It is well known that the associated functions are not defined uniquely. Functions of the form

$$\widetilde{y}_{k_0 j,1}(x) = y_{k_0 j,1}(x) + C_j y_{k_0 j}^{(1)}(x)$$

for any constants C_j are also associated functions of problem (5) to (6) corresponding to the eigenvalues $\lambda_{k_0 j}^{(1)}$ and the eigenfunctions $y_{k_0 j}^{(1)}(x)$. "Problem of choosing associated functions" is also well known. This problem is related to the fact that with one choice of the constants C_j the system can form a basis, and with other choice of these constants the system does not form an unconditional basis. To avoid this problem, we fix such a choice of associated functions by formula (24).

Lemma 7. *The system of eigen- and associated functions* (23) *to* (24) *of problem* (5) *to* (6) *is complete and minimal in* $L_2(-1,1)$.

The proof is similar to the proof of Lemma 4. Consider an arbitrary function $f(x)$ orthogonal to the system of functions (23) to (24). Since it is orthogonal to all functions $y_k^{(1)}(x)$, $k = 0, 1, 2, \ldots$, then, as in the proof of Lemma 4, we have that $f(x)+f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1, 1)$.

Further, from the orthogonality of $f(x)$ to all functions $y_n^{(2)}(x)$ from (23) we get that

$$\int_{-1}^{1} f(x)\sin(n\pi x)dx = 0 \tag{26}$$

for all $n = 1, 2, \ldots$, except the cases when $n = n_0 j$ for some j.

It follows from the oddness of $f(x)$ that it is orthogonal to the functions $x\sin(k_0 j\pi x)$. Therefore, from the orthogonality of $f(x)$ to all functions $y_{k_0 j,1}(x)$ from (24) we get that (26) holds and for the cases when $n = n_0 j$ for some j.

Since the system $\{\sin(n\pi x), n = 1, 2, \ldots\}$ forms the basis in $L_2(0, 1)$, then $f(x)-f(-x) = 0$ (that is, this function is even) holds almost everywhere on the interval $(-1, 1)$.

Thus, the function $f(x)$ turns out to be simultaneously even and odd almost everywhere on the interval $(-1, 1)$. Consequently, $f(x) = 0$ holds almost everywhere on the interval $(-1, 1)$. This proves the completeness of the system of functions (23) to (24) in $L_2(-1, 1)$.

Since the system under consideration (23) to (24) is the system of eigen- and associated functions of a linear operator, then it has a biorthogonal system consisting of eigen- and associated functions of an adjoint operator. We will not dwell here on a specific form of this system and the adjoint operator. But from the existence of the biorthogonal system follows the minimality of the system of functions (23) to (24) in $L_2(-1, 1)$. Lemma is proved.

Now let us prove that system (23) to (24) forms the unconditional basis in $L_2(-1, 1)$. For this we need a biorthogonal system. It is a system of eigen- and associated functions of the adjoint problem:

$$\mathcal{L}^* v\,(x) \equiv -v''\,(x) + \varepsilon v''\,(-x) = \lambda v\,(x)\,,\ -1 < x < 1, \tag{27}$$

$$\begin{cases} V_1\,(v) \equiv v'\,(-1) - v'\,(1) = 0, \\ V_2\,(v) \equiv (a-\varepsilon)v\,(-1) + (1-a\varepsilon)v\,(1) = 0. \end{cases} \tag{28}$$

Since the eigenvalues (16) of problem (5) to (6) are real, then they are also and the eigenvalues of the adjoint problem (27) to (28). The system of eigen- and associated functions of this problem can be constructed explicitly.

The eigenfunction

$$v_0(x) = \frac{1}{2} - \frac{(1+a)}{2(1-a)}r^2 x \tag{29}$$

corresponds to a zero eigenvalue.

By direct calculation it is easily shown that for those cases when $\frac{n}{k} \neq \frac{n_0}{k_0}$, problem (27) to (28) has the eigenfunctions

$$\begin{aligned} v_k^{(1)}(x) &= \cos(k\pi x) - \tfrac{1+a}{1-a}r^2 \tfrac{(-1)^k}{\sin(rk\pi)}\sin(rk\pi x), \\ v_n^{(2)}(x) &= -\tfrac{1}{1-a}\tfrac{1}{\sin\left(\frac{n\pi}{r}\right)}\sin(n\pi x), \end{aligned} \tag{30}$$

corresponding to the eigenvalues $\lambda_k^{(1)}$ and $\lambda_n^{(2)}$, where $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), problem (27) to (28) has the eigenfunctions $v_{n_0 j}^{(2)}(x)$ and the associated functions corresponding to them $v_{k_0 j,1}(x)$:

$$\begin{aligned} v_{n_0 j}^{(2)}(x) &= -k_0 j\pi(1-\varepsilon)\tfrac{1+a}{1-a} r(-1)^{(n_0+k_0)j} \sin(n_0 j\pi x), \\ v_{k_0 j,1}(x) &= -\tfrac{1+a}{1-a} r^2 (-1)^{(n_0+k_0)j} x\cos(n_0 j\pi x) + \cos(k_0 j\pi x). \end{aligned} \tag{31}$$

When constructing this system of eigen- and associated functions of the adjoint problem, we have normalized the eigenfunctions so that the biorthogonality conditions

$$\left\langle y_k^{(1)}, v_k^{(1)} \right\rangle = 1, \quad \left\langle y_n^{(2)}, v_n^{(2)} \right\rangle = 1,$$

hold for all $k = 0, 1, 2, ...$ and $n = 1, 2, ...$, except the cases when $k = k_0 j$, $n = n_0 j$ for some j.

And for the cases when $\frac{n}{k} = \frac{n_0}{k_0}$ (that is, when $k = k_0 j$, $n = n_0 j$ for some j), we have required the fulfilment of the biorthogonality conditions

$$\left\langle y_{k_0 j}^{(1)}, v_{k_0 j,1} \right\rangle = 1, \quad \left\langle y_{n_0 j,1}, v_{n_0 j}^{(2)} \right\rangle = 1.$$

Here by $\langle \cdot, \cdot \rangle$ we denote the inner product in $L_2(-1,1)$.

For what follows, we need to estimate the norms of the constructed eigen- and associated functions. By direct calculation we find

$$\left\| y_k^{(1)} \right\| = 1; \ \left\| y_n^{(2)} \right\|^2 = (1+a)^2 r^2 \left\{ 1 + \frac{r}{2n\pi} \sin\left(\frac{2n\pi}{r}\right) \right\} + (1-a)^2 \sin^2\left(\frac{n\pi}{r}\right);$$

$$\left\| v_k^{(1)} \right\|^2 = 1 + \left(\frac{1+a}{1-a}\right)^2 \frac{r^2}{\sin^2(rk\pi)}; \ \left\| v_n^{(2)} \right\|^2 = \frac{1}{(1-a)^2} \frac{1}{\sin^2\left(\frac{n\pi}{r}\right)};$$

$$\left\| y_{n_0 j}^{(1)} \right\| = 1; \ \left\| y_{n_0 j,1} \right\|^2 = \frac{1}{(2k_0 j\pi(1-\varepsilon))^2} \left\{ \frac{1}{3} - \frac{1}{2(k_0 j)^2} + \left(\frac{1-a}{1+a}\right)^2 \frac{1}{r^2} \right\};$$

$$\left\| v_{n_0 j}^{(2)} \right\|^2 = (2k_0 j\pi(1-\varepsilon))^2 \left(\frac{1+a}{1-a}\right)^2 r^2;$$

$$\left\| v_{k_0 j,1} \right\|^2 = 1 + \left(\frac{1+a}{1-a}\right)^2 r^4 \left\{ \frac{1}{3} + \frac{1}{2(k_0 j)^2} \right\}.$$

Analyzing these explicit formulas, we see that only the asymptotic behavior of multipliers $\sin\left(\frac{n\pi}{r}\right)$ and $\sin(rk\pi)$ is not obvious. Let us show that these multipliers are strictly separated from zero.

Lemma 8. *If r is a rational number: $r = \frac{n_0}{k_0}$, then for all values of the indices n and k, when $n \neq n_0 j$ and $k \neq k_0 j$, the inequalities hold*

$$\left|\sin\left(\frac{n\pi}{r}\right)\right| \geq \left|\sin\left(\frac{\pi}{n_0}\right)\right|, \quad \left|\sin(rk\pi)\right| \geq \left|\sin\left(\frac{\pi}{k_0}\right)\right|. \tag{32}$$

The proof will be carried out by the method used in [7], [8], [9]. Since $n \neq n_0 j$, then the representation $n = n_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq n_0 - 1$. Therefore, $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, then this number $\frac{n}{r} = k_0 j + \frac{k_0 i}{n_0}$ is not an integer. Consequently, we have:

$$\left|\sin\left(\frac{n\pi}{r}\right)\right| = \left|\sin\left(\pi\left(\frac{n}{r} - k_0 j\right)\right)\right| = \left|\sin\left(\frac{\pi}{n_0} k_0 i\right)\right| \geq \left|\sin\left(\frac{\pi}{n_0}\right)\right|.$$

The second inequality from (32) is proved similarly. Since $k \neq k_0 j$, then the representation $k = k_0 j + i$ holds for some $j, i \in \mathbb{N}$, $1 \leq i \leq k_0 - 1$. Therefore, $rk = n_0 j + \frac{n_0 i}{k_0}$. Since $\frac{n}{k} \neq \frac{n_0}{k_0}$, then this number $rk = n_0 j + \frac{n_0 i}{k_0}$ is not an integer. Hence we have:

$$|\sin(rk\pi)| = |\sin(\pi(rk - n_0 j))| = \left|\sin\left(\frac{\pi}{k_0} n_0 i\right)\right| \geq \left|\sin\left(\frac{\pi}{k_0}\right)\right|.$$

Lemma 9. *If r is a rational number: $r = \frac{n_0}{k_0}$, then each of the systems* (23) *to* (24) *and* (29) *to* (31)*, after the normalization in $L_2(-1,1)$, satisfies a Bessel type inequality and hence forms an unconditional basis in $L_2(-1,1)$.*

Note that the system $\{\varphi_j\}$ has the Bessel property in a Hilbert space H, if there exists a constant $B > 0$ such that the Bessel type inequality

$$\sum_j |\langle f, \varphi_j \rangle|^2 \leq B \|f\|^2$$

holds for all elements $f \in H$.

Proof By virtue of the above estimates of the eigen- and associated functions, to justify the Bessel property, it suffices to prove the Bessel property of the following three type of systems ($j \in \mathbb{N}$):

$$\cos(j\pi x), \quad \sin(j\pi x); \tag{33}$$

$$\cos\left(\frac{k_0}{n_0} j\pi x\right), \quad \sin\left(\frac{k_0}{n_0} j\pi x\right); \tag{34}$$

$$x\cos(j\pi x), \quad x\sin(j\pi x). \tag{35}$$

System (33) is orthonormal in $L_2(-1,1)$ and hence satisfies the Bessel type inequality with constant $B = 1$. The Bessel property of system (35) follows from the Bessel property of system (33), because the multiplier x is bounded. Finally, system (34) is a Bessel system by virtue of the following assertion proved in [7], [8], [9].

Lemma 10. ([7], [8], [9]) *Let $\{\gamma_j\}$ be a sequence of complex numbers such that*

$$\sup_j |Im(\gamma_j)| < \infty, \quad \sup_{t\geq 1} \sum_{j:\left|Re(\gamma_j) - t\right| \leq 1} 1 < \infty. \tag{36}$$

Then each of the systems $\{\sin(\gamma_j x)\}$ and $\{\cos(\gamma_j x)\}$ is a Bessel system in $L_2(-1,1)$.

System (34) satisfies condition (36) because

$$\operatorname{Im}(\gamma_j) = 0, \qquad \sum_{j:\left|\operatorname{Re}(\gamma_j) - t\right| \leq 1} 1 \leq 2m_0 + 1.$$

The unconditional basis property of the systems (23) to (24) and (29) to (31) follows from the well-known Bari theorem [21]. The proof of Lemma 9 is complete.

8 Formulation of main result

Combining all the results, we formulate them together in the form of one theorem.

Theorem *Let $a \neq -1$. Then the spectral problem* (5) *to* (6) *has the following properties.*
$\star$ *For $a = 1$ each number λ will be an eigenvalue of problem* (5) *to* (6)*. Corresponding eigenfunctions are of the form* (15)*.*
$\star$ *Problem* (5) *to* (6) *has double eigenvalues if and only if the number $r = \sqrt{(1-\varepsilon)/(1+\varepsilon)}$ is rational.*
$\star$ *If r is an irrational number, then all eigenvalues of problem* (5) *to* (6) *are simple, and its system of eigenfunctions* (17) *is complete and minimal but does not form an unconditional basis in $L_2(-1,1)$.*
$\star$ *If r is a rational number, then there exists an infinite countable subsequence of eigenvalues of problem* (5) *to* (6) *which are double. The rest of the eigenvalues of problem* (5) *to* (6) *(there are also infinite countable number of them) are simple. One eigenfunction and one associated function correspond to each double eigenvalue. The system of eigen- and associated functions* (23) *to* (24) *of problem* (5) *to* (6) *is complete and minimal in $L_2(-1,1)$. The associated functions of problem* (5) *to* (6) *can be chosen in such a special way that this special system of eigen- and associated functions forms an unconditional basis in $L_2(-1,1)$.*

9 Conclusions

Thus, in this paper, we consider one class of spectral problems for a nonlocal ordinary differential operator (with involution in the main part) with nonlocal boundary conditions of periodic type. The main result of the work is to study the questions of the unconditional basis property of the system of root vectors of the given differential operator. We have proved the criterion for the simplicity of the eigenvalues of the problem. In addition, it have been proved that the system of root vectors forms an unconditional basis only in the case of multiple eigenvalues. Therefore, (in the case of multiple eigenvalues) this system of root vectors can be further used to solve problems of nonlocal heat conduction with nonlocal boundary conditions of periodic type.

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References

[1] Cabada A., Tojo F., Adrian F. Differential equations with involutions // Workshop on Differential Equations. - 2015.

[2] Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012/ - Art. ID 576843/ P. 1-6. DOI: 10.1155/2012/576843.

[3] Kopzhassarova A.A., Lukashov A.L., Sarsenbi A.M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012. - Art. ID 590781. DOI: 10.1155/2012/590781.

[4] Kritskov L. V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differential Equations. - 2015. - V. 51, №8. - P. 984-990.

[5] Kritskov L. V., Sarsenbi A. M. Basicity in L_p of root functions for differential equations with involution // Electronic Journal of Differential Equations. - 2015. - V. 2015, №278. - P. 1-9.

[6] Baskakov A. G, Krishtal I. A., Romanova E. Y. Spectral analysis of a differential operator with an involution // Journal of Evolution Equations. - 2017. - V. 17, №2. - P. 669-684.

[7] Kritskov L. - V., Sarsenbi A. M. Riesz basis property of system of root functions of second-order differential operator with involution // Differential Equations. - 2017. - V. 53, №1. - P. 33-46.

[8] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Nonlocal spectral problem for a second-order differential equation with an involution // Bulletin of the Karaganda University-Mathematics. - 2018. - V. 91, №3. - P. 53-60.

[9] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Properties in L_p of root functions for a nonlocal problem with involution // Turkish Journal of Mathematics. - 2019. - V. 43, №1. - P. 393-401.

[10] Kirane M., Sadybekov M. A., Sarsenbi A. A. On an inverse problem of reconstructing a subdiffusion process from nonlocal data // Mathematical Methods in the Applied Sciences. - 2019. - V. 42, №6. - P. 2043-2052.

[11] Vladykina - V. E., Shkalikov A. A. Spectral Properties of Ordinary Differential Operators with Involution // Doklady Mathematics. - 2019. - V. 99, №1. - P. 5-10.

[12] Ashyralyev A., Sarsenbi A. M. Well-posedness of a parabolic equation with nonlocal boundary condition // Boundary Value Problems. - 2015. - V. 2015, №1.

[13] Ashyralyev A., Sarsenbi A. M. Well-Posedness of a Parabolic Equation with Involution // Numerical Functional Analysis and Optimization. - 2017. - V. 38, №1. - P. 1-10.

[14] Orazov I., Sadybekov M. A. One nonlocal problem of determination of the temperature and density of heat sources // Russian Mathematics. - 2012. - V. 56, №2. - P. 60-64.

[15] Orazov I., Sadybekov M. A. On a class of problems of determining the temperature and density of heat sources given initial and final temperature // Siberian Mathematical Journal. - 2012. - V. 53, №1. - P. 146-151.

[16] Kurdyumov - V. P.„ Khromov A. P. The Riesz bases consisting of eigen and associated functions for a functional differential operator with variable structure // Russian Mathematics. - 2010. - V. 54, №2. - P. 39-52.

[17] Sarsenbi A. M., Tengaeva A. On the basis properties of root functions of two generalized eigenvalue problems // Differential Equations. - 2012. - V. 48, №2. - P. 306-308.

[18] Sadybekov M. A., Sarsenbi A. A. Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution // Differential Equations. - 2012. - V. 48, №8. - P. 1112-1118.

[19] Ruzhansky M., Sadybekov M. A., Suragan D. Spectral Geometry of Partial Differential Operators // New York: Taylor & Francis Group. - 2020.

[20] Schmidt W. M. Diophantine Approximations and Diophantine Equations // Mathematics. - 1991.

[21] Bari N. K. Biorthogonal Systems and Bases in Hilbert Space // Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta. - 1951. - V. 148, №4 - P. 69-107. [in Russian]

Список литературы

[1] Cabada A., Tojo F., Adrian F. Differential equations with involutions // Workshop on Differential Equations. - 2015.

[2] Kopzhassarova A. A., Lukashov A. L., Sarsenbi A. M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012/ - Art. ID 576843/ P. 1-6. DOI: 10.1155/2012/576843.

[3] Kopzhassarova A.A., Lukashov A.L., Sarsenbi A.M. Spectral Properties of non-self-adjoint perturbations for a spectral problem with involution // Abstract and Applied Analysis. - 2012. - V. 2012. - Art. ID 590781. DOI: 10.1155/2012/590781.

[4] Kritskov L. - V., Sarsenbi A. M. Spectral properties of a nonlocal problem for the differential equation with involution // Differential Equations. - 2015. - V. 51, №8. - P. 984-990.

[5] Kritskov L. - V., Sarsenbi A. M. Basicity in L_p of root functions for differential equations with involution // Electronic Journal of Differential Equations. - 2015. - V. 2015, №278. - P. 1-9.

[6] Baskakov A. G, Krishtal I. A., Romanova E. Y. Spectral analysis of a differential operator with an involution // Journal of Evolution Equations. - 2017. - V. 17, №2. - P. 669-684.

[7] Kritskov L. - V., Sarsenbi A. M. Riesz basis property of system of root functions of second-order differential operator with involution // Differential Equations. - 2017. - V. 53, №1. - P. 33-46.

[8] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Nonlocal spectral problem for a second-order differential equation with an involution // Bulletin of the Karaganda University-Mathematics. - 2018. - V. 91, №3. - P. 53-60.

[9] Kritskov L. - V., Sadybekov M. A., Sarsenbi A. M. Properties in L_p of root functions for a nonlocal problem with involution // Turkish Journal of Mathematics. - 2019. - V. 43, №1. - P. 393-401.

[10] Kirane M., Sadybekov M. A., Sarsenbi A. A. On an inverse problem of reconstructing a subdiffusion process from nonlocal data // Mathematical Methods in the Applied Sciences. - 2019. - V. 42, №6. - P. 2043-2052.

[11] Vladykina - V. E., Shkalikov A. A. Spectral Properties of Ordinary Differential Operators with Involution // Doklady Mathematics. - 2019. - V. 99, №1. - P. 5-10.

[12] Ashyralyev A., Sarsenbi A. M. Well-posedness of a parabolic equation with nonlocal boundary condition // Boundary Value Problems. - 2015. - V. 2015, №1.

[13] Ashyralyev A., Sarsenbi A. M. Well-Posedness of a Parabolic Equation with Involution // Numerical Functional Analysis and Optimization. - 2017. - V. 38, №1. - P. 1-10.

[14] Orazov I., Sadybekov M. A. One nonlocal problem of determination of the temperature and density of heat sources // Russian Mathematics. - 2012. - V. 56, №2. - P. 60-64.

[15] Orazov I., Sadybekov M. A. On a class of problems of determining the temperature and density of heat sources given initial and final temperature // Siberian Mathematical Journal. - 2012. - V. 53, №1. - P. 146-151.

[16] Kurdyumov - V. P.,, Khromov A. P. The Riesz bases consisting of eigen and associated functions for a functional differential operator with variable structure // Russian Mathematics. - 2010. - V. 54, №2. - P. 39-52.

[17] Sarsenbi A. M., Tengaeva A. On the basis properties of root functions of two generalized eigenvalue problems // Differential Equations. - 2012. - V. 48, №2. - P. 306-308.

[18] Sadybekov M. A., Sarsenbi A. A. Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution // Differential Equations. - 2012. - V. 48, №8. - P. 1112-1118.

[19] Ruzhansky M., Sadybekov M. A., Suragan D. Spectral Geometry of Partial Differential Operators // New York: Taylor & Francis Group. - 2020.

[20] Schmidt W. M. Diophantine Approximations and Diophantine Equations // Mathematics. - 1991.

[21] Бари Н. К. Биортогональные системы и базисы в гильбертовом пространстве // Учение записки Московского государственного университета. - 1951. - Т. 148, №4 - С. 69-107.

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INFINITE DISCRETE CHAINS AND THE MAXIMAL NUMBER OF COUNTABLE MODELS

The paper is aimed at studying the countable spectrum of small linearly ordered theories. The objectives of the research are to study the structural properties of countable linearly ordered theories, as well as to promote the solution to the well-known open problem of model theory, Vaught's conjecture, which assumes that the number of countable models of a countable complete first-order theory cannot be equal to $\aleph_1$. An important step in solving Vaught's conjecture is the search for conditions under which the theory has the maximal number of countable pairwise non-isomorphic models. By limiting ourselves to linearly ordered theories we do not get special advantages from the viewpoint of studying their countable spectrum. Therefore, in the article, a restriction on 1-types and 1-formulas of the theory is introduced. It is proved that a small countable linearly ordered theory that satisfies the restriction and has an infinite discrete chain has the maximal number of countable non-isomorphic models. To build models, the authors use the method of constructing countable models over countable sets, based on the Tarski-Vaught criterion. It is shown that it is possible to carry out the construction in such a way that the types of unnecessary elements in the resulting model are omitted, what guarantees non-isomorphism of the models and their maximal number.
Key words: small theory, linear order, countable model, number of countable models, discrete chain, omitting types.

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Шектеусіз дискретті тізбектер және саналымды модельдердің максималды саны

Бұл мақала шағын сызықтық реттелген теориялардың саналымды спектрін зерттеуге бағытталған. Зерттеудің мақсаты саналымды сызықтық реттелген теориялардың құрылымдық қасиеттерін зерттеу, сонымен қатар модельдер теориясының белгілі ашық проблемасы – бірінші ретті толық теорияның саналымды модельдерінің саны $\aleph_1$-ге тең болмайды деп болжайтын Воот гипотезасын шешуді алға жылжыту. Теорияның екеуара изоморфты емес саналымды модельдердің саны максималды болатын шарттарды іздеу Воот гипотезасын шешудегі маңызды қадам. Сызықтық реттелген теориялармен шектеле отырып, біз олардың саналымды спектрін зерттеу тұрғысынан ерекше артықшылықтарға ие болмаймыз. Сондықтан осы теорияның 1-типтері мен 1-формулаларына шектеу енгізіледі. Мақалада осы шектеуді қанағаттандыратын және шектеусіз дискретті тізбегі бар, саналымды сызықтық реттелген шағын теориясының изоморфты емес саналымды модельдердің максималды саны бар екендігі дәлелденді. Модельдерді құру үшін авторлар Тарский-Воот өлшемшартына негізделген саналымды жиындарының үстінен саналымды модельдерін құру әдісін қолданады. Қыурылысты алынған модельдегі қажет емес элементтердің типтерін түсіріп жасауға болатындығы көрсетілген, бұл модельдердің изоморфизм болмауына және олардың максималды саны бар екендігіне кепілдік береді.
Түйін сөздер: шағын теория, сызықтық рет, саналымды модель, саналымды модельдердің саны, дискретті тізбе, типтерді төмендету.

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Бесконечные дискретные цепи и максимальное число счётных моделей

Данная статья направлена на изучение счётного спектра малых линейно упорядоченных теорий. Целями исследования являются изучение структурных свойств счётных линейно упорядоченных теорий, а также, продвижение решения известной открытой проблемы теории моделей – гипотезы Воота, которая предполагает, что числе счётных моделей счётной полной теории первого порядка не может равняться $\aleph_1$. Важным шагом в решении гипотезы Воота является поиск условий, при которых теория имеет максимальное число счётных попарно неизоморфных моделей. Ограничиваясь линейно упорядоченными теориями, мы не получаем особых преимуществ с точки зрения изучения их счётного спектра. Поэтому, в статье, будет введено ограничение на 1-типы и 1-формулы данной теории. В статье доказывается, что малая счётная линейно упорядоченная теория, удовлетворяющая данному ограничению и имеющая бесконечную дискретную цепь, имеет максимальное число счётных неизоморфных моделей. Для построения моделей авторы применяют метод построения счётных моделей над счётными множествами, основанный на критерии Тарского-Воота. Показывается, что можно провести построение таким образом, что типы ненужных элементов в полученной модели опускаются, что гарантирует не изоморфизм моделей и их максимальное количество.

Ключевые слова: малая теория, линейный порядок, счётная модель, число счётных моделей, дискретная цепь, опускание типов.

1 Introduction

Vaught's conjecture states that if the continuum hypothesis fails, for a countable complete theory T $I(T,\aleph_0)$ is either finite, $\aleph_0$ or $2^{\aleph}_0$. Vaught's conjecture was confirmed for various classes of theories: [1–6]. But for countable theories in general, this question is still open.

A theory T is said to be small if $|\bigcup\limits_{n<\omega} S_n(T)| = \aleph_0$. If a countable theory is not small, it has the maximal number of countable non-isomorphic models. Therefore, in the article, we restrict to studying small theories. We want to find theories that have the maximal number of countable models.

Theorem 1 *[7–9] Every countable model $\mathfrak{M}$ of a small theory T can be represented as a union of some elementary chain $(\mathfrak{M}(\bar{a}_i))_{i\in\omega}$ of prime models over the tuples $\bar{a}_i$.*

In Theorem 1 K.Zh. Kudaibergenov and S.V. Sudoplatov used a special method to inductively reconstruct a countable model of a small theory. The authors applied modified versions of this method to construct new models of small theories [10–13] as elementary submodels of an $\aleph_1$-saturated model. In the article, we consider one more application of such construction and prove that a small ordered theory that satisfies a special restriction and has a model with an arbitrarily long finite, and, therefore, with an infinite, discrete chain has the maximal countable spectrum.

2 Main Part

We use Gothic letters $\mathfrak{A}$, $\mathfrak{M}$, $\mathfrak{N}$, ... to denote structures, and we use capital letters (A, M, N, ...) for universes of those structures.

For subsets $A, B \subseteq M$ of a structure $\mathfrak{M}$ of a linearly ordered theory T we use the following notations:

$$A^+ := \{\gamma \in M \mid \text{for all } a \in A,\ \mathfrak{M} \models a < \gamma\};$$
$$A^- := \{\gamma \in M \mid \text{for all } a \in A,\ \mathfrak{M} \models \gamma < a\}.$$

We write $A < B$ if for all $a \in A$, $b \in B$ $\mathfrak{M} \models a < b$. If A and B are C-definable ($C \subseteq M$), then A^+, A^- and $A < B$ are also C-definable.

For a 2-formula $\varphi(x,y)$ denote $\varphi(x,y)^- := \forall z\big(\varphi(z,y) \to z > x\big)$, and $\varphi(x,y)^+ := \forall z\big(\varphi(z,y) \to z < x\big)$.

Definition 1 *The set A is convex in a set $B \supseteq A$ if for all $a, b \in A$ and all $c \in B$, $a < c < b$ implies $c \in A$. The set A is convex if it is convex in M.*

Definition 2 *A formula $\varphi(x, \bar{y}, \bar{a})$ is said to be a convex formula, if for every $\bar{b} \in M$ $\varphi(M, \bar{b}, \bar{a})$ is convex in every model of $\mathfrak{M} \models T$ containing $\bar{b}$ and $\bar{a}$.*

Definition 3 *[14] 1) A convex closure of a formula $\varphi(x, \bar{a})$ is the formula:*

$$\varphi^c(x, \bar{a}) := \exists y_1 \exists y_2 \big(\varphi(y_1, \bar{a}) \wedge \varphi(y_2, \bar{a}) \wedge (y_1 \leq x \leq y_2)\big).$$

2) A convex closure of a type $p(x) \in S_1(A)$ is the following type:

$$p^c(x) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in p\}.$$

Similarly, $tp^c(\alpha/A) := \{\varphi^c(x, \bar{a}) \mid \varphi(x, \bar{a}) \in tp(\alpha/A)\}$.

In weakly o-minimal theories, and, therefore, in o-minimal theories, $p^c(\mathfrak{M}) = p(\mathfrak{M})$ for every $p \in S_1(A)$.

Definition 4 *[15, 16] Let $\mathfrak{M}$ be a linearly ordered structure, $A \subseteq M$, $\mathfrak{M}$ be $|A|^+$-saturated, and $p \in S_1(A)$ be a non-algebraic type.*

1) An A-definable formula $\varphi(x, y)$ is p–preserving if there are α, γ_1, $\gamma_2 \in p(\mathfrak{M})$ such that $p(\mathfrak{M}) \cap \big(\varphi(M, \alpha) \setminus \{\alpha\}\big) \neq \emptyset$ and $\gamma_1 < \varphi(M, \alpha) < \gamma_2$.

3) A p-preserving formula $\varphi(x, y)$ convex to the right (left) on the type p if there is $\alpha \in p(\mathfrak{M})$ for which $p(\mathfrak{M}) \cap \varphi(M, \alpha)$ is convex, α is the left (right) endpoint of the set $\varphi(M, \alpha)$, and $\alpha \in \varphi(M, \alpha)$.

We introduce special notations for the following formulas:

$$S(x, y) := x < y \wedge \forall t\big(x \leq t \leq y \to t = x \vee t = y\big);$$

$$S_0(x, y) := (x = y);$$

and for all $n \geq 1$,

$$S_n(x,y) := \exists z_1, \exists z_2, ..., \exists z_{n+1}\big(x = z_1 < z_2 < ... < z_{n+1} = y \wedge \bigwedge_{1\leq i\leq n} S(z_i, z_{i+1})\big);$$

$$S_{-n}(x,y) := \exists z_1, \exists z_2, ..., \exists z_{n+1}\big(x = z_1 < z_2 < ... < z_{n+1} = y \wedge \bigwedge_{1\leq i\leq n} S(z_{i+1}, z_i)\big).$$

Definition 5 *Let $p_1, p_2 \in S_1(A)$. We say that p_2 is attached to p_1, if there exists $n \in \mathbb{Z}$ such that for some $\delta_1 \in p_1(\mathfrak{M})$ the element $\delta_2 \in S_n(\delta_1, M)$ realizes p_2.*

The relation of attachment of types is an equivalence relation on the set $S_1(A)$.

Restriction 1 *We restrict to theories T such that for every $M \models T$ and every $A \subseteq M$, where $\mathfrak{M}$ is $|A|^+$-saturated, there is no infinite family $P \subseteq S_1(A)$ of pairwise attached non-isolated types such that for every A-definable formula φ such that φ and $\neg\varphi$ belong to at least one type of P each,*

1) each φ and $\neg\varphi$ is in an infinite number of types from P;

2) for every δ realizing some type from P, for every $n \in \mathbb{Z}$ there exist $m_n^1, m_n^2 > n$ and $m_n^3, m_n^4 < n$ such that $S_{m_n^1}(\delta, M) \cap \varphi(M) \neq \emptyset$, $S_{m_n^2}(\delta, M) \cap \neg\varphi(M) \neq \emptyset$, $S_{m_n^3}(\delta, M) \cap \varphi(M) \neq \emptyset$, and $S_{m_n^4}(\delta, M) \cap \neg\varphi(M) \neq \emptyset$.

All weakly o-minimal theories satisfy the Restriction 1.

Theorem 2 *Let $\mathfrak{M}$ be a sufficiently saturated model of a small linearly ordered theory T that satisfies Restriction 1. If in $\mathfrak{M}$ there exists an infinite discrete chain, then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof of Theorem 2. For all $n \geq 1$ denote

$$P_n(x,y) := \exists z_1 \exists z_2 ... \exists z_n \Big(x = z_1 < z_2 < ... < z_n = y\Big) \wedge \forall z\,(x < z < y \rightarrow \\ \rightarrow \exists z_1 \exists z_2 \big(S(z_1, z) \wedge S(z, z_2)\big)\big)\,.$$

Consider the consistent set $p_0 := \{P_n(x,y)\}_{n\geq 1}$, and let $(\alpha_0, \beta_0) \in q_0(M)$. For $n < \omega$ denote $Q_n(z, \alpha_0, \beta_0) := \exists x_1 \exists x_2 ... \exists x_n \exists y_1 \exists y_2 ... \exists y_n (\alpha_0 = x_1 < x_2 < ... < x_n < z < y_n < ... < y_2 < y_1 = \beta_0)$. Then $q(z) := \{Q_n(z, \alpha_0, \beta_0)\}_{n<\omega}$ is locally consistent.

The given theory T is small, therefore the theory $T \cup tp(\alpha_0\beta_0)$ is small as well. Since in a countable model there is only a countable number of places to choose a realization of a 2-type, if we prove that $T \cup tp(\alpha_0\beta_0)$ has $2^{\aleph_0}$ countable models, then T also has. So, for convenience, we add the elements α_0 and β_0 to our language and replace T with the theory $T \cup tp(\alpha_0\beta_0)$.

There are two cases.

1) Suppose that there exist $\delta_1, \delta_2 \in q(\mathfrak{M})$ with $tp(\delta_1) = tp(\delta_2) =: r$, and there exists $c \geq 1$ such that $\mathfrak{M} \models S_c(\delta_1, \delta_2)$. Then for every $m \in \mathbb{Z}$, $S_{mc}(\delta_1, M) \subseteq r(\mathfrak{M})$.

For $m \in \mathbb{N}$ denote $S^m(x,y) := \forall z(S_{mc}(y,z) \rightarrow y \leq x \leq z)$.

Let $p_1(x,y) := p_0(x,y) \cup r(x) \cup r(y) \cup \Big\{\varphi_R(y,x) \mid \varphi_R$ is a convex to the right on the type r formula such that for all $m \in \mathbb{N}$ and all $\alpha \in r(\mathfrak{M})$, $\big((S^m(\alpha, M) \cap r(\mathfrak{M})) \subseteq \varphi_R(M, \alpha)\big)\Big\} \cup$

$\Big\{\varphi_L(x,y) \mid \varphi_L(x,y)$ is a convex to the left on r formula such that for all $n \in \mathbb{N}$ and all $\beta \in r(\mathfrak{M})$, $\big((S^{-m}(\beta,M) \cap r(\mathfrak{M})) \subseteq \varphi_L(M,\beta)\big)\Big\}$. Consistence of $p_1(x,y)$ can be verified directly. If T is weakly o-minimal, then p_1 is a complete 2-type.

For $\gamma, \gamma_1, \gamma_2 \in q(\mathfrak{M})$ and $n \in \mathbb{Z}$ we define neighborhoods of elements and intervals between neighborhoods:

$$V_{q(\mathfrak{M})}(\gamma) := \{\gamma' \in q(\mathfrak{M}) \mid S_m(\gamma,\gamma') \text{ for some } m \in \mathbb{Z}\};$$
$$V^n_{q(\mathfrak{M})}(\gamma) := \{\gamma' \in q(\mathfrak{M}) \mid S_{mn}(\gamma,\gamma') \text{ for some } m \in \mathbb{Z}\};$$
$$(V_q(\gamma_1), V_p(\gamma_2))_{q(\mathfrak{M})} := \{\gamma' \in q(\mathfrak{M}) \mid V_{q(\mathfrak{M})}(\gamma_1) < \gamma' < V_{q(\mathfrak{M})}(\gamma_2)\}.$$

Then $V_{r(\mathfrak{M})}(\delta_1) = V^n_{q(\mathfrak{M})}(\delta_1) \subseteq r(\mathfrak{M})$. When no ambiguity appears, we omit $\mathfrak{M}$ from the indexes.

Lemma 1 *There exists a complete type $p(x,y) \supseteq p_1(x,y)$ such that for all $(\alpha,\beta) \in p(\mathfrak{M})$ and all $\delta_1, \delta_2 \in (V_r(\alpha), V_r(\beta))_{r(\mathfrak{M})}$, $tp^c(\delta_1/\alpha\beta) = tp^c(\delta_2/\alpha\beta)$.*

Proof of Lemma 1. Towards a contradiction suppose that the lemma is not true. Let $p(x,y)$ be a type extending p_1, and let (α,β) be a tuple realizing p. Then there exist a convex $\{\alpha,\beta\}$-formula $\psi(x,y,\alpha,\beta)$ and $\delta_1, \delta_2 \in (V_r(\alpha), V_r(\beta))_{r(\mathfrak{M})}$ such that $\delta_1 \in \psi(M,\alpha,\beta) < \delta_2$. We can choose this formula so that the set $\psi(M,\alpha,\beta)$ coincides with the set $(\psi(M,\alpha,\beta)^+)^-$ and has δ_1 as its left endpoint. Notice that there are δ_1' and δ_2' such that $\delta_1' \in \psi(M,\alpha,\beta) < \delta_2'$ and $\delta_2' \in S(\delta_1', M)$.

For $P(x,y) \in p$, $R(x) \in r$, $k,l,m < \omega$ such that $l + m < k$ denote

$$\Lambda_{P,R,k,l,m}(x,y) := \Big(x < y \wedge \neg\, S^k(x,y) \wedge P(x,y) \wedge$$
$$\exists z_1 \exists z_2 \big(\psi(z_1,x,y) \wedge \neg\psi(z_2,x,y) \wedge S^1(z_1,z_2)\big)\Big) \to \exists z_1 \exists z_2 \Big(x < z_1 < z_2 < y \wedge$$
$$\wedge R(z_1) \wedge R(z_2) \wedge S(z_1,z_2) \wedge \neg S^l(x,z_1) \wedge \neg S^m(z_2,y) \wedge \psi(z_1,x,y) \wedge \neg\psi(z_2,x,y)\Big).$$

Let $\langle P_i \rangle_{i<\omega}$ and $\langle R_j \rangle_{j<\omega}$ be two infinite sequences of formulas from the types p and r respectively such that

$$\mathfrak{M} \models \forall x \forall y \big(P_{i+1}(x,y) \to P_i(x,y)\big),\ \mathfrak{M} \models \forall x \big(R_{j+1}(x) \to R_j(x)\big),$$
$$p(\mathfrak{M}) = \bigcap_{i<\omega} P_i(M), \text{ and } r(\mathfrak{M}) = \bigcap_{j<\omega} R_j(M).$$

By compactness, we can see the following:

There are increasing sequences $\langle i(n) \rangle$, $\langle j(n) \rangle_{n<\omega}$, $\langle k(n) \rangle_{n<\omega}$, $\langle l(n) \rangle_{n<\omega}$, $\langle m(n) \rangle_{n<\omega}$ such that for every $m < \omega$ $\mathfrak{M} \models \forall x \forall y \Lambda_{P_{i(n)},R_{j(n)},k(n),l(n),m(n)}(x,y)$.

Then the formula $\psi(x,\alpha,\beta)$ should divide some neighborhood in r: there exists $\gamma \in r(\mathfrak{M})$ such that $V_r(\alpha) < V_r(\gamma) < V_r(\beta)$ and $r(\mathfrak{M}) \cap \psi(M,\alpha,\beta) \cap V_r(\gamma) \neq \emptyset$.

Let $G(x,\alpha,\beta) := \psi(x,\alpha,\beta) \wedge \exists y \big(S^1(x,y) \wedge \neg\psi(y,\alpha,\beta)\big)$. If necessary, we can narrow down G by adding conjunction with a formula of $r(x)$. This guarantees that the future formulas ψ_τ will act the same way as $\psi(x,\alpha,\beta)$. Every first order property that holds for realizations of some type should hold for all realizations of some of its formula. The formula $G(x,\alpha,\beta)$ defines a single element $\psi(M,\alpha,\beta)\cap r(\mathfrak{M})$ whose S^1-successor is in $\neg\psi(M,\alpha,\beta)$. Without loss of generality, let this element be γ. Then the following two cases cases are possible:

Case A. $tp(\gamma/\alpha)=tp(\beta/\alpha)$ and $tp(\gamma/\beta)=tp(\alpha/\beta)$;

Case B. $tp(\gamma/\alpha)\neq tp(\beta/\alpha)$ or (and) $tp(\gamma/\beta)\neq tp(\alpha/\beta)$.

In Case A, for every $m<\omega$ there exist $n<\omega$, $m_1<\omega$, $m_1>m$, such that

$$\begin{aligned}&\mathfrak{M}\models\forall x\forall y\left(\left(P_{m_1}(x,y)\wedge\neg S^n(x,y)\right)\rightarrow\right.\\&\left.\rightarrow\exists z_1\exists z_2\left(P_m(x,z_1)\wedge P_m(z_1,y)\wedge S^1(z_1,z_2)\wedge\psi(z_1,x,y)\wedge\neg\psi(z_2,x,y)\right)\right).\end{aligned}$$

Then, the definable set of $\alpha<x<\beta$ is divided into three disjoint sets definable by formulas $\Psi(x,\alpha,\beta)$, $\varphi_0(x,\alpha,\beta):=\alpha<x<\Psi(M,\alpha,\beta)$ and $\varphi_1(x,\alpha,\beta):=\Psi(M,\alpha,\beta)<x<\beta$ with

$$\varphi_0(M,\alpha,\beta)<\Psi(M,\alpha,\beta)<\varphi_1(M,\alpha,\beta).$$

Define

$$\begin{aligned}\psi_0(x,\alpha,\beta)&:=\exists z(\psi(x,\alpha,z)\wedge\Psi(z,\alpha,\beta));\\\psi_1(x,\alpha,\beta)&:=\exists z(\psi(x,z,\beta)\wedge\Psi(z,\alpha,\beta)).\end{aligned}$$

The formulas ψ_0 and ψ_1 are defined correctly since, in Case A, $tp(\alpha\delta)=tp(\delta\beta)=p$. In Case B γ still realizes some complete extension of the type p_0. By supposing that Lemma 1 is not true, there exists a formula $\psi^0(x,\alpha,\gamma)$ and (or) a formula $\psi^1(x,\gamma,\beta)$ that has the same properties as $\psi(x,\alpha,\beta)$. Then, in the definitions of ψ_0 and ψ_1 replace $\psi(x,y)$ with $\psi^0(x,y)$ and $\psi^1(x,y)$ respectively. Then ψ_0 divides the interval $(V_q(\alpha),V_q(\gamma))_p$, and ψ_1 divides $(V_q(\gamma),V_q(\beta))_q$.

Let

$$\begin{aligned}\Psi_0(x,\alpha,\beta)&:=\psi_0(x,\alpha,\beta)\wedge\exists y(\neg\psi_0(y,\alpha,\beta)\wedge S^1(x,y));\\\Psi_1(x,\alpha,\beta)&:=\psi_1(x,\alpha,\beta)\wedge\exists y(\neg\psi_1(y,\alpha,\beta)\wedge S^1(x,y))\end{aligned}$$

.

The formulas Ψ_0 and Ψ_1 define singletons γ_0 and γ_1 in $\psi_0(M,\alpha,\beta)\cap r(\mathfrak{M})$ and $\psi_1(M,\alpha,\beta)\cap r(\mathfrak{M})$ whose S^1-successors are in $\neg\psi_0(M,\alpha,\beta)\cap r(\mathfrak{M})$ and $\neg\psi_1(M,\alpha,\beta)\cap r(\mathfrak{M})$ respectively. Each of those singletons can satisfy an analogue of either the Case A or the Case B. Suppose that they satisfy the Case A: $tp(\alpha,\gamma_0)=tp(\gamma_0,\gamma)=tp(\gamma,\gamma_1)=tp(\gamma_1,\beta)=p$. In Case B, replace ψ with suitable formulas the same way as in the definitions of ψ_0 and ψ_1.

We have $V_r(\alpha)<V_r(\gamma_0)<V_r(\gamma)<V_r(\gamma_1)<V_r(\beta)$.

Denote $\varphi_{00}(x,\alpha,\beta):=\alpha<x<G_0(M,\alpha,\beta)$, $\varphi_{01}(x,\alpha,\beta):=\Psi_0(M,\alpha,\beta)<x<\Psi(M,\alpha,\beta)$, $\varphi_{10}(x,\alpha,\beta):=\Psi(M,\alpha,\beta)<x<\Psi_1(M,\alpha,\beta)$, and $\varphi_{11}(x,\alpha,\beta):=\Psi_1(N,\alpha,\beta)<x<\beta$.

We continue defining such formulas by induction:

$$\begin{aligned}\psi_{\tau 0}(x)&:=\exists z_1\exists z_2\left(\Psi_\mu(z_1)\wedge\Psi_\tau(z_2)\wedge\psi(x,z_1,z_2)\right);\\\psi_{\tau 1}(x)&:=\exists z_1\exists z_2\left(\Psi_\tau(z_1)\wedge\Psi_\delta(z_2)\wedge\psi(x,z_1,z_2)\right).\end{aligned}$$

If τ consists only of zeros, denote

$$\psi_{\tau 0}(x) := \exists z_2 \big(\Psi_\tau(z_2) \wedge \psi(x, \alpha, z_2)\big);$$

If τ consists only of ones, denote

$$\psi_{\tau 1}(x) := \exists z_1 \big(\Psi_\tau(z_1) \wedge \psi(x, z_1, \beta)\big),$$

Above, μ (δ) is obtained by removing the one or more digit from τ such that Ψ_μ is the left (right) "closest" to Ψ_τ. In Case B, replace ψ in the definitions with a suitable formula as before.

Define

$\Psi_{\tau 0}(x) := \exists z(\psi_{\tau 0}(x) \wedge S(x, z) \wedge \neg\psi_{\tau 0}(z))$; $\Psi_{\tau 1}(x) := \exists z(\psi_{\tau 1}(x) \wedge S(x, z) \wedge \neg\psi_{\tau 1}(z))$; $\varphi_{\tau 0}(x) := \Psi_\mu(M) < x < \Psi_\tau(M)$ and $\varphi_{\tau 1}(x) := \Psi_\tau(M) < x < \Psi_\delta(M)$. Then we obtain $\varphi_{\tau 0}(M) < \Psi_\tau(M) < \varphi_{\tau 1}(M)$; $\varphi_{\tau 0}(M) \cup \varphi_{\tau 1}(x) \subset \varphi_\tau(M)$.

This way, for arbitrary $\nu \in 2^\omega$ we constructed the following set of 1-$\alpha\beta$-formulas: $r_\nu := \{\varphi_{\nu(n)}(x) \mid n < \omega\}$. It is obviously consistent. But this is impossible in a small theory. A contradiction. $\square$ Lemma 1

A 1-type $r \in S_1(A)$ is said to be irrational if $r^c(\mathfrak{M})^+$ and $r^c(\mathfrak{M})^-$ are both non-definable sets. By Lemma 1 there is a type $q(x, y)$ such that for every $\alpha, \beta \in p(\mathfrak{M})$ if $tp(\alpha, \beta) = q(x, y)$, for every $\gamma \in (V_{p,\varphi}(\alpha), V_{p,\varphi}(\beta))_p$, $tp^c(\gamma/\alpha\beta)(\mathfrak{M}) = (V_{p,\varphi}(\alpha)^+ < x < V_{p,\varphi}(\beta)^-)(\mathfrak{M})$ is an irrational type. From this follows that the type $tp(\gamma/\alpha\beta)$ is non-principal.

It follows from Lemma 1, that for every formula $\psi(z, x, y)$ such that $\mathfrak{M} \models \forall x \forall y \big(\psi(M, x, y) < y \wedge \forall z(\psi(z, x, y) \leftrightarrow \psi(z, x, y)^{+-})\big)$, there are the following two possibilities:

1) There exists $k_0 < \omega$ for which $\forall z\big(\neg S^{k_0}(x, z) \to \neg\psi(z, x, y)\big) \in p(x, y)$, or, equivalently, for $T_1[\psi](x, y) := \forall x_1 \Big(S^1(x_1, x) \to \exists z\big(\psi(z, x, y) \wedge \neg\psi(z, x_1, y)\big)\Big) \wedge \forall y_1 \Big(S^1(y, y_1) \to \forall z\big(\neg\psi(z, x, y) \leftrightarrow \neg\psi(z, x, y_1)\big)\Big)$, $T_1[\psi](x, y) \in p$.

2) There exists $l_0 < \omega$ for which $\forall z\Big(\big(x < z \wedge \neg S^{l_0}(z, y)\big) \to \psi(z, x, y)\Big) \in p(x, y)$, or, equivalently, for $T_2[\psi](x, y) := \forall y_1 \Big(\varphi(y_1, y) \to \exists z\big(\psi(z, x, y) \wedge \neg\psi(z, x, y_1)\big)\Big) \wedge \forall x_1 \Big(S^1(x_1, x) \to \forall z\big(\psi(z, x, y) \leftrightarrow \psi(z, x_1, y)\big)\Big)$, $T_2[\psi](x, y) \in p$.

The formulas $\forall x \forall y (T_1[\psi](x, y) \to \neg T_2[\psi](x, y))$ and $\forall x \forall y \big(T_2[\psi](x, y) \to \neg T_1[\psi](x, y)\big)$ are in the theory T. Therefore, $\big(T_1[\psi](x, y) \vee T_2[\psi](x, y)\big) \in p$.

Then $p_1(x, y) := p_0(x, y) \cup \Big\{T_1[\psi](x, y) \vee T_2[\psi](x, y) \mid \mathfrak{M} \models \forall x \forall y \Big(\psi(M, x, y) < y \wedge \forall z\big(\psi(z, x, y) \leftrightarrow \psi(z, x, y)^{+-}\big)\Big)\Big\}$ should be consistent, and every its extension to a complete type should satisfy the property from Lemma 1.

We can generalize Lemma 1 the following way:

Lemma 2 *For every n ($n < \omega$), there exists an n-type $p^n(x_1, \ldots, x_n)$ such that for every increasing sequence $\alpha_1, \alpha_2, \ldots, \alpha_n$ of realizations of p in $\mathfrak{M}$ such that $tp(\alpha_1, \alpha_2, \ldots, \alpha_n) = p$, for every i ($1 \le i < n$), and every $\delta_1, \delta_2 \in (V_{q,\varphi}(\alpha_i), V_{q,\varphi}(\alpha_{i+1}))_q$,*

$$tp^c(\delta_1/\bar\alpha) = tp^c(\delta_2/\bar\alpha), \text{ and}$$
$$tp^c(\delta_1/\bar\alpha)(\mathfrak{M}) = \{m \in q(\mathfrak{M}) \mid V_q(\alpha_i)^+ < m < V_q(\alpha_{i+1})^-\}.$$

Here $\bar\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Let $tp(\bar{\delta}) = p(\bar{x})$ satisfy Lemma 2. Then for every $i (1 \le i < n)$, and every formula $\psi_i(z, \bar{\delta})$ with $\mathfrak{M} \models \delta_i < \psi_i(M, \bar{\delta}) < \delta_{i+1}$ and

$$\mathfrak{M} \models \forall x \forall y \Big(\exists z \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \to \forall z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \psi_i(z, x, y, \bar{\delta}_n^{i,i+1})^{+-} \big) \Big)$$

one of the following two cases is possible:

1) There is $k_0 < \omega$ for which $\forall z \big(\neg S^{k_0}(x, z) \to \neg \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \big) \in p(x, y, \bar{\delta}_n^{i,i+1})$, or, equivalently, for

$$T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) := \forall x_1 \Big(S^1(x_1, x) \to \exists z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \wedge \neg \psi_i(z, x_1, y, \bar{\delta}_n^{i,i+1}) \big) \Big) \wedge \\ \forall y_1 \Big(\varphi(y_1, y) \to \forall z \big(\neg \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \neg \psi_i(z, x, y_1, \bar{\delta}_n^{i,i+1}) \big) \Big),$$

$T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

2) There exists $l_0 < \omega$ for which $\forall z \Big(\big(x < z \wedge \neg S^{l_0}(z, y) \big) \to \psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \Big) \in p(x, y, \bar{\delta}_n^{i,i+1})$, or, equivalently, for

$$T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) := \forall y_1 \Big(S^1(y, y_1) \to \exists z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \wedge \neg \psi_i(z, x, y_1, , \bar{\delta}_n^{i,i+1}) \big) \Big) \wedge \\ \forall x_1 \Big(S^1(x_1, x) \to \forall z \big(\psi_i(z, x, y, \bar{\delta}_n^{i,i+1}) \leftrightarrow \psi(z, x_1, y) \big) \Big),$$

$T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

Because

$$\mathfrak{M} \models \forall x \forall y \big(T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \to \neg T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big) \wedge \forall x \forall y \big(T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \to \\ \neg T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big),$$

$\big(T_1[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \vee T_2[\psi_i](x, y, \bar{\delta}_n^{i,i+1}) \big) \in p(x, y, \bar{\delta}_n^{i,i+1})$.

Therefore, the following set is consistent:

$$p_1^n(\bar{x}) := p_0^n(\bar{x}) \cup \Big\{ T_1[\psi_i](\bar{x}) \vee T_2[\psi_i](\bar{x}) \mid \mathfrak{M} \models \forall x \forall y \Big(\psi_i(x, x, y, \bar{x}_n^{i,i+1}) \wedge \psi_i(M, \bar{x}) < \\ y \wedge \forall z \big(\psi(z, x, y, \bar{x}_n^{i,i+1}) \leftrightarrow \psi(z, x, y, \bar{x}_n^{i,i+1})^{+-} \big) \Big) \Big\}.$$

Every complete extension of p_1^n should satisfy the property in Lemma 2.

Denote $\Delta_n := \{ \langle i_1, i_2, \ldots, i_n \rangle \mid i_1 < i_2 < \cdots < i_n < \omega \}$. For every $\mu \in S_n$ let $\bar{x}_\mu := x_{\mu(1)} x_{\mu(2)} \ldots x_{\mu(n)}$. Lemma 2 implies consistence of the set $\bigcup\limits_{n < \omega,\ \mu \in S_n} q_1^n(\bar{x}_\mu) := \Gamma$.

Let $\mathfrak{N}$ be an $\aleph_1$-saturated elementary extension of $\mathfrak{M}$. There exists a countable ordered set $D \subset N$ that satisfies Γ and is ordered by the type of ω. Consider the Ehrefeucht-Mostovski type $EM(\omega/D) = \{ \varphi(x_1, ..., x_n) \mid \text{for every } \mu \in S_n, n < \omega, \mathfrak{N} \models \varphi(\bar{a}_\mu) \}$. Then, $\Gamma(x_1, x_2, \ldots, x_n, \ldots) \subseteq EM(\omega/D)$. The Standard Lemma [17] implies that for every infinite linear ordering there is an indiscernible sequence $\langle d_j \rangle_{j \in J}$. Since $p_1^n(x_\mu) \subseteq EM(\omega/D)$, every finite sequence of J of length n forms a tuple $\bar{d}_\mu$ that satisfies the property from Lemma 2. This allows us to conduct the construction of $2^{\aleph_0}$ countable models.

Let $\mu := \langle \mu_1, \mu_2, ..., \mu_i, ... \rangle_{i < \omega}$, $\mu_i \in \{0, 1\}$, be an infinite sequence of zeros and ones. And let $K_\mu := \{ \kappa_1, \kappa_2 \} \cup \{ \kappa_{2i-1,j} \mid i \in \mathbb{N}, j \in \mathbb{Q} \} \cup \{ \kappa_{2i,1}, \kappa_{2i,2} \mid i \in \mathbb{N}, \mu_i = 0 \} \cup \{ \kappa_{2i,1}, \kappa_{2i,2}, \kappa_{2i,3} \mid i \in \mathbb{N}, \mu_i = 1 \}$ be an indiscernible subset of $r(\mathfrak{N})$ that exists by the previous statements, and such that the sets $V_{r(\mathfrak{N})}(\kappa_{i,j})$ are disjoint and ordered lexicographically by the indices i, j, and $\kappa_1 < \kappa_{i,j} < \kappa_2$ for all i and j. Fix some enumeration $K_\mu = \{\kappa_1, \kappa_2, ..., \kappa_i, ...\}$. For $n < \omega$ denote $\bar{\kappa}_n := (\kappa_1, \kappa_2, ..., \kappa_n)$. We use an analogical notation $\bar{c}_n := (c_1, c_2, ..., c_n)$ later as well.

Now we construct a countable model $\mathfrak{A}_\mu \prec \mathfrak{N}$ such that $K_\mu \subseteq A_\mu$. We want the models obtained for different sequences μ to be non-isomorphic.

Step 1. Let $\Phi_1 := \{\varphi_{1,i}(x) \mid i < \omega\}$ be the set of all non-equivalent 1-formulas over $\emptyset$ such that $\mathfrak{N} \models \exists x \varphi_{1,i}(x)$ for all $i < \omega$. First we find a witness for $\varphi_{1,1}(x)$. Since the theory T is small, $\varphi_{1,1}$ has a principal over $\emptyset$ subformula $\varphi_{1,1,0}$. The formula $\varphi_{1,1,0}$, in turn, has a principal subformula $\varphi_{1,1,1}$ over κ_1. The formula $\varphi_{1,1,1}$ has a principal subformula over κ_2, and so on. We obtain a principal over parameters nested sequence of formulas $\varphi_{1,1,i}$, that has to be realized in the $\aleph_1$-saturated model $\mathfrak{N}$. Denote this realization by c_1. Denote $C_1 := \{c_1\}$, and $K_1 := \{\kappa_1\}$.

Next, we continue the same procedure. To satisfy the Tarski-Vaught criterion, on each step we form a new set of parameters, and realize one formula over each of the existing sets of parameters.

At the end of step n we have defined the finite nested sets $C_1 \subseteq C_2 \subseteq ... \subseteq C_n$, and, for every i, $2 \leq i \leq n$, the family Φ_i of all $C_{i-1} \cup K_{i-1}$-definable 1-formulas that have witnesses in $\mathfrak{N}$.

Step n+1. Firstly, realize one new formula from Φ_1, then from $\Phi_2, ..., \Phi_{n-1}$. For $1 \leq m \leq n$ let i_m be the smallest index such that $\varphi_{m,i_m} \in \Phi_m$ was not considered before. Construct a nested sequence of principal over parameters formulas: $\varphi_{m,i_m}(N, \bar{\kappa}_m, \bar{c}^{m_1}) \supseteq \varphi_{m,i_{m,0}}(N, \bar{\kappa}_i, \bar{c}^{m_1}) \supseteq \varphi_{m,i_{m,1}}(N, \bar{\kappa}_{m+1}, \bar{c}^{m_2}) \supseteq ... \supseteq \varphi_{m,i_{m,j}}(N, \bar{\kappa}_{m+j}, \bar{c}^{m_2}) \supseteq ...$, where $\bar{c}^{m_1}$ is the tuple enumerating the set C_m ($\bar{c}_{\frac{(m+1)m}{2}}$ to be exact), and $\bar{c}^{m_2} = \bar{c}_{\frac{(n+1)n}{2}+m-1}$ is the tuple enumerating all the c's obtained so far. Choose a realization $c_{\frac{(n+1)n}{2}+m} \in N$ of this sequence. Then $c_{\frac{(n+1)n}{2}+m}$ is principal over K_{m-1} and the c_j's for $j < \frac{(n+1)n}{2} + m$.

Let Φ_{n+1} be the family of all $K_n \cup C_n$-definable 1-formulas that are satisfiable in $\mathfrak{N}$. Choose $c_{\frac{(n+1)n}{2}+n+1}$ as before, as a realization of a chain of nested principal subformulas. Denote $C_{n+1} := \{c_1, c_2, ..., c_{\frac{(n+1)n}{2}+n+1}\}$.

Let $A_\mu := K \cup \bigcup\limits_{i<\omega} C_i$. By Tarski-Vaught criterion A_μ is a universe of an elementary substructure of $\mathfrak{N}$.

It can be easily verified by induction that for every $i < \omega$ the type $tp(c_i/K_n)$ is principal starting from a sufficient $n < \omega$. For instance, for every $n \geq i-1$. Then $r(\mathfrak{A}_\mu) \setminus \bigcup\limits_{\kappa \in K_\mu} V_{p(\mathfrak{M})}(\kappa) = \emptyset$, since otherwise, if some element of $\mathfrak{A}_\mu$ was in this set, it would have a principal type over some tuple from K_μ, but, by Lemma 2, this is impossible.

Since the theory T is small, the theory $T \cup tp(\alpha_0\beta_0, \kappa_1, \kappa_2)$ is small as well. The number of different infinite sequences μ of zeros and ones equals to $2^{\aleph_0}$, thereofre $I(T \cup tp(\alpha_0\beta_0, \kappa_1, \kappa_2), \aleph_0) = 2^{\aleph_0}$, and $I(T, \aleph_0) = 2^{\aleph_0}$.

2) Suppose that for all $\delta_1, \delta_2 \in q(\mathfrak{M})$ with $tp(\delta_1) = tp(\delta_2)$, and all $n \in \mathbb{Z}$, $\mathfrak{M} \models \neg S_n(\delta_1, \delta_2)$. Then for every complete type $q_1 \supseteq q$ over $\{\alpha, \beta\}$ and every $\delta \in q_1(\mathfrak{M})$, δ is the only realization of q_1 in its neighborhood $V(\delta) := \bigcup\limits_{n \in \mathbb{N}} S_n(\delta, M)$.

2.1) Suppose that there exists $q_1 \in S_1(T)$ such that $q \subseteq q_1$, and q_1 has exactly n attached types for some $n \in \mathbb{N}$. This case contradicts with 2): for every $\delta \in q_1(\mathfrak{M})$ its neighborhood

$V(\delta)$ is infinite, but every its element should have a type attached to q_1 and realized by a singleton.

2.2) Let Π be the set of all types from $S_1(T)$ that extend q and have an infinite number of attached types. We consider the case when Π is infinite. Let $\Pi = P \cup R$, where P contains all the principal types from Π, and R – all non-principal types from Π. Since T is small, P and R can be either countable, finite or empty. Fix enumerations $P = \{p_1, p_2, ..., p_i, ...\}_{i<\omega}$, $R = \{r_1, r_2, ..., r_i, ...\}_{i<\omega}$.

2.2.1) Suppose that P is not empty. Consider, for instance, $p_1 \in P$. Since all the types of Π have unique realizations inside of a single neighbourhood, each r_n will be different from the other types of R by a formula $\psi_n(x) := \exists x_1(\varphi_1(x_1) \wedge S_n(x, x_1))$, where φ_1 is the isolating formula of p_1, and n is the distance between realizations of p_1 and r_n.

2.2.2) Suppose that $P = \emptyset$.

2.2.2.1) Suppose there exists a formula φ that belongs to only a finite number of types from Π. Then, analogically with 2.2.1), for every $n < \omega$ we can find a formula that distinguishes r_n from all other types form Π: $\psi_n(x) := \exists x_1 \exists x_2 ... \exists x_m(\bigwedge\limits_{1\leq i\leq m-1} S_{n_i}(x_i, x_{i+1}) \wedge \bigwedge\limits_{1\leq i\leq n} \varphi(x_i) \wedge S_n(x, x_1))$, where m is the number of types φ belongs to, and $n, n_1, ..., n_{m-1}$ are sufficient integers.

In cases 2.2.1) and 2.2.2.1), the set $\Gamma := \{\neg\varphi_i \mid i < \omega,\ \varphi_i$ is an isolating formula of $p_i\} \cup \{\neg\psi_i \in r_i \mid i < \omega\}$ is a locally consistent set of negations of representatives of all types of Π, and every its completion to a 1-type is not in Π. Then, by 2.1), the proof is done.

2.2.2.2) Let 2.2.2.1) be not true. Then every $\emptyset$-definable formula φ will belong either to no types from Π, or to all of them, or both φ and $\neg\varphi$ belong to an infinite number of types from Π. This contradicts with Restriction 1.

□

Corollary 1 *Let $\mathfrak{M}$ be a model of a small linearly ordered theory T that satisfies Restriction 1. Let for every $n < \omega$ there is $m_n \geq n$ such that in $\mathfrak{M}$ there exists a discrete chain of length m_n. Then $I(T, \aleph_0) = 2^{\aleph_0}$.*

Proof of Corollary 1. By compactness, there exists an infinite discrete chain in some elementary extension $\mathfrak{M}'$ of $\mathfrak{M}$. Then, by Theorem 2, $I(T, \aleph_0) = 2^{\aleph_0}$.

□

Corollary 2 *Let T be a small linearly ordered theory that satisfies Restriction 1 and such that $I(T, \aleph_0) < 2^{\aleph_0}$.*

1) There exists $n_T \in \mathbb{N}$ such that in every model of T length of every discrete chain is less than n_T.

2) If T has no finite models, every model of T is densely ordered up to finite discrete chains.

3 Conclusions

In the article, in order to avoid a fictitious linear order, a special restriction on theories was given. It was proved that if a small linearly ordered theory satisfies this restriction and has an infinite discrete chain, then it has $2^{\aleph_0}$ countable non-isomorphic models. Two corollaries of this theorem were given. Description of all cases of maximality of the countable spectrum ultimately leads to consideration of all possible countable spectra of complete theories with a definable linear order.

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References

[1] M. Rubin, "Theories of linear order", *Israel Journal of Mathematics* Vol. 17 (1974): 392–443.

[2] L. Mayer, "Vaught's conjecture for o-minimal theories", *Journal of Symbolic Logic* Vol. 53, No. 1(1988): 146–159.

[3] B.Sh. Kulpeshov, S.V. Sudoplatov, "Vaught's conjecture for quite o-minimal theories", *Annals of Pure and Applied Logic* Vol. 168, No. 1 (2017): 129–149.

[4] A. Alibek, B.S. Baizhanov , B.Sh. Kulpeshov , T.S. Zambarnaya, "Vaught's conjecture for weakly o-minimal theories of convexity rank 1", *Annals of Pure and Applied Logic* Vol. 169, No. 11 (2018): 1190–1209.

[5] S. Moconja, P. Tanovic, "Stationarily ordered types and the number of countable models", *Annals of Pure and Applied Logic* Vol. 171, No. 3 (2019): 102765.

[6] B.Sh. Kulpeshov, "Vaught's conjecture for weakly o-minimal theories of finite convexity rank", *Izvestiya: Mathematics* Vol. 84, No. 2 (2020): 324–347.

[7] Kudaibergenov K.Zh., "O konechno-porozhdennyh modelyah [On finitely generated models]", *Siberian Mathematical Journal* Vol. 27, No. 2 (1986): 208–209.

[8] Sudoplatov S.V., "Polnye teorii s konechnym chislom schetnyh modelej [Complete theories with finite number of countable models]", *Algebra and Logic* Vol. 43, No. 1 (2004): 110–124.

[9] Sudoplatov S.V., *Classification of countable models of complete theories: Part 1*, (Novosibirsk: Edition of NSTU, 2018): 326.

[10] Alibek A.A., Baizhanov B.S., Zambarnaya T.S., "Discrete order on a definable set and the number of models", *Matematicheskij zhurnal [Mathematical Journal]* Vol. 14, No. 3 (2014): 5–13.

[11] Baizhanov B., Baldwin J.T., Zambarnaya T., "Finding $2^{\aleph_0}$ countable models for ordered theories", *Siberian Electronic Mathematical Reports* Vol. 15, No. 7 (2018): 719–727.

[12] Baizhanov B.S., Umbetbayev O.A., Zambarnaya T.S., "On a criterion for omissibility of a countable set of types in an incomplete theory", *Kazakh Mathematical Journal* Vol. 19, No. 1 (2019): 22–30.

[13] Baizhanov B., Umbetbayev O., Zambarnaya T., "Non-existence of uniformly definable family of convex equivalence relations in an 1-type of small ordered theories and maximal number of models", *Kazakh Mathematical Journal* Vol. 19, No. 4 (2019): 98–106.

[14] Baizhanov B.S., Verbovskiy V.V., "Uporyadochenno stabil'nye teorii [Ordered stable theories]", *Algebra and Logic* Vol. 50, No. 3 (2011): 303–325.

[15] Baizhanov B.S., "Orthogonality of one-types in weakly o-minimal theories", *Algebra and Model Theory 2. Collection of papers*, eds.: A.G. Pinus, K.N. Ponomaryov. Novosibirsk, NSTU (1999): 5–28.

[16] Baizhanov B.S., Kulpeshov B.Sh., "On behaviour of 2-formulas in weakly o-minimal theories", *Mathematical Logic in Asia*, Proceedings of the 9th Asian Logic Conference, eds.: S. Goncharov, R. Downey, H. Ono, World Scientific, Singapore (2006): 31–40.

[17] Tent K., Ziegler M., *A Course in Model Theory*, (Cambridge: Cambridge University Press, 2012): x + 248.

2-бөлім Раздел 2 Section 2

Механика Механика Mechanics

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RESONANT PHENOMENA IN NONLINEAR VERTICAL ROTOR SYSTEMS

In this paper, the study of the dynamics of a rotor system mounted on an elastic foundation rotating in rolling bearings is considered. To describe the bearing model, the Hertz theory was used, linking radial loads acting on the bearing and deformation at the points of contact between the movable foundation and the bearing rings. In the bearing model, it is assumed that there are no types of sliding of bodies and rolling surfaces. The obtained differential equations of the rotor and the foundation do not have a common solution. Therefore, the study was conducted using numerical methods. In order to simplify the problem and increase the accuracy in solving the obtained differential equations, dimensionless quantities were used. With the increase and decrease of dimensionless quantities, the amplitudes of the rotor and the foundation are constructed. As a result, two resonances were formed: the main resonance and the second resonance. The work is connected with the physical meaning of the process considered in the problem the results obtained are the basis for the application of this mathematical model in the design of a rotary system rotating in rolling bearings.
Key words: Hertz theory, rolling bearings, numerical methods, "rotor-foundation" system, nonlinear rotary system.

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Бейсызық роторлық жүйелердеги резонанстық құбылыстар

Бұл жұмыста домалау мойынтіректерінде айналатын серпімді негізге орнатылған ротор жүйесінің динамикасын зерттеу қарастырылады. Мойынтірек моделін сипаттау үшін Герцті мойынтірекке әсер ететін радиалды жүктемелерді байланыстыратын теориясы және жылжымалы негіз мен мойынтірек сақиналары арасындағы байланыс нүктелеріндегі деформация қолданылды. Мойынтірек моделінде жылжымалы денелер мен жылжымалы беттер жоқ деп болжанады. Алынған ротор мен фундаменттің дифференциалдық теңдеулерінде ортақ шешім жоқ. Сондықтан зерттеу сандық әдістерді қолдану арқылы жүргізілді. Есепті жеңілдету және алынған дифференциалдық теңдеулерді шешудің дәлдігін арттыру үшін өлшемсіз шамалар қолданылды. Өлшемсіз шамалардың ұлғаюымен және азаюымен қозғалтқыш пен фундаменттің амплитудасы құрылады. Нәтижесінде екі резонанс пайда болды: бас резонанс және екінші резонанс. Жұмыс тапсырмада қарастырылған процестің физикалық мағынасымен байланысты. Алынған нәтижелер осы математикалық модельді домалау мойынтіректерінде айналатын роторлы жүйені жобалау кезінде қолдануға негіз болып табылады.
Түйін сөздер: Герц теориясы, домалау мойынтіректері, сандық әдістер, "ротор-фундамент" жүйесі, бейсызық роторлық жүйе.

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Резонансные явления в нелинейных вертикальных роторных системах

В данной работе рассматривается исследование динамики роторной системы, установленной на упругом основании, вращающемся в подшипниках качения. Для описания модели подшипника использовалась теория Герца, связывающая радиальные нагрузки, действующие на подшипник, и деформацию в точках контакта между подвижным основанием и кольцами подшипника. В модели подшипника предполагается, что не существует типов скольжения тел и поверхностей качения. Полученные дифференциальные уравнения ротора и фундамента не имеют общего решения. Поэтому исследование проводилось с использованием численных методов. Для упрощения задачи и повышения точности решения полученных дифференциальных уравнений использовались безразмерные величины. С увеличением и уменьшением безразмерных величин строятся амплитуды двигателя и фундамента. В результате образовались два резонанса: главный резонанс и второй резонанс. Работа связана с физическим смыслом рассматриваемого в задаче процесса. Полученные результаты являются основой для применения данной математической модели при проектировании вращающейся системы, вращающейся в подшипниках качения.

Ключевые слова: теория Герца, подшипники качения, численные методы, система "ротор-фундамент" , нелинейная роторная система.

1 Introduction

Currently, most of the rotary machines used in industry, manufacturing and mechanical engineering rotate in rolling bearings [1]. As a mathematical model of rolling bearings, it is important to choose models that most fully reflect the features of rolling bearings, in particular, such as the number of holes, the influence of geometric errors, as well as the properties of nonlinear stiffness; the influence of centrifugal forces, mutual displacement and mismatch of bearing rings; gyroscopic phenomena [2].

In the proposed work, the nonlinear dynamics of a rotor system mounted on an elastic foundation rotating in rolling bearings is investigated. Due to the increased requirements for the accuracy of rotation and an increase in the speed of rotation of the rotors, it becomes necessary to take into account the elastic nonlinear properties of rolling bearings.

2 Review

Currently, rotating machines, widely used in industry, mainly work with rolling bearings [3] and [4]. The use of rolling bearings as supports for high-speed rotors is limited by their speed and strength, therefore, sliding bearings are widely used to ensure reliable rotation of the rotor in a wide range of speeds and loads. These bearings have smaller dimensions in the radial direction, greater rigidity, low sensitivity to shocks and temporary loads, unlike rolling bearings, which makes them suitable for use in high-speed turbomachines.

According to the number of rotor supports, the turbomachine layout schemes used can be two- and three-support. Three-support rotor schemes are used in rare cases when a two-support scheme leads to an unacceptably large decrease in the bending stiffness of the rotor. The use of a three-support circuit makes the rotor statically indeterminate, which makes it difficult to assemble the turbomachine due to the difficulty of ensuring an accurate fit of the rotor in the foundation on three surfaces [5].

Also, for the most complete description of the process, it is important to take into account the influence of factors such as imbalance, asymmetry of the rotor installation on the shaft, external friction, changes in inertial parameters and positional forces of various kinds [6]- [8].

Such complications of the model in the analysis of dynamics make it possible to investigate the effect of the gap size, rotation frequency on the frequency spectra and amplitude-frequency characteristics for any rotary system on rolling bearings.

Mathematical models of bearings that take into account non-linearity factors are distinguished by complexity and, first of all, by the loads they take into account. In our case, the Hertz contact theory is used to describe the bearing model, which relates radial loads acting on the bearing and deformation at the points of contact between the rolling body and the bearing rings [9]. When describing the bearing model, it is assumed that there are no any types of slippage of rolling bodies and surfaces. Damping is considered in the formulation of equivalent viscous and linear friction.

3 Problem statement and equation of motion

Consider a vertical rotary system (Fig. 1). The damping foundation on elastic supports moves in a horizontal plane. The rotor has a static imbalance. The rotor performs a plane-parallel motion, and rotation around the coordinate axes does not occur. The motion of the rotor and the foundation is considered relative to the fixed coordinate system of the Oxy. The nonlinear regenerative force of the bearing is described as (1) in accordance with the Hertz contact theory.

$$F_C = C_b \delta_r^{\frac{3}{2}} \tag{1}$$

where F_C is a component of the restoring force in the radial direction (N), δ_r is the deformation in the radial direction (m), C_b is the stiffness coefficient $(\frac{N}{m}^{\frac{3}{2}})$.

In order to solve the equations of motion of the system and qualitative analysis, the restoring force of a bearing of type (1) can be approximated by a degree series of type (2) in accordance with [10], [13]:

$$F_C = c_0 \delta_r + c_1 \delta_r^3 \tag{2}$$

where c_0 and c_1 are stiffness coefficients for the linear and cubic terms, respectively. This expansion for $\delta_r < 1000\mu m$ with a sufficient degree of accuracy is in agreement with the experiments [13].

The geometric coordinates of the rotor center are denoted by $O_1(x_1, y_1)$, and its center of mass is denoted by $O_s(x, y)$. The center of mass of the foundation is $O_2(x_2, y_2)$ (Fig.1).

The kinetic energy of the system is defined as (3) :

$$U = \frac{m}{2}(x'^2 + y'^2) + \frac{J}{2}\Omega_0^2 + \frac{M}{2}(x_1'^2 + y_1'^2) \tag{3}$$

where m is the mass of the rotor, J is the moment of polar inertia of the rotor, and M is the mass of the foundation.

Considering that the potential energy of the isotropic elastic nonlinear field of rolling bearings depends on the radially directed deformation of rolling bearings, that is

$$\delta_r^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \tag{4}$$

Where is the potential energy of rolling bearings and elastic supports:

$$W = \frac{c_2}{2}(x_2^2 + y_2^2)^2 + \frac{c_0}{2}((x_1 - x_2)^2 + (y_1 - y_2)^2) + \frac{c_1}{4}((x_1 - x_2)^4 + (y_1 - y_2)^4)) + ... \quad (5)$$

Accordingly, the dissipation function:

$$R = \frac{\chi}{2}(x'^2 + y'^2) + \frac{\chi_0}{2}(x_2'^2 + y_2'^2) \quad (6)$$

The center of mass of the rotor relative to a fixed coordinate system is determined as follows:

$$x = x_1 + e\cos\Omega_0 t, y = y_1 + e\sin\Omega_0 t. \quad (7)$$

where e is the magnitude of the deviation of the center of mass of the rotor from the geometric center.

The equation of motion of the system has the form (8)

$$\begin{cases} m\frac{d^2x_1}{dt^2} + c_0(x_1 - x_2) + c_1(x_1 - x_2)^3 + \chi\frac{dx_1}{dt} = me\Omega_0^2\cos\Omega_0 t, \\ m\frac{d^2y_1}{dt^2} + c_0(y_1 - y_2) + c_1(y_1 - y_2)^3 + \chi\frac{dy_1}{dt} = me\Omega_0^2\sin\Omega_0 t, \\ M\frac{d^2x_2}{dt^2} + c_2x_2 - c_0(x_1 - x_2) - c_1(x_1 - x_2)^3 + \chi_0\frac{dx_2}{dt} = 0, \\ M\frac{d^2y_2}{dt^2} + c_2y_2 - c_0(y_1 - y_2) - c_1(y_1 - y_2)^3 + \chi_0\frac{dy_2}{dt} = 0, \\ x_1(0) = e, \quad x_2(0) = 0.1e, \quad y_1(0) = 0, \quad y_2(0) = 0, \\ \frac{dx_1}{dt}|_{t=0} = 0, \frac{dx_2}{dt}|_{t=0} = 0, \frac{dy_1}{dt}|_{t=0} = 0, \frac{dy_2}{dt}|_{t=0} = 0, \end{cases} \quad (8)$$

(8) – a system of equations describes the movement of the rotor and the foundation on unbalanced, nonlinear supports.

We reduce the system of equations (8) to a system of dimensionless equations of the form (9), i.e

$$\begin{cases} \frac{d^2f_1}{d\tau^2} + 2\zeta_1\frac{df_1}{d\tau} + (f_1 - f_2) + \varepsilon(f_1 - f_2)^3 = \eta^2\cos(\eta\tau), \\ \frac{d^2\nu_1}{d\tau^2} + 2\zeta_1\frac{d\nu_1}{d\tau} + (\nu_1 - \nu_2) + \varepsilon(\nu_1 - \nu_2)^3 = \eta^2\sin(\eta\tau), \\ \frac{d^2f_2}{d\tau^2} + 2\mu\zeta_2\frac{df_2}{d\tau} - \mu(f_1 - f_2) - \mu\varepsilon(f_1 - f_2)^3 + \mu\lambda f_2 = 0, \\ \frac{d^2\nu_2}{d\tau^2} + 2\mu\zeta_2\frac{d\nu_2}{d\tau} - \mu(\nu_1 - \nu_2) - \mu\varepsilon(\nu_1 - \nu_2)^3 + \mu\lambda\nu_2 = 0, \end{cases} \quad (9)$$

where,

$$x_1 = ef_1, x_2 = ef_2, y_1 = e\nu_1, y_2 = e\nu_2,$$

$$\mu = \frac{m}{M}, \omega_1^2 = \frac{c_0}{m}, \omega_2^2 = \frac{c_2}{m}, \tau = \omega_1 t, \Omega_0 = \omega_1\eta,$$

$$\zeta_1 = \frac{\chi}{2m\omega_1}, \zeta_2 = \frac{\chi_0}{2m\omega_1}, \varepsilon = \frac{c_1e^2}{m\omega_1^2}, \lambda = \frac{\omega_2^2}{\omega_1^2}.$$

We introduce complex variables in the form (10)

$$z_1 = f_1 + i\nu_1, z_2 = f_2 + i\nu_2. \quad (10)$$

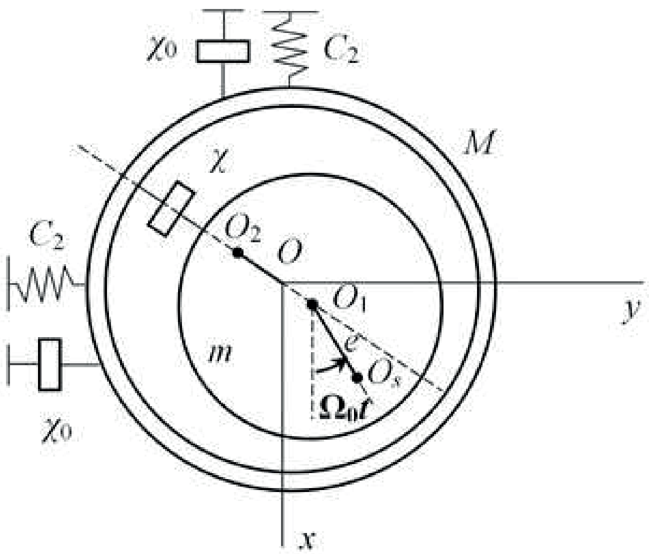

Figure 1: Vertical rotor system.

Then, given (10) from (9), we write the equation of motion in the complex plane in the form (11).

$$\begin{cases} z_1'' + (z_1 - z_2) + \varepsilon(z_1 - z_2)^3 + 2\zeta_1 z_1' = \eta^2 e^{i\eta\tau}, \\ z_2'' + \mu\lambda z_2 - \mu(z_1 - z_2) - \mu\varepsilon(z_1 - z_2)^3 + 2\mu\zeta_2 z_2' = 0. \end{cases} \tag{11}$$

The approximated solution of the system of equations (11) can be searched analytically.

$$z_1 = A_1 e^{-i\eta\tau} + B_1 e^{-2i\eta\tau} + ..., \tag{12}$$

$$z_2 = A_2 e^{-i\eta\tau} + B_2 e^{-2i\eta\tau} + ... \tag{13}$$

Studies in this direction can be found in the works [11], [12]. The parametric analysis of a given rotor system in the presented work is based on the results of numerical methods.

4 Results and discussion

In Figure 2, there is one resonance and one autotherm zone. In the case of a head resonance, i.e., at $\eta = 1.49$, $f_1 = 1.789523788$ is equal to when the rotor amplitude $\mu = 10$. For the rotor, the autothermic zone is generally observed in the range of $0.01 < \eta < 0.99$. In this case, $f_1 = 0.364555924$, when the maximum amplitude of the rotor is $\mu = 10$. With a decrease in the mass of the foundation, the value of the amplitudes of the rotor and the autotherm of the foundation decreases to 35-50%. With an increase in μ, there is a shift of the head resonance to the right in the direction of frequency growth and an increase in the autothermic zone. A decrease in the mass of the foundation and the movement of the rotor and the foundation in the reverse phase in this interval leads to the disconnection of the autotherms.

In Figure 3, the region of bass resonance and autotherms is observed at low frequencies and occurs in a wide range of ?. In the case of a general resonance, i.e. at $\eta = 1.13$, $f_2 = 0.809546423$ is equal to when the amplitude of the base $\mu = 5$ at $\mu = 10$, there is a shift of the head resonance to the right in the direction of frequency growth. If the mass of the foundation decreases, i.e. the value μ increases, then the amplitude of the rotor increases during the head resonance, and vice versa, if the value μ decreases, the area of autotherms and leads to the attenuation of the head resonance.

In Figure 4, two resonances occur. At the head nonlinear resonance, i.e. $\eta = 1.13$, the rotor amplitude is equal to $f_1 = 1.382985602$ for $\varepsilon = 1$. The second (left) resonance is observed in the range of $0.01 < \eta < 0.32$. In this case, the rotor amplitude at all values of ε is $f_1 = 0.080489253$. When the rigidity of the rotor decreases, the value of the left resonant amplitudes of the rotor and the foundation becomes the same. At $\varepsilon = 50$, the amplitude of the rotor reaches $f_1 = 2.146287817$, at $\eta = 1.77$, there is a break in the amplitudes, and at $\eta > 1.77$, the amplitude values decrease sharply. At $\varepsilon = 100$, the rotor amplitude is equal to $f_1 = 3.372849069$, and there is a shift of the head resonance to the right in the direction of frequency growth. If the rigidity of the rotor decreases, i.e. the value of ε decreases, then the value of the resonant amplitudes on the left will have the same value, and vice versa, if the value of ε increases, the amplitude of the rotor at the head resonance will increase.

In Figure 5, there is a general resonance. At the main resonance, i.e. $\eta = 1.21$, $f_2 = 0.130031134$, when the amplitude of the base is $\varepsilon = 2$. At $\varepsilon = 50$, the amplitude of the foundation is equal to $f_2 = 0.182760123$. At $\varepsilon = 50$, the amplitude of the foundation reaches $f_2 = 0.19322489$, at $\eta = 1.73$, there is a break in the amplitudes, and at $\eta > 1.73$, the amplitude values decrease sharply. At $\varepsilon = 100$, the amplitude of the base is equal to $f_2 = 0.343379276$, and there is a shift of the head resonance to the right in the direction of frequency growth. If the value ε increases during the head resonance, then the rotor amplitude increases and there is a shift of the head resonance to the right in the direction of frequency increase.

In Figure 6, two resonances occur. At the head resonance, i.e. $\eta = 1.2$, the rotor amplitude $\lambda = 1$ is equal to $f_1 = 1.38132898$. The second (left) resonance is observed in the range of $0.01 < \eta < 0.52$. In this case, the value $f_1 = 0.201216136$ when the rotor amplitude is $\lambda = 10$. With a decrease in the rigidity of the foundation, the value of the left resonant amplitudes of the rotor and the foundation decreases to 15-20%, but at $\lambda = 0.1$, the left resonant amplitude is equal to $f_1 = 0.231227653$. If the rigidity of the foundation decreases, i.e. the value λ decreases, then the value of the resonant amplitudes on the left decreases, and vice versa, if the value λ increases, then the amplitude of the rotor at the head resonance will have the same value.

In Figure 7, one resonance occurs. The second (left) resonance is observed in the range of $0.01 < \eta < 0.35$. In this case, when the amplitude of the foundation is $\lambda = 10$, $f_2 = 0.203282867$. With a decrease in the rigidity of the Foundation, the value of the left resonant amplitudes of the rotor and the foundation decreases to 10-37%, but at $\lambda = 0.1$, the left resonant amplitude is equal to $f_2 = 0.243458891$. If the hardness of the foundation decreases, i.e. the value λ decreases (except $\lambda = 0.1$), then the value of the left resonance amplitudes decreases, and vice versa, if the value λ increases, this leads to a head resonance shutdown.

In Figure 8, two resonances occur. At the head resonance, i.e. $\eta = 1.26$, $f_1 = 9.182929085$ is equal to when the rotor amplitude is $\zeta_1 = 0.1$. The second (left) resonance is observed in the range of $0.01 < \eta < 0.15$. In this case, $f_1 = 0.306469753$ is equal to when the rotor amplitude is $\zeta_1 = 0.2$. With a decrease in the internal coefficient of friction, the value of the left resonant amplitudes of the rotor and the foundation increases to 30-45%. With an increase in the coefficient of internal friction, i.e. when $\zeta_1 > 2$, the zone of autotherms and leads to the deactivation of the head resonance. If the internal coefficient of friction decreases, i.e. the value of ζ_1 increases, then this leads to a shutdown of the left and head resonance, and vice versa, if the value of ζ_1 decreases, the amplitude of the rotor at the head resonance increases.

In Figure 9, two resonances occur. At the main resonance, i.e. $\eta = 1.26$, $f_2 = 1.022983788$, when the amplitude of the base is $\zeta_1 = 0.1$. The second (left) resonance is observed in the range of $0.01 < \eta < 0.35$. In this case, at $\zeta_1 = 0.1$, the amplitude of the foundation is equal to $f_2 = 0.53389977$. With a decrease in the internal coefficient of friction, the value of the left resonant amplitudes of the rotor and the foundation increases to 40-60% at $\zeta_1 < 1$. With an increase in the coefficient of internal friction, i.e. when $\zeta_1 > 2$, it leads to the attenuation of the head resonance. If the internal coefficient of friction decreases, i.e. the value of ζ_1 increases, then this leads to a shutdown of the head resonance, and vice versa, if the value of ζ_1 decreases, the amplitude of the rotor at the head resonance increases.

If the external coefficient of friction ζ_2 decreases or increases, then the value of the amplitudes of the rotor and the foundation at the head resonance will have the same value.

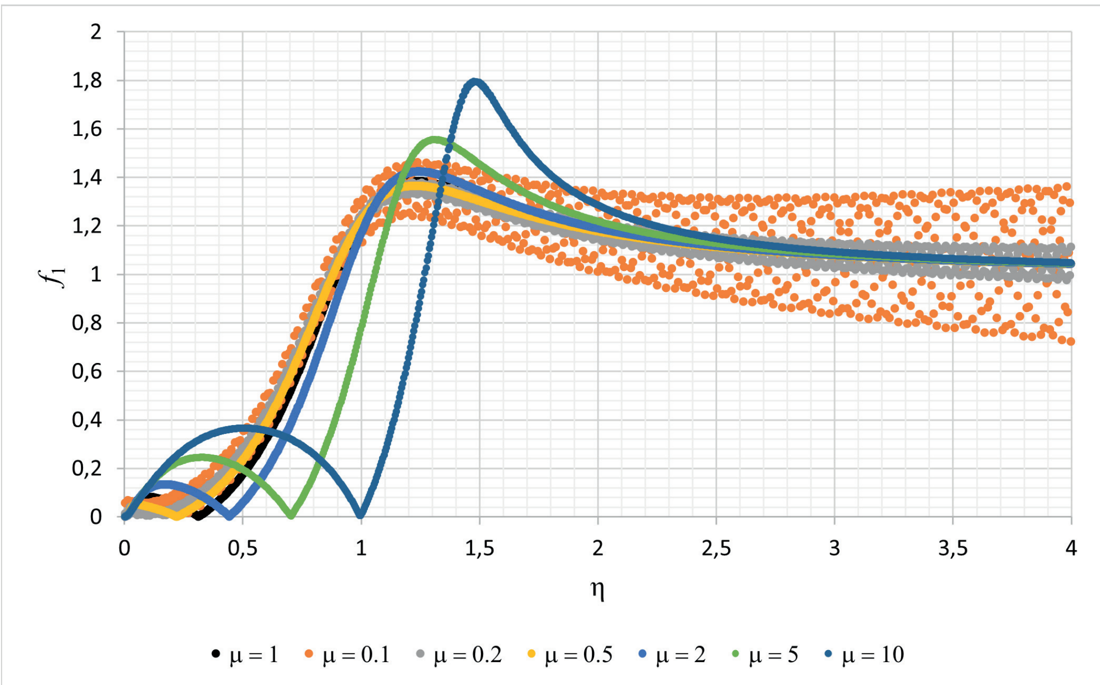

Figure 2: The amplitude of the rotor at different values of the ratio of the mass of the rotor and the foundation – μ.

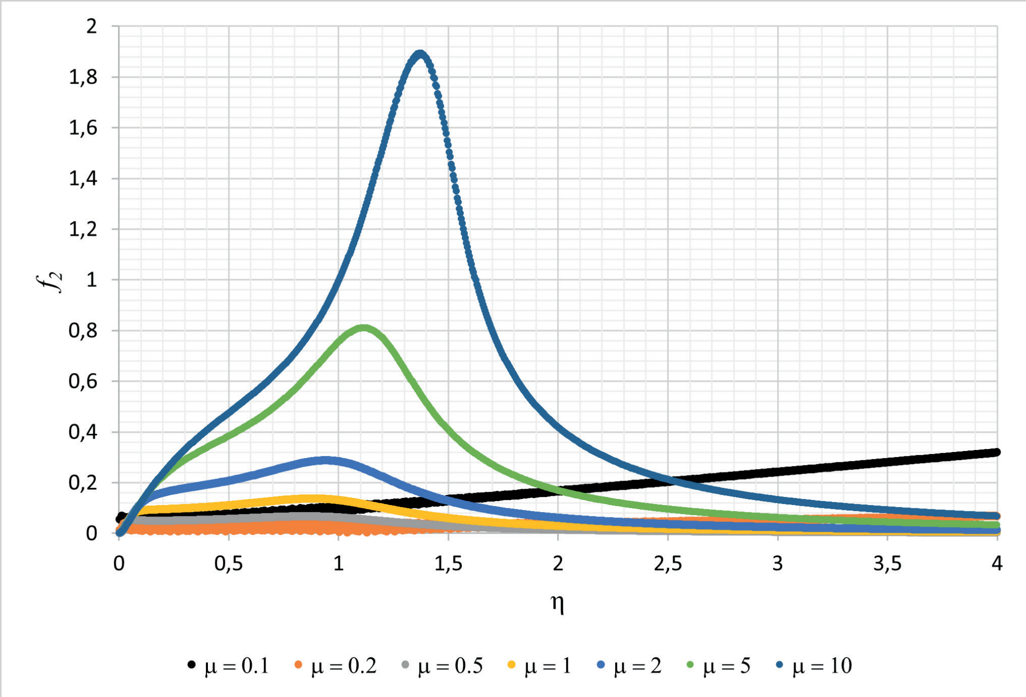

Figure 3: The amplitude of the foundation at different values of the ratio of the mass of the rotor and the foundation – μ.

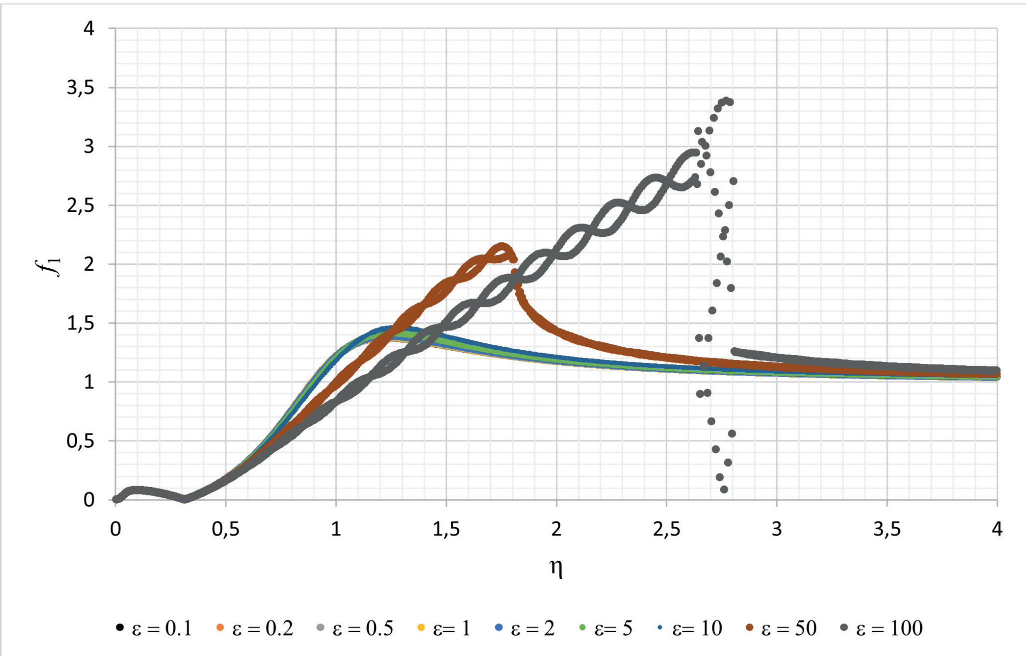

Figure 4: The amplitude of the rotor at different values of the stiffness of the rotor – ε.

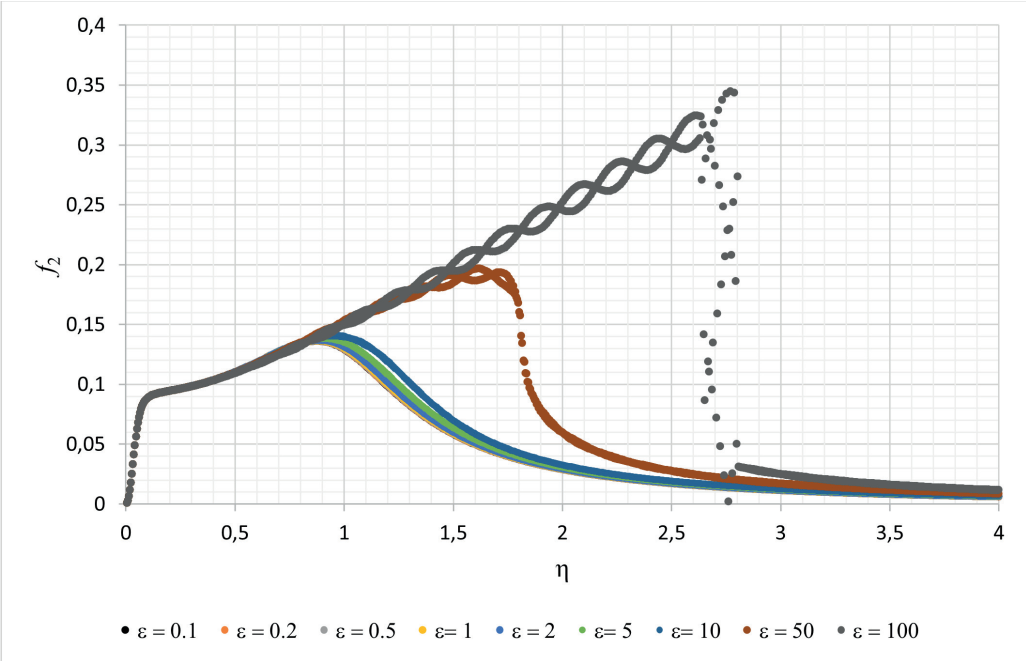

Figure 5: The amplitude of the foundation at different values of the stiffness of the rotor – ε.

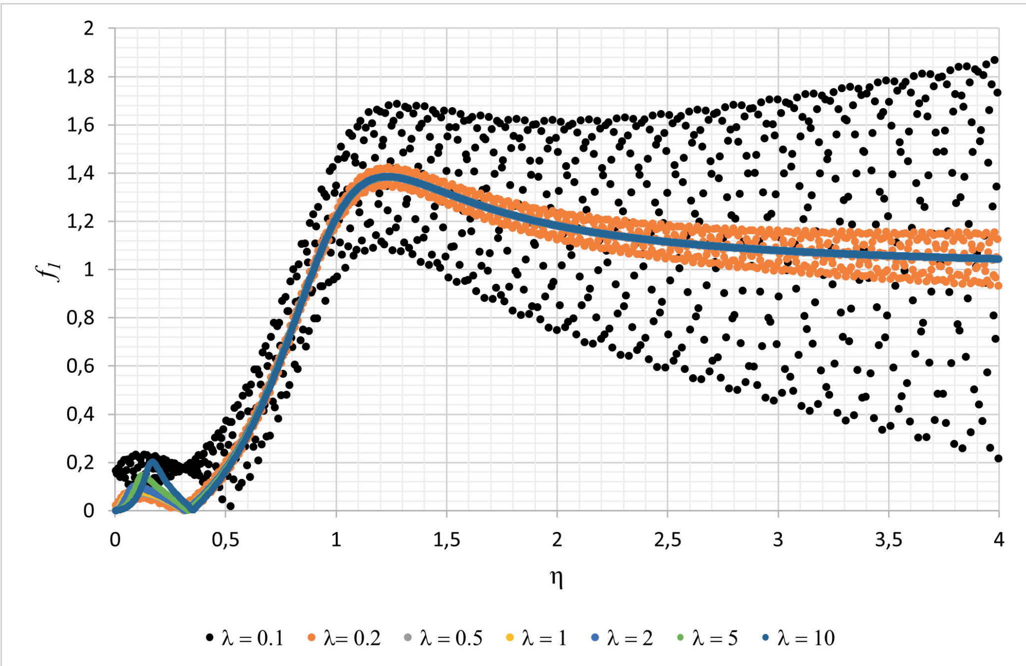

Figure 6: The amplitude of the rotor at different values of the rigidity of the foundation – λ.

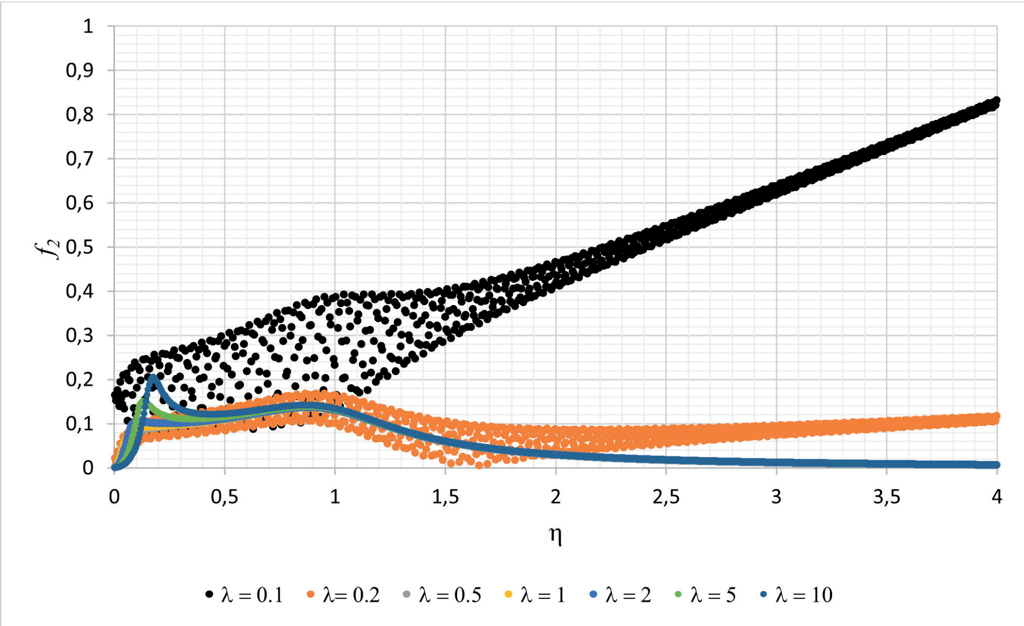

Figure 7: The amplitude of the foundation at different values of the rigidity of the foundation – λ.

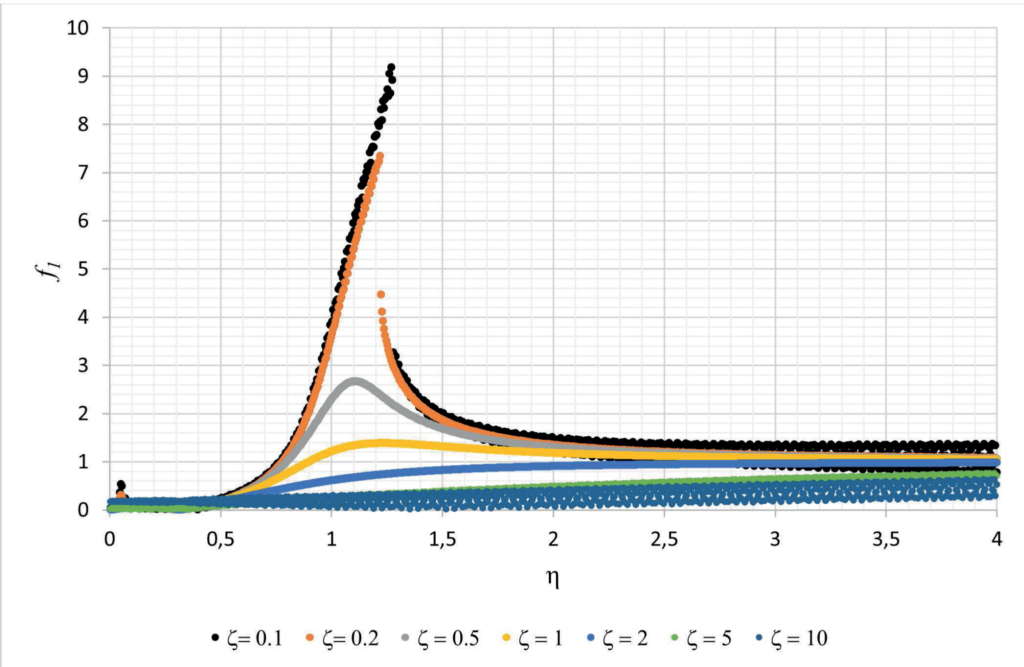

Figure 8: The amplitude of the rotor at different values of the coefficient of internal friction – ζ.

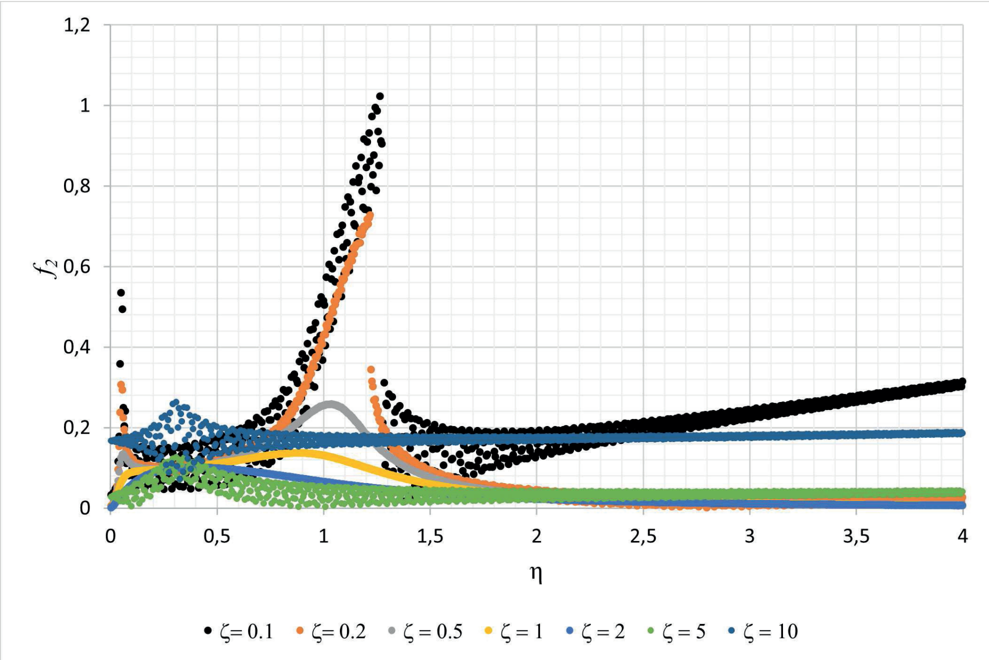

Figure 9: The amplitude of the foundation at different values of the coefficient of internal friction – ζ.

5 Conclusion

In this paper, a generalized dynamic model of the "rotor-foundation" system on elastic supports has been developed, the description of which is nonlinear. A method for determining the amplitude of forced oscillations of the system has been developed. The resonant frequencies are determined, as well as the frequency range in which the answering machines occur. The features of the coupled "rotor-foundation" system are shown when taking into account the movement of the foundation. The values of the coefficients of imbalance, foundation mass, stiffness and damping, providing optimal values of amplitudes, are determined. The results of the work performed prove the physical meaning of the task, and this, in turn, can serve as the basis for the introduction and application of this mathematical model in production. Disabling dangerous rotor vibrations by selecting system parameters is cost-effective and technically easy. The results of the work make it possible to conduct engineering and computational experiments with minimal costs, give qualitative and quantitative characteristics and reduce the design time of new vertical rotary machines, improve the quality and safety of their work.

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References

[1] Sharma A., Upadhyay N., Kankar P.K., Amarnath M., "Nonlinear dynamic investigations on rolling element bearings: A review" , *Advances in Mechanical Engineering,* 10 (3) (2018): 1–15. DOI: 10.1177/1687814018764148.

[2] Li Z., Li J., Li M., "Nonlinear dynamics of unsymmetrical rotor-bearing system with fault of parallel misalignment" , *Advances in Mechanical Engineering,* 10(5) (2018): 1–17. DOI: 10.1177/1687814018772908.

[3] Sun L., "Active Vibration Control of Rotor-Bearing System. PhD Thesis" , *University of Melbourn, Melbourn* (1995).

[4] Bai C., Zhang H., Xu Q. "Experimental and numerical studies on nonlinear dynamic behavior of rotor system supported by ball bearings" , *Journal of Engineering for Gas Turbine and Power,* 132(8) (2010): 082502. DOI:10.1115/1.4000586.

[5] Xia Z., Qiao G., Zheng T., Zhang W., "Nonlinear modelling and dynamic analysis of the rotor-bearing system" , *Nonlinear Dynamics,* 57(4) (2009): 559–577. DOI: 10.1007/s11071-008-9442-3.

[6] Harris T.A., *Rolling Bearing Analysis* (John Wiley & Sons, Inc., 2001).

[7] David P.F., Poplawski J.V., "Transient vibration prediction for rotors on ball bearings using load-dependent non-linear bearing stiffness" , *International Journal of Rotating Machinery,* 10(6) (2002): 489–494. DOI: 10.1080/10236210490504102.

[8] Yamamoto T., Ishida Y., "Transient vibration prediction for rotors on ball bearings using load-dependent non-linear bearing stiffness" , *International Journal of Rotating Machinery,* 10(6) (2002): 489–494. DOI: 10.1080/10236210490504102.

[9] Changsen W., *Analysis of rolling element bearings* (Mechanical Engineering Publications Limited, 1991).

[10] Kel'zon A.S., Zhuravlev Yu.N., YAnvarev N.V., *Raschet i konstruirovanie rotornyh mashin [Calculation and design of rotary machines]* (L.: Mashinostroenie, 1977): 288.

[11] Kydyrbekuly A., Khajiyeva, L., Gulama-Garyp A.Y., Kaplunov J., "Nonlinear Vibrations of a Rotor-Fluid-Foundation System Supported by Rolling Bearings" , *Strojniski Vestnik/Journal of Mechanical Engineering* 62(6) (2016): 351–362.

[12] Kydyrbekuly A., Ibrayev G., Ospan T., Nikonov A., "Multi-parametric Dynamic Analysis of a Rolling Bearings System" , *Strojniski Vestnik/Journal of Mechanical Engineering* 67(9) (2021): 421–432.

[13] Jin Y., Lu Z., Yang R., Hou L., Chen Y. "A new nonlinear force model to replace the Hertzian contact model in a rigid-rotor ball bearing system" , *Applied Mathematics and Mechanics* 39(3) (2018): 365–378.

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ANALYSIS OF THE HETEROGENEITY INFLUENCE ON MAIN PARAMETERS OF POROUS MEDIA AT THE PORE SCALE

The study to analyze the heterogeneity of porous medium, especially its influence of main characteristics is conducted in this paper. For this goal computer versions of real porous models, which were available in open source, were used. These models contain several slices at three directions. Re-build and processing of models were performed using the Avizo Software. For analyzing the influence of the heterogeneity of porous medium on basic parameters, each model were divided into several geometrical pieces. These parameters were computed using computational method of pore network modelling (PNM) and Kozeny-Carman (KC) method with purpose of collation. Also, these two methods were collated with available data from direct numerical computation method (DNC). According to analyzing it was set a good match between PNM and DNC for sandstone models, also KC method showed a divergence with DNC. For carbonate models, a divergence was seen between PNM and DNC, and KC method showed very good match with DNC. The relationship between the parameters for each of piece shows inhomogeneous character for the carbonate model.
Key words: Pore Network Modelling, Direct Numerical Computation, Kozeny-Carman equation, permeability, porosity.

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Кеуекті ортаның біркелкі еместігінің оны сиппаттайтын параметрлерге әсерін кеуек масштабын зерттеу

Бұл мақалада кеуекті ортаның біртексіздігінің, әсіресе оның негізгі сипаттамаларға әсерін талдау үшін зерттеу жүргізілді. Осы мақсатта нақты кеуекті модельдердің компьютерлік нұсқалары пайдаланылды. Бұл үлгілерде үш бағытта бірнеше тілім бар. Үлгілерді қайта құру және өңдеу Avizo бағдарлама арқылы жүзеге асырылды. Ортаның біртексіздігінің негізгі параметрлерге әсерін талдау үшін әрбір модель бірнеше геометриялық бірдей бөліктерге бөлінді. Бұл параметрлер салыстыру үшін кеуек желісін модельдеу (КЖМ) және Козени-Карман (КК) әдістерімен есептелді. Сонымен қатар, бұл екі әдіс тікелей сандық модельдеу (ТСМ) әдісінің қолда бар деректерімен салыстырылды. Талдауға сәйкес, құмтас әлгілері үшін КЖМ және ТСМ арасында жақсы сәйкестік табылды, ал КК әдісі ТСМ әдісімен сәйкессіздікті көрсетті. Карбонатты модельдер үшін КЖМ және ТСМ әдісітірінің арасында сәйкессіздік байқалды, ал КК әдісі ТСМ өте жақсы сәйкестік көрсетті. Әрбір бөлік үшін параметрлер арасындағы байланыс карбонатты модель үшін гетерогенді сипат көрсетті.
Түйін сөздер: кеуекті желіні модельдеу, тікелей сандық модельдеу, Козени-Карман теңдеуі, өткізгіштік, кеуектілік.

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Анализ влияния неоднородности на основные параметры пористых сред в масштабе пор

В данной статье было проведено исследование по анализу неоднородности пористой среды, особенно ее влияния на основные характеристики. Для этой цели использовались компьютерные версии реальных пористых моделей, которые были доступны в открытом доступе. Эти модели содержат несколько срезов в трех направлениях. Перестроение и обработка моделей производились с помощью програмного обеспечния Avizo. Для анализа влияния неоднородности пористой среды на основные параметры каждая модель разбивалось на несколько геометрически одинаковых частей. Эти параметры были рассчитаны с использованием вычислительного метода моделирования поровой сети (МПС) и метода Козени-Кармана (КК) с целью сопоставления. Кроме того, эти два метода были сопоставлены с доступными данными из метода прямого численного моделирования (МЧМ). Согласно анализу, было установлено хорошее соответствие между МПС и МЧМ для моделей песчаника, в то время как метод КК показал расхождение с МЧМ. Для карбонатных моделей было замечено расхождение между МПС и МЧМ, в то время как метод КК показал очень хорошее совпадение с МЧМ. Соотношение между параметрами для каждых частей показало неоднородный характер для карбонатной модели.

Ключевые слова: моделирование поровой сети, прямое численное моделирование, уравнение Козени-Кармана, проницаемость, пористость.

1 Introduction

Determination of main parameters of pore media are important for different areas, for example, understanding the effect of various oil recovery factors with their further development. Researches on the core size models, for example, $\sim$ 4–6 cm are carried out on the basis of laboratory experiments. But laboratory tests have a number of disadvantages [1-4]. Real models of pore medium are very complex, what makes experimental tests inconvenient. It is not possible to carry out different tests on the same core models since experimental studies on the core are destructive [5,6].

Thus, with the development of computerized technologies, in the last 10 to 20 years, the idea of a computer version of core (obtaining properties based on modeling at the pore scale) has received more and more attention. The computerized models of porous structure are used to determine properties by simulating processes on these models [7-9].

In the theory of soils, the concepts of fictitious and ideal soils are introduced. A fictitious soil is understood as a porous medium formed by solid balls (ball filling). An ideal soil is understood as a medium with pores in the form of a bundle of capillaries. The KC equation reflects the relationship between the porosity of the layer, the specific surface of the particles in the medium, the pressure difference, the length of the medium, and the filtration rate for this ideal structure. But this method is very inaccurate in order to their simplification of porous models into ideal cylinders (capillaries) which does not consider inhomogeneous character of the medium [10].

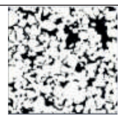
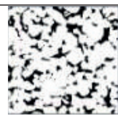
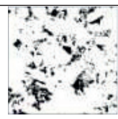
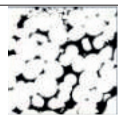
The most usable and developing methods at last decades for computation geometrical and hydrodynamic properties are PNM, DNC and Lattice-Boltzmann method from real computerized models of void material. Determining the main parameters by PNM method, the void material reconstructs into spherical and cylindrical system. But, reconstruction of real core models into cylinders and spheres includes inaccuracy in this method. Direct numerical computation is the most accurate numerical method. It is solved directly from the rendered computer versions of pore material and consider all the complexities of the geometry of the pore structure. The disadvantage of this method is that it requires large computer resources and the complexity of determining the boundary conditions [11-16]. The purpose of this work is to analyze the heterogeneity of pore structure of real porous material by dividing into identical pieces in four computer models. For each piece the computation of basic parameters of porous medium is performed using the PNM and KC methods. The characteristics of full-sized models are also computed and compared with direct numerical computation.

2 Materials and methods

2.1 Materials

The x-ray images of real porous models were obtained using micro-computed tomography. These samples are available on the website of the Imperial College of London [17]. Each sample contained 1000 slices in three directions with the appropriate resolution (Table 1).

Table 1: Properties of samples

Core models	**Voxels in 3 directions**	**Resolution**	**2D slice of core**
Bentheimer sandstone	1000	3.0035 μm	
Doddington sandstone	1000	2.6929 μm	
Estaillades carbonate	1000	3.31136 μm	
Ketton carbonate	1000	3.00006 μm	

The obtained data has already been pre-processed in order to reduction of noise and corner smoothing, therefore, in this study, image processing was reduced to building a 3D version of the samples. Avizo software was used to build a 3D model. The construction of a 3D model was carried out using the data, described in table-1. Number of voxels is 3d pixel and each voxel have their own value as the resolution. After the construction of the 3D model, the “axis connectivity” operation was performed in the Avizo software. Axis Connectivity generates a binary image containing all paths connecting the two planes. This operation makes it possible to obtain information about the connected pores. After separation of pore spaces, the pore network model was constructed. A Pore Network Model is designed with store data that can be represented as linear lines in 3D space and that may be organized in networks of such multiple lines. Branching or endpoints of the network are called pores, the lines connecting pores are called throats. For each pore and throat, one or more scalar data items can be stored. According to the given data in Table 1, each of throats and pores have their own size. Pores and throats are represented as spheres and cylinders, respectively [19].

2.2 Calculation of properties with different methods

Calculation of the main parameters of the porous samples based on the network model was carried out with Avizo software. In this regard, it is assumed that the network is completely filled with only one phase. In a steady flow of an incompressible fluid, the mass conservation for each pore is described as [18]:

$$\sum q_{ij} = 0, \tag{1}$$

where q_{ij} is the flow rate between pores i and j, when the summation is performed over the entire pore j connected to the pore i.

The relationship between pressure difference and flow rate is linear for laminar flow conditions:

$$q_{ij} = q_{ij}(P_i - P_j), \tag{2}$$

where g_{ij} is the conductivity of the throat between pores i and j. Since the conducting throats are represented by cylindrical pipes of radius r_{ij} and length l_{ij}, the hydraulic conductivity is determined by Poiseuille's law, where μ is the viscosity of the fluid:

$$q_{ij} = \frac{\pi}{8\mu}\frac{r_{ij}^4}{l_{ij}}. \tag{3}$$

By creating a pressure difference in the network, we obtain a linear system of equations, which is solved numerically: equations (1) and (2) lead to the following matrix equation:

$$G \cdot P = S, \tag{4}$$

where G is the conductivity matrix, symmetric matrix of dimension N, where N is the number of pores in the network; P is the vector of size N corresponding to the pressure in each pore; S is the vector of size N, limited by the boundary conditions of the pressure at the inlet and outlet of the system. The total flow rate can then be calculated as:

$$Q = (P_i - P_j)q_{ij}. \tag{5}$$

The permeability of the network is finally calculated from Darcy's law:

$$k = \frac{Q}{\Delta P}\frac{\mu L}{A}, \tag{6}$$

where ΔP is the pressure gradient, applied to the boundary ($\Delta P = P_{inlet} - P_{outlet}$), L is the length of the network in the direction of flow.

Figure 1 illustrates demanding boundary conditions.

The hydraulic tortuosity is based on velocities derived from the previous calculation described above. Knowing the velocities in each throat, tortuosity is calculated by summing the lengths of all velocities divided by the sum of the projected velocities along the direction of flow:

$$\tau = \frac{\sum_{i=0}^{n} \|v_i\|}{\sum_{i=0}^{n} \|vx_i\|}, \tag{7}$$

where n is the number of channels, v_i is the velocity of the fluid, passing though the throat i, and vx_i is the projection of the velocity along the flow direction of the fluid passing through the throat.

The Kozeny-Carman equation of dependency relationship is written[20]:

$$k = \frac{\phi^3}{c\tau^2 S^2}, \tag{8}$$

where S is the specific surface area(1/m); ϕ is the porosity; c shows the Kozeny constant, for our case c=2.5 (for the cylinder), τ is the tortuosity. The values of porosity, tortuosity and specific surface area were obtained from calculations carried out in Avizo.

2.3 Influence of the heterogeneity

To assess the influence of heterogeneity on main parameters of porous media, the 3D rendered images were divided into several same pieces. The division is illustrated on Figure 2.

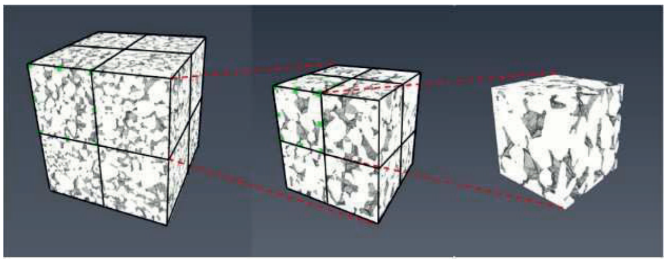


Figure 1: Dividing samples to the pieces

For each sample, calculations were carried out to determine the basic parameters using the methods described above. Also, for each piece of the sample, pore-network models and pore space separation were built.

3 Results

3.1 Image reconstruction and comparison of methods

The results of image processing and pore network generation are presented on Figure 3. The PNM method gives a good information about real characteristics of core sample, such as porosity, tortuosity, specific surface area.

There are results of computation for full sized computer models of real samples with Δ P=0.03MPa and μ=0.001Pa*s in the Table 2.

The porosity, tortuosity, and specific surface area were used for the KC method. After that, two methods were compared with the value of DNC results from Imperial College of London [17] for the Bentheimer sandstone, Doddington sandstone and Estaillades carbonate samples (Figure 4).

3.2 Results of computation on the divided pieces of the models

Each model was divided into 8 identical pieces. In turn, each piece was divided into 8 more pieces. However, the division into more and more pieces turned out to be difficult to assess the dependence of main parameters of the pore space among themselves due to the absence of connected pores and a decrease in the total porosity of small pieces. In this regard, the results of 8 pieces were used. For each piece, calculations to determine basic parameters were carried out to using the KC equation and pore network modeling method. Also, for each of the pieces, calculations of such parameters as porosity, tortuosity, specific surface area of the

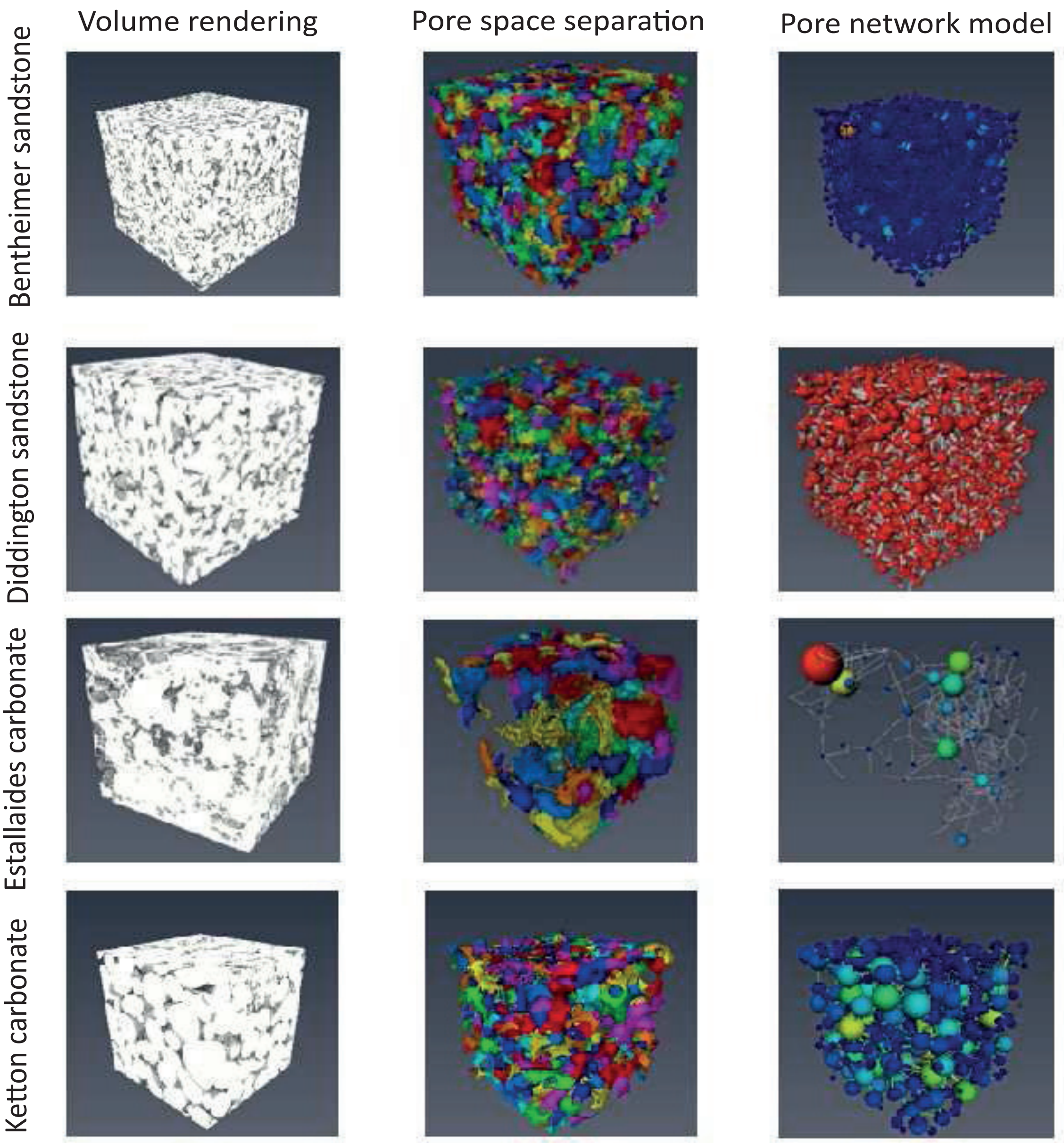


Figure 2: Results of basic steps of generating pore network model

Table 2: Results of pore network modelling

Type of core	Permeability (μm^2)	Total flow rate ($\mu l/s$)	Tortuosity	Porosity	Specific Surface Area ($1/\mu m$)
Bentheimer sandstone	4,26	380,61	1,68	0,22	0,02
Doddington sandstone	4,15	331,58	1,63	0,19	0,01
Estaillades carbonate	0,33	32,11	1,58	0,10	0,01
Ketton carbonate	7,34	656,36	1,57	0,13	0,01

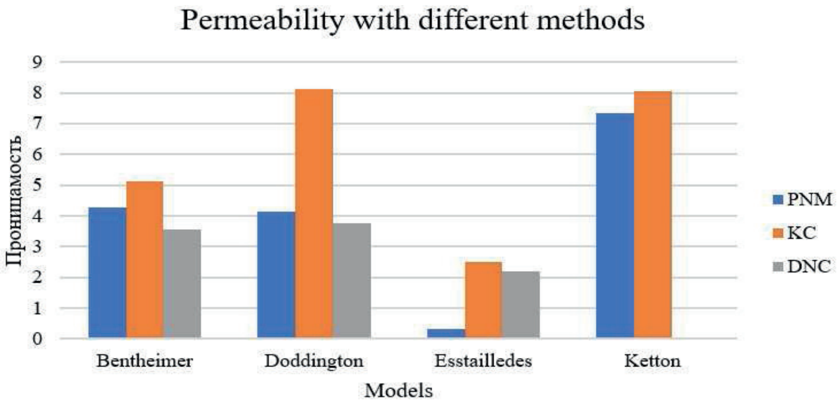


Figure 3: Comparison of different methods in computation

pore space were carried out. Figure 5 shows results of the dependence of permeability and porosity for each piece with two methods.

4 Conclusions

In this research results of analyzing of the effect of heterogeneity on basic parameters of real core models are presented. For that goal each of the computer model of real porous samples were divided into several parts (pieces), which was direct to asses the heterogeneity on properties of the medium. Results of calculation of these properties for each of the pieces, PNM method shown good match with the DNC for the Bentheimer and Doddington rocks, whereas in the empirical model, an increase in the value was observed. For the Estaillades, KC method was in a good match with the DNC, while PNM, on the contrary, began to move away from the value of the DNC method. This is clearly seen on Figure 5, where there was a division of the samples into pieces. The reason of deviation values for the carbonate Estaillades related with inhomogeneous character of the model which is shown in each piece. For example, for some pieces the estimation of parameters shown that there were no connected porous planes. These methods have its own advantages and disadvantages and at the same

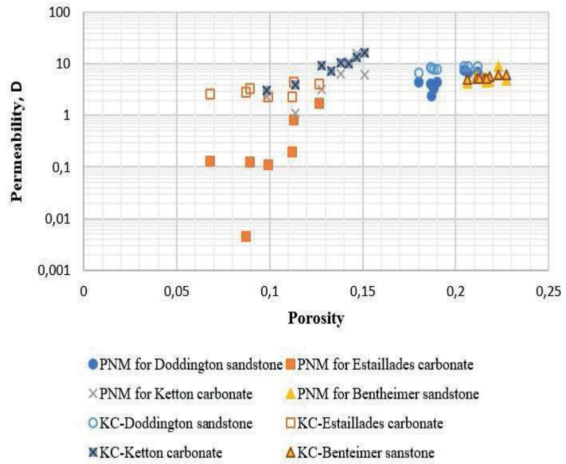


Figure 4: Dependence of porosity-permeability for each rock models' pieces computed with PNM and KC equation

time is actively developing, so it is not possible to establish the optimal one. The choice of method should be determined by the specific problem to be solved. Further works will be directed to the determination and parametrization of the factors for that influence to the development of the KC relationship by performing numerical experiments for computer models of real porous samples.

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References

[1] Huang S.Y., Wu Y.Y., Meng X.B., "Recent advances on microscopic pore characteristics of low permeability sandstone reservoirs" *Adv. Geo-Energy Res.,* (2018): 122-134, https://doi.org/10.26804/ager.2018.02.02.

[2] Rodriguez E.F., Giacomelli F., Vazquez A., "Permeability–porosity relationship in RTM for different fiberglass and natural reinforcements" *J. Compos. Mater.,* 38 (2004): 259–268, https://hdl.handle.net/2122/2258.

[3] Lai J., Wang G.W., Cao J., "Investigation of pore structure and petrophysical property in tight sandstones" *Mar. Pet. Geol.,* 91 (2018): 179-189.

[4] Mavko G., Nur A., "The effect of a percolation threshold in the Kozeny–Carman relation" *Geophysics,* 62 (1997): 1355-1673, https://doi.org/10.1190/1.1444251.

[5] Lai J., Wang G.W., Fan Z., "Insight into the pore structure of tight sandstones using NMR and HPMI measurements" *Energy Fuels,* 30 (2016): 13159–13178, https://doi.org/10.1021/acs.energyfuels.7b01816.

[6] Pape H., Clauser C., Iffland J., "Variation of permeability with porosity in sandstone diagenesis interpreted with a fractal pore space model" *Pure Appl. Geophys.,* 157 (2000): 603–619, https://doi.org/10.1007/PL00001110.

[7] Civan F., "Scale effect on porosity and permeability: kinetics, model and correlation" *AIChE J.,* 47 (2001): 271–287, https://doi.org/10.1002/aic.690470206.

[8] Knackstedt M.A., Latham S., Madadi M., "Digital rock physics: 3D imaging of core material and correlations to acoustic and flow properties" *Lead. Edge,* 28(1) (2009): 28-33, https://doi.org/10.1190/1.3064143.

[9] Taron J., Elsworth D., Min K.B., "Numerical simulation of thermal-hydrologic-mechanical-chemical processes in deformable, fractured porous media" *Int. J. Rock Mech. Min. Sci.,* 46(5) (2009): 842-854, https://doi.org/10.1016/j.ijrmms.2009.01.008.

[10] Blunt M.J., Branko B., Dong H., "Pore-scale imaging and modelling" *Adv. Water Resour.,* 51(1) 2013: 197-216, https://doi.org/10.1016/j.advwatres.2012.03.003.

[11] Song R., Wang Y., Liu J., Cui M., Lei Y., "Comparative analysis on pore-scale permeability prediction on micro-CT images of rock using numerical and empirical approaches" *Energy Sci. Eng.,* 7 (2019): 2842-2854, https://doi.org/10.1002/ese3.465.

[12] Dong H., Blunt M. J., "Pore-network extraction from micro-computerized-tomography images" *Phys. Rev. E.,* 80 (2009): 036307, https://doi.org/10.1103/PhysRevE.80.036307.

[13] Dong H., Fjeldstad S., Alberts L., Roth S., Bakke S., Oren P.-E., "Pore network modelling on carbonate: a comparative study of different micro-ct network extraction methods" *International symposium of the society of core analysts, Society of Core Analysts,* (2008): 1-12.

[14] Delerue J.-F., Lomov S. V., Parnas R., Verpoest I., Wevers M., "Pore network modeling of permeability for textile reinforcements" *Polym. Compos.,* 24 (3) (2003): 344-357, https://doi.org/10.1002/pc.10034.

[15] Balhoff M.T., Wheeler M.F., "A predictive pore-scale model for non-Darcy flow in porous media" *SPE,* 14(04) (2009): 579-587, https://doi.org/10.2118/110838-PA.

[16] Xiong Q.R., Todor B., Andrey P.J., "Review of pore network modelling of porous media: experimental characterisations, network constructions and applications to reactive transport" *J. Contam. Hydrol.,* 192 (2016): 101-117, https://doi.org/10.1016/j.jconhyd.2016.07.002.

[17] Imperial College of London. Micro-CT Images and Networks. https://www.imperial.ac.uk/earth-science/research/research-groups/pore-scale-modelling/micro-ct-images-and-networks/.

[18] Dong H., "Micro-CT imaging and pore network extraction" *J (Doctor Thesis, London, UK: Imperial College London.,* (2007), https://www.imperial.ac.uk/media/imperial-college/faculty-of-engineering/earth-science-and-engineering/recovered-files/33551696.PDF.

[19] Nordahl K., Ringrose P.S., "Identifying the representative elementary volume for permeability in heterolithic deposits using numerical rock models" *Math. Geosci.,* 40 (2008): 753-771, https://doi.org/10.1007/s11004-008-9182-4.

[20] Latief F., Fauzi U., "Kozeny-Carman and empirical formula for the permeability of computer rock models" *Int. J. Rock Mech. Min. Sci.,* 50 (2012): 117-123, https://doi.org/10.1016/j.ijrmms.2011.12.005.

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NUMERICAL SIMULATION OF CARBONATE ROCKS DISSOLUTION NEAR THE WELLBORE

This paper examines the process of wormholes formation during hydrochloric acid treatment of well bottom-hole zone in carbonate formations. An algorithm for solving the problem of wormhole formation in a porous medium for a two-dimensional case was developed. Two-scale model (pore-scale and Darcy's scale) taking into account convection, diffusion, and chemical reaction were used in this work in order to describe the dissolution of carbonates with hydrochloric acid (hydrochloric acid treatment). The initial distribution of the porosity field was generated as a distribution of random numbers around some mean value. Based on the distribution of the initial porosity field, the initial permeability field of the rock was calculated. The random distribution was used to describe the heterogeneity of the actual rock. The rest of the study parameters were taken from known experiments on the dissolution of carbonate cores. The numerical model was built for solving the system of equations for acid dissolution, and carbonate dissolution modes with hydrochloric acid were obtained depending on the Damkohler number, Thiele modulus on the pore-scale and Darcy's scale as a result of this research. Also, the optimal Damkohler numbers (injection rates) were found. The computer code for the problem of the development/growth of wormholes in a porous medium based on the developed algorithm was built using the C++ programming language.
Key words: acid treatment, carbonate core, dissolution mode, Damkohler number, Thiele modulus.

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Ұңғыманың түптік аймағындағы карбонатты тау жыныстарының еру процесін сандық моделдеу

Бұл жұмыста карбонатты жер қабаттарындағы тұз қышқылымен өңдеу кезінде канал тесіктерінің пайда болуы зерттеледі. Екі өлшемді жағдайға арналған кеуекті ортада канал тесіктерінің пайда болу мәселесін шешу алгоритмі жасалды. Бұл жұмыста тұз қышқылымен карбонаттардың еруін сипаттау үшін (тұз қышқылымен өңдеу) конвекция, диффузия және химиялық реакцияны ескеретін екі масштабты модельді қолданылды (кеуек шкаласы мен Дарси шкаласы бойынша). Кеуектілік өрісінің бастапқы таралуы кездейсоқ сандардың орташа мөннің айналасында таралуы ретінде құрылды. Бастапқы кеуектілік өрісінің таралуы негізінде жыныстыө бастапқы өткізгіштік өрісі есептелді. Кездейсоқ бөлу нақты жыныстың біртекті емпестігін сипаттау үшін қолданылды. Зерттеудің қалған параметрлері карбонатты керндердің еруі бойынша белгілі тәжірибелерден алынды. Тұз қышқылымен еру процесін сиппаттайтын теңдеулер жүйесін шешу үшін сандық модель құрылды және осы зерттеудің нәтижесінде карбонаттардыө тұз қышқылымен еру режимдері Дамколлер санына, кеуек масштабында және Дарси масштадбында Тиле модуліне байланысты алынды. Дамколердің оңтайлы сандары (айдау жылдамдығы) да табылды. Құрылған алгоритмге негізделген кеуекті ортада канал тесіктерінің құрылуы мен өсуі мәселесінің компьютерлік коды C ++ программалау тілінің көмегімен құрылды.
Түйін сөздер: қышқылмен өңдеу, карбонатты керн, еру режимі, Дамколлер саны, Тиле модулі.

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Численное моделирование растворения карбонатных пород вблизи скважины

В данной работе исследуется процесс образования червоточин при соляно-кислотной обработке призабойной зоны скважины в карбонатных пластах. Был разработан алгоритм решения задачи образования червоточин в пористой среде для двумерного случая. Для описания растворения карбонатов соляной кислотой использовалась модель двух масштабов (в масштабе пор и в масштабе Дарси), учитывающая конвекцию, диффузию и химическую реакцию. Начальное распределение поля пористости генерировалось как распределение случайных чисел вокруг некоторого среднего значения. На основании распределения начального поля пористости было рассчитано начальное поле проницаемости породы. Случайное распределение использовалось для описания неоднородности реальной породы. Остальные параметры исследования были взяты из известных экспериментов по растворению карбонатных кернов. Численная модель была построена для решения системы уравнений растворения, и в результате данного исследования были получены режимы растворения карбонатов соляной кислотой в зависимости от числа Дамколера, модуля Тиле в масштабе пор и в масштабе Дарси. Также были найдены оптимальные числа Дамколера (скорости закачки). Компьютерный код для задачи развития/роста червоточин в пористой среде был построен с использованием языка программирования C++.

Ключевые слова: кислотная обработка, карбонатный керн, режим растворения, число Дамколера, модуль Тиле.

1 Introduction

Evidently that over the time, the bottom-hole zone of production and injection wells may be damaged, as a result of the reduction in productivity/injectivity of wells. Well drilling, perforating, cementing, fine particles migration and other industrial operations can cause pore plugging. In this case, an acid treatment is one of the most effective and widely used methods in stimulation techniques for the restore of permeability in carbonate reservoirs. After the acid treatment, the improved formation permeability can increase by several orders of magnitude, for this reason the topic of this study is actual for oil industry.

The literature review on mathematical and numerical modeling of the formation of dissolution channels in a porous medium during the treatment with hydrochloric acid was done in this work. The mathematical and numerical formulation of a two-scale model, dissolution process modeling in a rectangular region and the influence of core size on dissolution were studied as well.

Also, an algorithm for solving the system of equations for the dissolution problem of carbonate cores in a two-scale was developed. Moreover, the computer code based on the created solution algorithm for the 2D case was developed in order to study the modes of carbonate cores dissolution at different injection rates (from 5×10^{-5} cm/s to 5×100 cm/s).

2 Literature review

The experimental work on the study of core dissolution process by acid in laboratory conditions was carried out by many authors [1-8]. These works were devoted to determining the optimal injection rate (or the optimal value of the Damkohler number), developing a mechanistic model of wormhole growth, studying the effect of rock and acid types on core dissolution modes at different injection rates, and developing a 2D network model to describe the wormholes formation. As a result of these experiments, the change in the pressure drop until the moment of an acid breakthrough at the right end of the core was determined. In the work [6], the Darcy-Brinkman and Stokes equations, together with the reagent concentration equation are used to describe the rock dissolution process, and the influence of the injection rate, temperature and the emulsion use on the dissolution modes is determined. The work [7] is devoted to the development of a model for the wormholes growth in the case of using a self-diverting acid together with an ordinary acid during the cylindrical core testing initially filled with a solution. Authors formally divided the core into three potential zones: the wormholes are formed in the first zone; the second zone is contaminated by the ingress of acid; and the third zone is remaining untouched by acid. And also, the influence of the end effect on the pressure drop change along the core length was studied.

There are several methods for the acidizing process modeling. The work [9] studied the formation of wormholes in carbonate rocks. The concentration equation without a chemical reaction was used to describe the process. The reaction rate was taken into account in the form of a boundary condition on the walls of wormholes, and the acid transfer rate inside the wormhole obeyed the Poiseuille's parabolic law. Research results show that the growth rate of wormholes and the geometry of wormholes during the breakthrough depends on the duration of the contact of the rock with the acid and the acid injection rate.

A semi-empirical model for a quick assessment of the growth rate of wormholes during the acidizing of carbonate rocks is described in [10]. This model includes such two empirical parameters, as an efficiency factor and a parameter that is the inverse of the second power of the optimal averaged fluid velocity in pores (which is experimentally determined). According to this model, the growth rate of wormholes is proportional to the average flow rate to the power of $2/3$ for high values of the acid injection rate. This model does not take into account the diffusion of a chemical reaction. It is possible to translate the results of rock dissolution from laboratory conditions to industrial conditions by using this semi-empirical model.

In the work [11] the strong influence of heterogeneity of a carbonate formation on its filtration characteristics during the acid treatment was proved. A model of acid dissolution of the carbonate matrix was constructed, taking into account possible sedimentation. The mathematical model includes an empirical expression for determining the rate of change in the concentration of deposited nanosized particles and the permeability changes of a porous medium. The model also takes into account the moving boundary of the dissolution channel, on which the particles are deposited. The permeability increases with increasing pore size and pore opening due to the mineral dissolution. In the presented work, macroscopic equations can be used to simulate the acidizing process of wells at the field scale.

The results of work [12] are in a good agreement with the experimental data obtained in the work [4]. Numerical calculations were carried out in a cylindrical region using the Navier-Stokes equations taking into account the resistance of the porous medium instead of the Darcy's equation. The influence of mineral composition of the core on its solubility was investigated by authors. It was found by numerical experiments that in the case of a homogeneous mineral, the acid breakthrough faster than in the case of a sample with inclusions of a less soluble mineral.

In work [13], the influence of medium heterogeneity on the formation of wormholes was studied using a two-scale continuous model, which was first described in [14-16]. It was determined that rock heterogeneity affects not only the structure of the patterns formed during reactive dissolution, but also the amount of acid required to achieve a given increase in permeability. The acid volume decreases with an increase in the heterogeneity degree, as well as with the decrease of length scale. This is especially noticeable at high acid injection rates.

In work [14], an averaged two-scale model of carbonate rocks dissolution by hydrochloric acid in 1D formulation is described. The model describes the relationship between transport processes and reactions occurring at the pore and Darcy's scale. The coefficients of mass transfer and dispersion are estimated and the local equation and criteria are derived.

In [15], the two-scale model was extended for a 2D region. The paper studied the influence of Thiele modules for pore and Darcy's scales, acid concentration in solution, height-to-length ratio, and acid number on dissolution modes. The results of the two-scale model are compared with the results of the experiment and early work.

A detailed study of the carbonates dissolution process is given in [16]. In this work, a criterion for the qualitative prediction of the wormholes formation was found and it was shown that it is approximately equal to 1. Also, the asymptotic values of the breakthrough volume for low and high injection rates were found by authors. To study the effect of rock heterogeneity on dissolution, the concepts of the magnitude of heterogeneity and the dimensionless scale of heterogeneity were introduced. In addition, this paper shows that the optimal breakthrough volume decreases from the 1D case to the 3D case.

Study of the dissolution modes of carbonates with hydrochloric acid in works [17-19] were carried out in a polar coordinate system. In work [17], a criterion is found for qualitatively predicting the wormholes formation for a radial flow similar to the plane case. And also, the fractal size of the wormholes has been determined. In the work [18], the normal distribution was used to generate the initial porosity distribution. Thus, the influence of the normal distribution of porosity on the dissolution modes was investigated. It has been shown that in the case of a normal distribution of porosity, the acid breakthrough volume will be lower than in the case of an uniform distribution, and it has been proven that such distribution of porosity is close to the real distribution. The influence of the perforation length and the presence of cavities on the dissolution modes were investigated in the work as well.

In the work [19], the two-scale model expands from the laboratory scale to the scale of well bottom-hole zone. The study region consists of such two adjacent areas, as the contaminated area in the well bottom-hole zone and the area behind it, in which the fluid is considered to be weakly compressible. And also, a new criterion for acid breakthrough is introduced for calculating the concentration values at the right end of the region. If this concentration is 10% higher than the initial value, then the breakthrough is considered to occur. And also, the influence of the fluid compressibility, which is contained in the second region, on the growth of wormholes was investigated. It was found that the use of the normal distribution of porosity brings the results closer to the experimental data.

The influence of the region geometry on the rock dissolution was investigated in [20]. Based on the two-scale model, it was determined that if the core height is increased, the breakthrough volume decreases. The dependence of the optimal velocity on the shape function for dominant wormholes was found. The shape function is defined as the ratio of the height of the area of interest to its length.

3 Materials and methods

3.1 Physicochemical formulation of the problem

In acidizing, the acid reaction rate competes with the acid injection rate (inverse of the Damkohler number) and diffusion. Usually the diffusion coefficient is small compared to the injection rate [3]. When wormholes are formed, this mode is effective, since in this case the minimum acid volume leads to the wormholes formation connecting the well with the undisturbed region, and the permeability in the reaction zone increases by several orders of magnitude. The wormholes formation mode is observed at average values of the injection rate. When the injection rate is low or high, surface and uniform dissolution modes are observed, in these cases the injected acid volume will be much larger than in the case of wormholes.

Dissolution modes (surface, wormhole and uniform) are related to the Damkohler number, defined as the ratio of the rate of a chemical reaction to the rate of acid injection. Consequently, at large and small values of the Damkohler number, surface and uniform modes are observed, respectively, and at medium values, the formation mode. As a result of experiments, it was found that the optimal Damkohler number is 0.29 [4].

Important components in determining the optimal mode of hydrochloric acid treatment are the kinetic parameters of the reaction, which can be determined only from experiments. The main process taking place in the rock during acidizing is the dissolution of carbonates. It is known that more than half of the minerals that make up carbonate strata are represented by dolomites ($CaMg(CO_3)_2$) and calcite ($CaCO_3$). The chemical reactions of calcite and dolomite with hydrochloric acid are represented by the equations:

$$CaCO_3 + 2HCl = CaCl_2 + CO_2 + H_2O$$

$$CaMg(CO_3)_2 + 4HCl = CaCl_2 + MgCl_2 + 2H_2O + 2CO_2$$

3.2 Mathematical formulation of the problem

The acidizing process of carbonate formations is considered for single-phase multicomponent isothermal filtration in a heterogeneous formation, provided that the reaction products are completely dissolved in the fluid phase. The simulation takes into account the processes of convection and diffusion, and a chemical reaction. Acid, getting into the pores, corrodes their walls, thereby increasing the pore space. An increase in pore space leads to an increase in permeability. The mathematical model of the dissolution process includes Darcy's law, the equations of continuity, concentration, porosity in dimensionless form (Figure 1):

Figure 1: Region of study

$$\nabla \cdot (\boldsymbol{k} \nabla p) = Da_{eff} N_{ac} c_f, \tag{1}$$

$$\vec{u} = -\boldsymbol{k} \nabla p, \tag{2}$$

$$\frac{\partial \pi}{\partial t} + \nabla \cdot (c_f \vec{u}) = \nabla \cdot (\boldsymbol{D_e} \nabla c_f), \tag{3}$$

$$\frac{\partial \varepsilon}{\partial t} = Da_{eff} N_{ac} c_f, \tag{4}$$

$$k_x = k_y = k = \frac{\varepsilon}{\varepsilon_0} \left(\frac{\varepsilon(1-\varepsilon_0)}{\varepsilon_0(1-\varepsilon)} \right)^{2\beta}, \tag{5}$$

$$r = \sqrt{\frac{k\varepsilon_0}{\varepsilon}}, \tag{6}$$

$$A_\nu = \frac{\varepsilon}{\varepsilon_0 r}, \tag{7}$$

$$Sh = Sh_\infty + \frac{0.7}{m^{1/2}} Re_p^{1/2} Sc^{1/3}, \tag{8}$$

$$D_{eX} = \frac{\alpha_{0s} Da \varepsilon}{\Phi^2} + \lambda_X |\vec{u}| r \eta, \tag{9}$$

$$D_{eT} = \frac{\alpha_{0s} Da \varepsilon}{\Phi^2} + \lambda_T |\vec{u}| r \eta, \tag{10}$$

$$Da_{eff} = \frac{Da A_\nu}{\left(1 + \frac{\phi^2 r}{Sh}\right)}, \pi = \varepsilon \left(c_f + \frac{1}{N_{ac}} \right), \tag{11}$$

$$-k \frac{\partial p}{\partial x}\bigg|_{x=0} = 1, \tag{12}$$

$$p|_{x=1} = 0, \tag{13}$$

$$\frac{\partial p}{\partial y}\bigg|_{y=0} = \frac{\partial p}{\partial y}\bigg|_{y=\alpha_0} = 0, \tag{14}$$

$$\left(c_f - \varepsilon D_{eX} \frac{\partial c_f}{\partial x} \right)\bigg|_{x=0} = 1, \tag{15}$$

$$\frac{\partial c_f}{\partial x}\bigg|_{x=1} = \frac{\partial c_f}{\partial y}\bigg|_{y=0} = \frac{\partial c_f}{\partial y}\bigg|_{y=\alpha_0} = 0, \tag{16}$$

$$c_f|_{t=0} = 0, \tag{17}$$

$$\varepsilon|_{t=0} = \varepsilon_0 + \hat{f}, \tag{18}$$

where α_0 is the ratio of the height and length of the core, where $\vec{u}$ is the velocity vector, $\boldsymbol{k} = (k_x, k_y)$ is the permeability tensor, p is the pressure, ε is the porosity, t is the time, c_f is the acid concentration in the fluid phase, $\boldsymbol{D_e} = (D_{eX}, D_{eT})$ is the effective dispersion tensor, A_ν is the surface area per unit volume of rock available for reaction, Sh is the Sherwood number; Sh_∞ is the asymptote of the Sherwood number; m is the ratio of the pore length to its diameter; $Re_p = 2|\vec{u}|r_p/\nu$ is the pore Reynolds number; $|\vec{u}|$ is the velocity module; ν is the kinematic viscosity of the fluid; $S_c = \nu/D_m$ is the Schmidt number; λ_x, λ_T, and a_{0s} are the constants that depend on the type of rock, $\hat{f}$ is a fluctuation that artificially creates porosity heterogeneity of the reservoir, and lies in the interval $[-\Delta\varepsilon_0, \Delta\varepsilon_0]$, where $\Delta\varepsilon_0$ is a given number.

Fugure 1 presents the region of study. For the pressure equation the Neumann condition is set on the left, upper and lower boundaries, the Dirichlet condition is set on the right boundary. For the equation of acid concentration the Dankvert condition is set on the left boundary, and the Neumann condition is set on the other boundaries. We assume that at the initial moment of time, there is no acid in the study region. The porosity is given by the initial distribution. To create the rock heterogeneity, the initial distribution of porosity is set as random with an uniform distribution law around the average value.

Dimensionless parameters that are included in the mathematical model are defined as follows:

$$\eta = \frac{2r_0}{L}; \phi^2 = \frac{2k_s r_0}{D_m}; N_{ac} = \frac{\alpha C_0}{\rho_s}; Da = \frac{k_s \alpha_0 L}{u_0}; Pe = \frac{u_0 L}{D_m}; \Phi^2 = DaPe = \frac{k_s \alpha_0 L^2}{D_m}, \quad (19)$$

where L is the core length; C_0 is the inlet acid concentration; α is the degree of dissolution of the acid; ρ_s is the density of the solid phase; D_m is the effective molecular diffusion coefficient; u_0 is the injection rate; ϕ^2, Φ^2 are the Thiele modulus for the scale of pores and core; N_{ac} is the acid number; Da is the Damkohler number; Pe is the Peclet number.

3.3 Numerical formulation of the problem

To numerically solve the system of equations (1)-(4), together with equations (5)-(11) and the initial-boundary conditions (12)-(18), we use the difference grid, which is shown in Figure 2a. We integrate equations over the control volume, which is shown in Figure 2b, taking into account the boundary conditions.

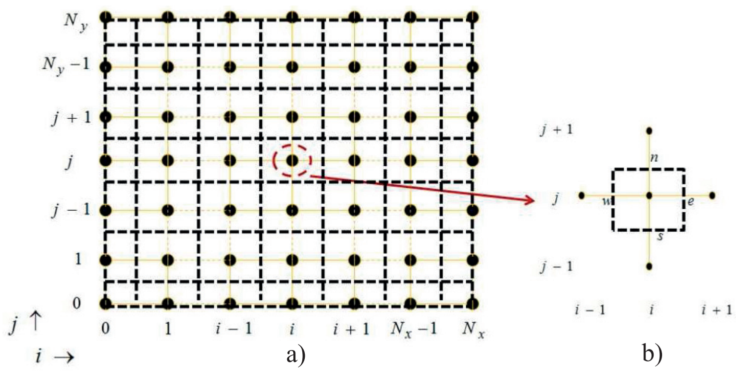

Figure 2: The difference grid: a) (circles are the area where the pressure, concentration and porosity are determined, the dotted lines are the area where the velocity components are determined); b) the control volume

The algorithm for solving the acid treatment problem: from the known initial distribution of concentration and porosity, we iteratively find the distribution of the pressure field, then from the known pressure field we determine the components of the filtration rate according to Darcy's law.

Next, we determine the intermediate $(c^*_{fij},\ \varepsilon_{ij}{}^*)$ and final $(c^{n+1}_{fij},\ \varepsilon^{n+1}_{ij})$ values of concentration and porosity. Then the increase in concentration and porosity is compared with a given small number, if the largest of them does not exceed a given number, then we go to a new time layer. Otherwise, the time step is artificially reduced and all equations are re-solved on the same layer until the above increments are less than a given number.

Thus, this procedure continues until the acid breakthrough at the right end of the core. The criterion for stopping the entire calculation: there is an acid breakthrough at the right end of the core if the average permeability increases 100 times over its original value.

4 Numerical results

This section presents the results of numerical calculations of the development/growth of wormholes in porous media, depending on the parameters of the porous medium, acid type and acid injection rate. The dimensions of carbonate cores are L×H=10cm×4cm and 4cm×10cm, respectively. The other parameters are shown in tables 1 and 2.

4.1 Results for the carbonate core with dimensions L×H=10cm×4cm

Below in Figures 3-6, the results of numerical calculations of acid treatment of a carbonate core with the size L×H=10cm×4cm are shown.

The acid injection rate in all calculations varied from 5×10^{-5} cm/s to 5×100 cm/s. Depending on the above parameters, the optimal injection rates were determined, at which the smallest volume of acid is required to achieve the desired increase in the average core permeability. And also, the modes of rock dissolution were determined.

Table 1: The values of the dimensionless parameters used

Parameter	Value	Parameter	Value
Da	40-40000	ϕ^2	0,1; 2; 10
N_{ac}	0,05; 0,1; 0,5	S_c	10^3
Φ^2	10^4; 10^5; 10^6	η	2×10^{-5}
ε_0	0.2	$\Delta\varepsilon_0$	0,01; 0,15

Figures 3 and 4 show the results of the numerical solution at different injection rates, when the Thiele modulus for the pore-scale ϕ^2 takes the value 10^{-2}, 2×10^{-1} and 10^1, and the remaining problem parameters remain unchanged.

The Thiele pore-scale modulus ϕ^2 characterizes the relationship between diffusion and reaction times.

Figure 3 shows the fields of porosity and pressure at the breakthrough moment at different values of ϕ^2. Figure 4 shows acid breakthrough curves depending on the change in injection rate for different values of ϕ^2.

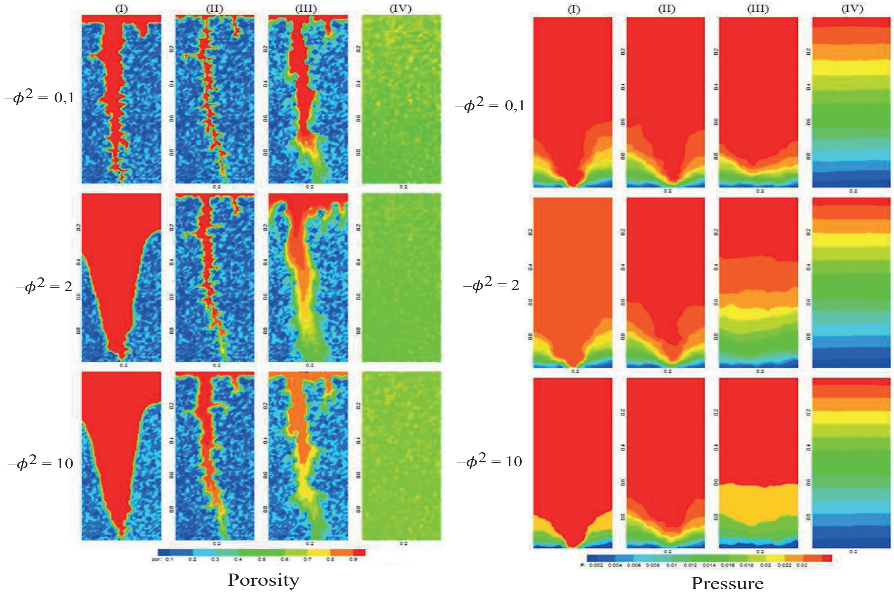

Figure 3: Distribution of porosity and pressure at the breakthrough moment at different values of ϕ^2

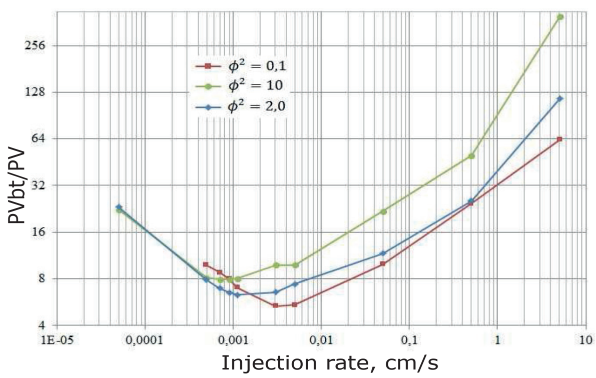

Figure 4: Acid breakthrough curves for different values of ϕ^2

When the reaction at rock surface begins to proceed faster than the diffusion process, a minimum acid volume is required to breakthrough. This process can be seen from Figure 3a. For ϕ^2=0.1 at the same values of the injection rate, the width of the dissolution channels will be less than for ϕ^2=2 and ϕ^2=10.

When changing the Thiele modulus for a pore scale ϕ^2, the pressure field straightens faster (Fig. 3b). In the case of ϕ^2=0.1, the perturbation created by the acid injection reaches the right end faster than for the cases of ϕ^2=2 and ϕ^2=10.

From Figure 4 it can be seen that for different values of ϕ^2, the values of the optimal injection rate lie in different intervals (for ϕ^2=0.1 ϕ^2=2 and ϕ^2=10, the optimal injection rates lie in the intervals 1.1×10^{-3}, -5×10^{-3}, 9×10^{-4}, -3×10^{-3} and 5×10^{-4}, -9×10^{-4}). With a decrease in ϕ^2, the optimal point shifts to the right and is located below.

The results of studying the effect of changing the Thiele modulus for the core scale on the carbonate dissolution process are shown below in Figures 5 and 6.

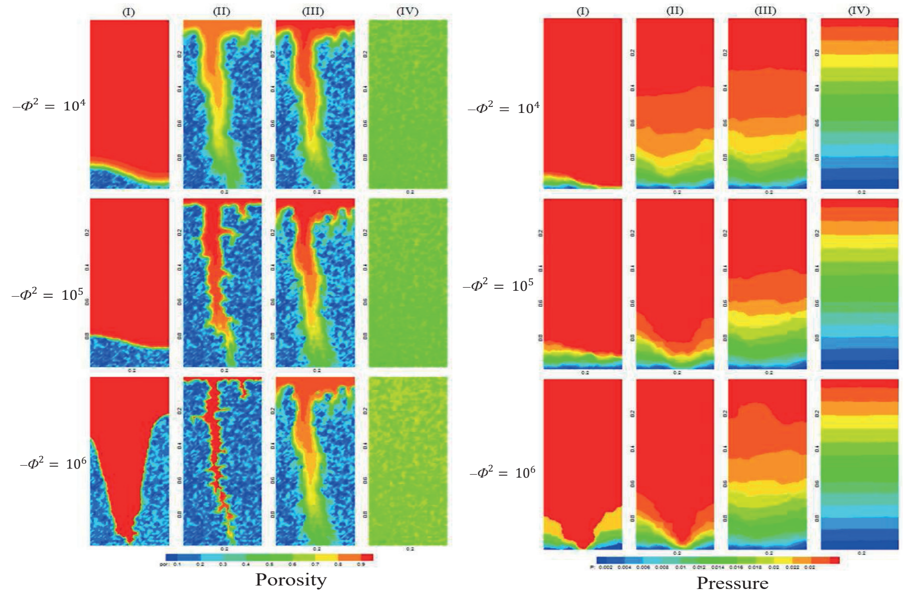

Figure 5: Distribution of porosity and pressure at the breakthrough moment at different values of Φ^2

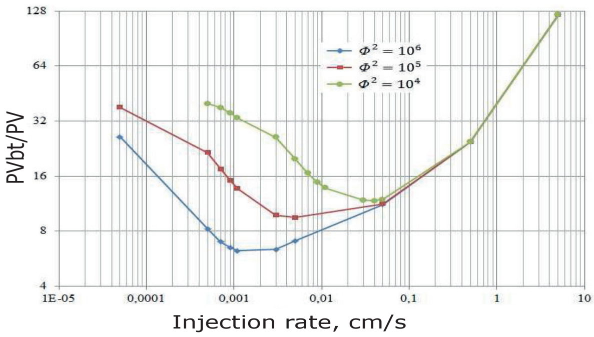

Figure 6: Acid breakthrough curves for different values of Φ^2

Figures 5a and 5b show the distribution of the porosity field and pressure for different values of Φ^2, when the acid breakthrough at the right end of the core. From the distribution of the porosity field (Fig. 5a), it can be seen that with a decrease in Φ^2, the dissolution channels, especially under optimal conditions (i.e., in the case of wormholes), become wider and branched.

Figure 6 shows the distribution of acid breakthrough curves versus injection rate. In this case, the Thiele modulus for the core scale takes the values $\Phi^2 = 10^4$, $\Phi^2 = 10^5$ and $\Phi^2 = 10^6$, respectively.

In work [8], analytical formulas for the asymptote for low and high acid injection rates were found, which are given below. The surface dissolution for low velocity is:

$$PV_F = \frac{1-\varepsilon_0}{N_{ac}\varepsilon_0}, \tag{20}$$

The uniform dissolution for high velocity is:

$$PV_U = \frac{1}{DaN_{ac}\varepsilon_0}\int\limits_{\varepsilon_0}^{\varepsilon_f} \frac{1+\dfrac{\phi^2 r}{Sh}}{A_\nu} d\varepsilon$$

where PV_F and PV_U are the volume of acid breakthrough in surface and uniform modes, respectively, ε_f is the porosity at the breakthrough moment. In accordance with the equation (19), it is possible to approximately calculate the breakthrough volume during surface dissolution. For example, it can be seen from Figure 6 that as the acid injection rate decreases, the breakthrough volume tends to the horizontal asymptote, in this case to $PV_F = 40$. For the uniform dissolution, you can also notice this phenomenon.

4.2 Results for the carbonate core with dimensions L×H=4cm×10cm

This choice can be explained by the fact that, in practice, acidizing is usually carried out in an area whose height (the length of the perforated zone of the well from which the acid is injected) is much greater than its length (the radius of the contaminated zone). Therefore, research in this area is of greatest interest. In this regard, the following are the results of numerical calculations of the dissolution of a carbonate core with dimensions L×H=4cm×10cm with a change in various operating parameters, such as injection rate, N_{ac}, ϕ^2, Φ^2. The parameters used in the calculations are shown in Table 2 below.

Table 2: The values of the dimensionless parameters used

Parameter	**Value**	**Parameter**	**Value**
Da	40-40000	ϕ^2	0,1; 2; 10
N_{ac}	0,05; 0,1; 0,5	S_c	10^3
Φ^2	1.6×10^4; 8×10^4; 1.6×10^5	η	2×10^{-5}
ε_0	0.2	$\Delta\varepsilon_0$	0,15

In work [20], the influence of the transition from the scale of the core to the scale of the bottomhole zone of the well on the dissolution process is investigated by changing the geometry of the study region (changing the length and height). As shown by the study results in this work, the density of wormholes increases with an increase in injection rate and with a decrease in the distance calculated from the wall where the acid is injected. The authors of this work, summarizing the results, obtained a correlation between the density of wormholes and the injection rate. This dependency is given below.

$$\rho_{wh}(x) = a(x)\ln(u_0) + b(x)$$

where $\rho_{wh}(x)$ is the number of wormholes per meter, $a(x)$, $b(x)$ are correlation parameters, u_0 is the acid injection rate.

The results of the study of the influence of ϕ^2 on the dissolution of carbonate rock at different injection rates are shown below in Figures 7 and 8.

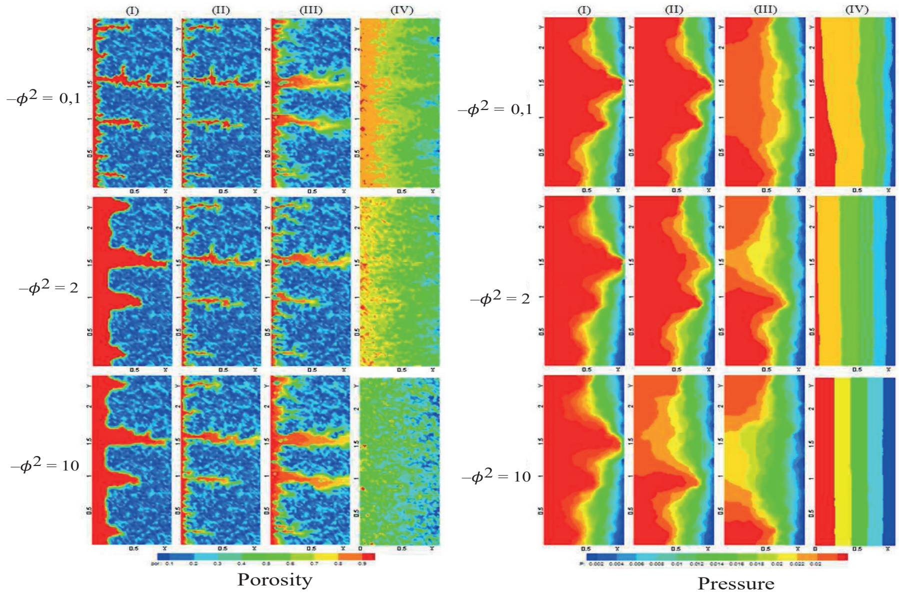

Figure 7: Distribution of porosity and pressure at the breakthrough moment at different values of ϕ^2

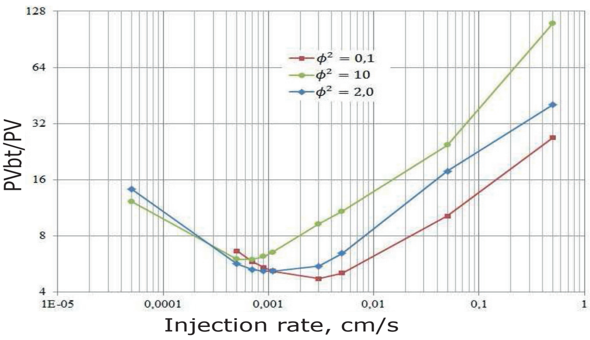

Figure 8: Acid breakthrough curves for different values of ϕ^2

From Figure 7, it is noticeable that with an increase in the height of the core sample from 4 cm to 10 cm, the density of the dissolution channels, i.e. the number of channels per unit height increased several times for the surface mode (I), the mode of formation of thin channels (wormholes) (II) and the mode of branched dissolution (III). Moreover, this tendency can be seen for all considered values of the microscopic Thiele modulus ϕ^2= 0.1; 2; 10. In addition, comparing Figures 8 and 4, we note that the amount of required volume of acid breakthrough became lower in the case of a height of 10 cm (Fig. 8) compared to the case of a height of 4 cm (Fig. 4). This can be explained by the shorter length in the case of the 10 cm height, which contributed to the rapid breakthrough of the acid.

Below in Figures 9 and 10 the dissolution results are shown at different values of Φ^2=1.6×10^4, 8×10^4, 1.6×10^5.

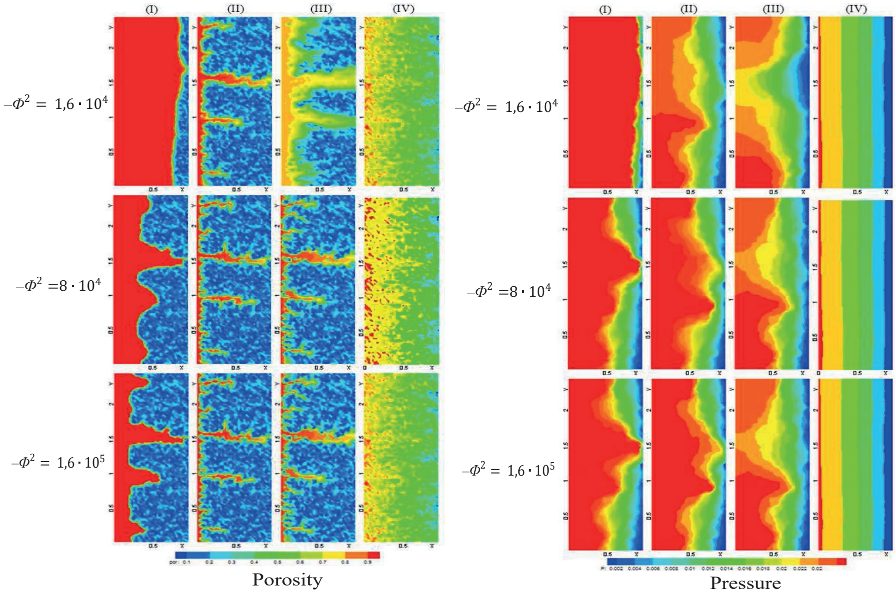

Figure 9: Distribution of porosity and pressure at the moment of breakthrough at different values of Φ^2

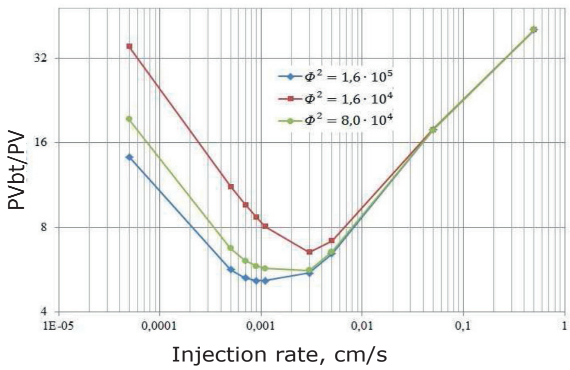

Figure 10: Acid breakthrough curves for different values of Φ^2

Figure 9 shows that with an increase in the macroscopic Thiele modulus, the shape of the dissolution channel changes: if at a low injection rate (I) of acid for Φ^2=1.6×10^4 the dissolution front is almost even, then for Φ^2= 8×10^4 and Φ^2=1.6×10^5 already clearly manifests itself in the formation of several dissolution channels, i.e. dissolution channels become thinner. This is also confirmed by Figure 10, in which the graph for Φ^2=1.6×10^4 is located above the graphs for the remaining values of Φ^2. It can also be seen from Figure 10 that the optimal injection rate, therefore, the optimal breakthrough acid volume shifts to the left with an increase in the macroscopic Thiele modulus Φ^2.

5 Conclusions

The process of wormholes formation during hydrochloric acid treatment of well bottom-hole zone in carbonate formations was examined in this paper. Numerical calculations were performed to study the dissolution modes of carbonate cores at different injection rates. The mathematical and numerical formulation of the two-scale model (pore-scale and Darcy's scale) has been implemented and the computer code of the problem was built on the basis of the created solution algorithm for the two-dimensional case using the C++ programming language. The dependence of the acid breakthrough volumes on the injection rate (Damkohler number Da) was obtained for various values of the acid capacity number N_{ac} and Thiele modulus (ϕ^2 and Φ^2). Also, the influence of the core size on the dissolution process was investigated. It has been shown that there are horizontal and oblique asymptotes at low and high injection rates. It was found that in the case when the core height increases, the density of dominant wormholes increases.

6 Acknowledgement

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References

[1] Wang Y., Hill A.D., Schechter R.S., "The Optimum Injection Rate for Matrix Acidizing of Carbonate Formations" *SPE,* (1993): 675-687, https://doi.org/10.2118/26578-MS.

[2] Hung K.M., Hill A.D., Sepehrnoori K., "A Mechanistic Model of Wormhole Growth in Carbonate Matrix Acidizing and Acid Fracturing" *J. of Petr. Tech.,* 41 (1989): 59-66, https://doi.org/10.2118/16886-PA.

[3] Fredd C.N., Fogler H.S., "Alternative Stimulation Fluids and Their Impact on Carbonate Acidizing" *SPE,* 3 (1998): 34-41, https://doi.org/10.2118/31074-PA.

[4] Fredd C.N., Fogler H.S., "Influence of Transport and Reaction on Wormhole Formation in Porous Media" *AIChE J.,* 44:9 (1998): 1933-1949, https://doi.org/10.1002/aic.690440902.

[5] Bazin B., Abdulahad G., "Experimental Investigation of Some Properties of Emulsified Acid Systems for Stimulation of Carbonate Formations" *SPE,* (1999): 1-10, https://doi.org/10.2118/53237-MS.

[6] Golfier F., Bazin B., Zarcone C., Lernormand R., Lasseux D., Quintard M., "Acidizing Carbonate Reservoirs: Numerical Modeling of Wormhole Propagation and Comparison to Experiments" *SPE,* (2001): 1-11, https://doi.org/10.2118/68922-MS.

[7] Tardy P., Lecerf B., Christanti Y., "An Experimentally Validated Wormhole Model for Self-Diverting and Conventional Acids in Carbonate Rocks Under Radial Flow Conditions" *SPE,* (2007): 1-17, https://doi.org/10.2118/107854-MS.

[8] Shedid Sh.A., "An Experimental Approach of Matrix Acidizing of Permeability-Damaged Carbonate Reservoirs" *SPE,* (2007): 1-9, https://doi.org/10.2118/106956-MS.

[9] Buijse M.A., "Understanding Wormholing Mechanisms Can Improve Acid Treatments in Carbonate Formations" *SPE Production&Facilities,* 15 (2000): 168-175, https://doi.org/10.2118/65068-PA.

[10] Buijse M., Glasbergen G., "A Semiempirical Model To Calculate Wormhole Growth in Carbonate Acidizing" *SPE,* (2005): 1-14, https://doi.org/10.2118/96892-MS.

[11] Izgec O., Zhu D., Hill A.D., "Numerical and experimental investigation of acid wormholing during acidization of vuggy carbonate rocks" *Elsevier: J. of Petr. Sci.& Eng.,* 74 (2010): 51-66, https://doi.org/10.1016/j.petrol.2010.08.006.

[12] De Oliveira T.J.L., Melo A.R., Oliveira J.A.A., Pereira A.Z.I., "Numerical Simulation of the Acidizing Process and PVBT Extraction Methodology Including Porosity/Permeability and Mineralogy Heterogeneity" *SPE,* (2012): 1-9, https://doi.org/10.2118/151823-MS.

[13] Kalia N., Balakotaiah V., "Effect of medium heterogeneities on reactive dissolution of carbonates" *Elsevier: J. of Chem. Eng. Sci.,* 64 (2009): 376-390, https://doi.org/10.1016/j.ces.2008.10.026.

[14] Panga M.K.R., Balakotaiah V., Ziauddin M., "Modeling, Simulation and Comparison of Models for Wormhole Formation during Matrix Stimulation of Carbonates" *SPE,* (2002): 1-19, https://doi.org/10.2118/77369-MS.

[15] Panga M.K.R., Ziauddin M., Gandikota R., Balakotaiah V., "A New Model for Predicting Wormhole Structure and Formation in Acid Stimulation of Carbonates" *SPE,* (2004): 1-11, https://doi.org/10.2118/86517-MS.

[16] Panga M.K.R., Ziauddin M., Balakotaiah V., "Two-Scale Continuum Model for Simulation of Wormholes in Carbonate Acidization" *AIChE J.* 51:12 (2005): 3231-3248, https://doi.org/10.1002/aic.10574.

[17] Kalia N., Balakotaiah V., "Modeling and analysis of wormhole formation in reactive dissolution of carbonate rocks" *Elsevier: J. of. Chem. Eng. Sci.,* 62 (2007): 919-928, https://doi.org/10.1016/j.ces.2006.10.021.

[18] Liu M., Zhang Sh., Mou J., "Effect of normally distributed porosities on dissolution pattern in carbonate acidizing" *Elsevier: J. of. Petr. Sci.& Eng.,* 94-95 (2012): 28-39, https://doi.org/10.1016/j.petrol.2012.06.021.

[19] Liu M., Zhang Sh., Mou J., "Wormhole Propagation Behavior Under Reservoir Condition in Carbonate Acidizing" *Springer: J. of. Trans. in Porous Media,* 96 (2013): 203-220, https://doi.org/10.1007/s11242-012-0084-z.

[20] Cohen Ch.E., Ding D., Quintard M., Bazin B., "From pore scale to wellbore scale: Impact of geometry on wormhole growth in carbonate acidization" *Elsevier: J. of. Chem. Eng. Sci.,* 63 (2008): 3088-3099, https://doi.org/10.1016/j.ces.2008.03.021.

3-бөлім | Раздел 3 | Section 3

Информатика | Информатика | Computer Science

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NATURAL LANGUAGE PROCESSING METHODS FOR CONCEPT MAP MINING: THE CASE FOR ENGLISH, KAZAKH AND RUSSIAN TEXTS

Concept maps are used for knowledge visualization via representing an input text or domain at the conceptual level. Concept maps reflect the systemic relations between key concepts of a text/ domain and thereby contribute to a deeper understanding of text/domain ideas, save time spent on reading and analysis. However, the process of concept maps construction is laborious and time consuming. Currently, there is a lot of research on the idea of automatic generation concept map from natural language texts. The problem has a high practical value, but in theoretical terms, methods for its solution are mainly language-dependent. Such methods require high-quality annotated linguistic resources, which is a serious problem for low-resource languages like Kazakh. In this work, we analyze the issues related to language-dependent approaches and present our experimental work on automatic generating concept maps from English, Kazakh and Russian texts. We use a well-known language-dependent method called ReVerb which was originally developed for English, and on the example of this method we explore the issues that we have encountered in the case of Kazakh and Russian languages.
Key words: concept maps, concept map mining, natural language processing, low-resource languages, R language.

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Концепт-карталарды өндіруге арналған табиғи тілді өңдеу әдістері: ағылшын, қазақ және орыс мәтіндерінің мысалында

Концепт-карталар концептуалды деңгейде кіріс мәтінін немесе пәндік аймақты ұсыну арқылы білімді визуализациялау үшін қолданылады. Концепт-карталар мәтіннің/пәндік аймақтың негізгі ұғымдары арасындағы жүйелік қатынасты көрсетеді және сол арқылы оқу мен талдауға кететін уақытты үнемдей отырып, пәндік аймақтың идеяларын тереңірек түсінуге ықпал етеді. Алайда, концепт-карталарды құру процесі еңбек пен және уақытты көп қажет етеді. Қазіргі уақытта табиғи тілдегі мәтіндерден концепт-карталарды автоматты түрде құру идеясына байланысты көптеген зерттеулер жүргізілуде. Мәселе жоғары практикалық құндылыққа ие, бірақ теориялық тұрғыдан оны шешу әдістері негізінен тілге тәуелді. Мұндай әдістер аннотациялары бар сапалы лингвистикалық ресурстарды талап етеді, бұл қазақ тілі сияқты ресурстары шектеулі тілдер үшін елеулі қиындық туғызады. Бұл жұмыста тілге тәуелді тәсілдерге байланысты проблемаларға талдау жасалған және ағылшын, қазақ,

орыс тілдеріндегі мәтіндерден концепт-карталарды автоматты түрде құру бойынша жасалған эксперименттік жұмыс ұсынылған. Көпшілікке белгілі бастапқыда ағылшын тілі үшін әзірленген тілге тәуелді ReVerb әдісін қолданамыз және осы әдістің мысалында оны қазақ және орыс тілдеріне аудару мәселелерін талдаймыз.
Түйін сөздер: концепт-карталар, концепт-картаны өндіру, табиғи тілді өңдеу, ресурстары шектеулі тілдер, R тілі.

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Методы обработки естественного языка для извлечения концепт-карт: кейс для текстов на английском, казахском и русском языках

Концепт-карты используются для визуализации знаний посредством представления входного текста или предметной области на концептуальном уровне. Концепт-карты отражают системные отношения между ключевыми понятиями текста/предметной области и тем самым способствуют более глубокому пониманию идей предметной области, экономя время, затрачиваемое на чтение и анализ. Однако сам процесс построения концептуальных карт трудоемок и требует много времени. В настоящее время проводится много исследований, связанных с идеей автоматической генерации концепт-карт из текстов на естественном языке. Задача имеет высокую практическую ценность, но теоретически методы ее решения в основном являются языко-зависимыми. Такие методы требуют качественных лингвистических ресурсов с аннотациями, что представляет серьезную трудность для таких малоресурсных языков, как казахский. В этой работе мы анализируем проблемы, связанные с языко-зависимыми подходами, и представляем нашу экспериментальную работу по автоматической генерации концептуальных карт из текстов на английском, казахском и русском языках. Мы используем хорошо известный, языко-зависимый метод ReVerb, который изначально был разработан для английского языка, и на примере этого метода анализируем проблемы его переноса на казахский и русский язык.

Ключевые слова: концепт-карты, извлечение концепт-карт, обработка естественного языка, малоресурсные языки, язык R.

1 Introduction

As powerful knowledge visualization tools, concept maps allow representing a text and its domain at a conceptual level. They link the key concepts and ideas of a text into a single conceptual framework, which is a kind of guide to a given text and contributes to a deeper understanding of it. Well-built concept maps allow reducing the mental and physical stress on a human, saving time spent on reading and analyzing. The latter actualizes the problem of automatic generation and embedding of concept maps into digital reading services. The problem is at the junction of three disciplines at once – human-computer interaction, natural language processing and digital reading, and largely inherits the challenges of each of them. First, there are challenges posed by the high creative variability inherent in the concept mapping process. There is no single correct way of constructing concept maps, and therefore no unambiguous assessment criteria: this is a creative process, during which new ideas and new, previously non-verbalized relations are generated [2]. Second, there are the challenges posed by traditional natural language processing issues. Generating concept maps from natural language texts involves solving problems such as text preprocessing, open information extraction, co-reference resolution, etc. [3]. Third, these are the challenges caused by the novelty of the problem of digital reading [4].

Taken together, all these challenges determine the scientific complexity of the problem of automatic generation of concept maps from natural language texts. They determine the fact that, despite the enormous practical significance, in theoretical terms, this problem is not fully resolved. In this article, we review 20 years of research it the field of automatic concept map generation and analyze the issues related to language-dependent approaches such as ReVerb [5]. We use a well-known language-dependent method called ReVerb which was originally developed for English, and on the example of this method we explore the issues that we have encountered in the case of Kazakh and Russian languages.

2 Related work

During the period from 2001 to 2020, about fifty works were published, directly related to the topic of automatic construction of concept maps based on texts in natural language. Table 1 presents 45 of the most famous publications, of which 24 are conference reports, 20 are journal articles, and 1 is a doctoral dissertation. Previously, these publications were compared with publications included in the review of related works given in [44]. The authors of this thorough and in-depth review covered the period from 2001 to 2016 and highlighted 30 relevant publications; this table supplements their overview with uncovered years and works. Most of the publications listed in the table use methods of morphological and syntactic analysis to identify concepts and relations contained in the text [6, 7], [11, 12], [18], [20], [22], [25–27], [29, 30], [36], [38], [40–42]. Typically, these are techniques such as POS tagging, syntax tree building, and morpho-syntactic patterns extraction. These methods are often supplemented by co-reference resolution, synonym extraction, and named entity recognition [6, 18, 26, 27, 30, 36, 40]. Several publications use the search for association rules to extract relations [9, 28, 39, 50].

The extracted concepts and relations are often grouped into larger categories and/or ranked in order of importance. For grouping, as a rule, clustering methods are used [6, 16, 30, 32], and for ranking – statistical methods such as TF-IDF [32, 36, 42, 48, 49], LSA [6, 17, 32], PCA [16], HARD [39], VF-ICF [30], two articles use a method for measuring bursts in text streams [42, 49].

Ultimately, the most significant concepts and the relations connecting them are combined into a single map, which from the mathematical point of view is a graph [39, 49]. In most publications, this stage is not described or described superficially, with the exception of [49], in which the construction of the graph is considered as NP-complete optimization problem with constraints imposed on the size and connectivity of the graph, and with an objective function that maximizes the total significance of vertices and edges included in the graph. It should be noted that [49] is notable not only for the detailed consideration of the final stage of assembling a concept map from previously extracted fragments (by constructing a graph). The author of this work also painstakingly considers all stages of generating concept maps, and combines them into a single logical scheme, consisting of five subtasks of the first level and eight subtasks of the second level (Figure 1). Through its comprehensive decomposition, the scheme provides a universal basis for comparing different approaches.

Table 1: List of related research for the period of 2001-2020

Year	Publication title, reference
2001	Automatic reading and learning from text [6]
2002	Knowledge discovery from texts: a concept frame graph approach [7]
2003	Concept maps as visual interfaces to digital libraries: summarization, collaboration, and automatic generation [8]
2004	A new approach for constructing the concept map [9]
2005	Using concept maps in digital libraries as a cross-language resource discovery tool [10]
2006	Jump-starting concept map construction with knowledge extracted from documents
	Concept mining for indexing medical literature [12]
2007	A new approach for constructing the concept map [13]
2008	Automatically constructing concept maps based on fuzzy rules for adapting learning systems [14]
	Building domain ontologies from text for educational purposes [15]
	Mining e-Learning domain concept map from academic articles [16]
	Concept map mining: A definition and a framework for its evaluation [17]
	Mining knowledge from natural language texts using fuzzy associated concept mapping [18]
2009	Concept extraction from student essays, towards concept map mining [19]
	Toward a fuzzy domain ontology extraction method for adaptive e-learning [20]
	A concept map extractor tool for teaching and learning [21]
2010	Concept Maps core elements condidates recongnition from text [22]
	Mining concept maps from news stories for measuring civic scientific literacy in media [23]
	Analysis of a Gold Standard for Concept Map Mining – How Humans Summarize Text Using Concept Maps [24]
2011	Generating concept map exercises from textbooks [25]
2012	The automatic creation of concept maps from documents written using morphologically rich languages [26]
	English2mindmap: An automated system for mindmap generation from English text [27]
2013	Constructing concept maps for adaptive learning systems based on data mining techniques [28]
	Document analysis based automatic concept map generation for enterprises [29]
	Concept map construction from text documents using affinity propagation [30]

Year	Publication title, reference
2014	A practical approach for automatically constructing concept map in e-learning environments [31]
	Automatic concept maps generation in support of educational processes [32]
	Burst analysis of text document for automatic concept map creation [33]
	Evaluation of concept importance in concept maps mined from lecture notes [34]
2015	Burst analysis for automatic concept map creation with a single document [35]
	Implementation of method for generating concept map from unstructured text in the Croatian language [36]
	An automatic construction of concept maps based on statistical Text Mining [37]
	Exploiting concept map mining process for e-content development [38]
2016	Using prerequisites to extract concept maps from textbooks [39]
	Automatic construction of concept maps from texts [40]
2017	Bringing structure into summaries: crowdsourcing a benchmark corpus of concept maps [41]
	Utilizing automatic predicate-argument analysis for concept map mining [42]
2018	Research on a new automatic generation algorithm of concept map based on text clustering and association rules mining [43]
	Towards technological approaches for concept maps mining from text [44]
2019	Improving an AI-based algorithm to automatically generate concept maps [45]
	Concept map mining approach based on the mental models retrieval [46]
	Fuzzy concept map generation from academic data sources [47]
	Using a recommender system to suggest educational resources and drawing a semi-automated concept map to enhance the learning progress [48]
	Automatic structured text summarization with concept maps [49]
2020	Research on a new automatic generation algorithm of concept map based on text analysis and association rules mining [50]

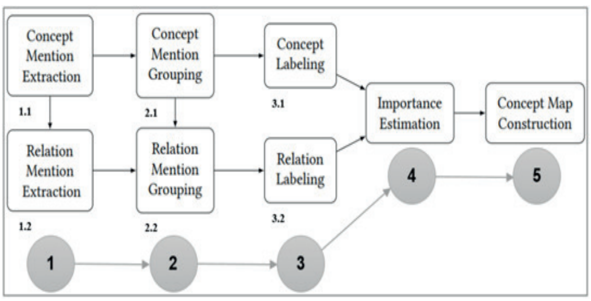

Figure 1: General scheme for solving the problem of automatic generation of concept maps from texts in natural language [49] (task numbering is ours)

3 Material and methods

3.1 ReVerb relation extraction method

Over the past two decades, the solution to the problem of automatic generation of concept maps from natural language texts has been largely based on the methods of parsing. Since these methods are language-dependent, the degree of their elaboration directly depends on the status and resource availability of the language used. Most of the cited publications use language-dependent methods designed for English language [39, 48]. In other words, there is a clear imbalance not only between language-dependent and language-independent methods of generating concept maps (not in favor of the latter), but also between high-resource and low-resource languages (also not in favor of the latter).

ReVerb which we use in this work, is exactly the kind of such language-dependent methods. It takes as input a POS-tagged sentence and returns a set of *(x,r,y)* extraction triples [5]. The method first identifies relation phrases that satisfy syntactic and lexical constraints, and then finds a pair of entities (noun phrases) for each relation phrase. The method retrieves only sequences of tokens expressing a verb relation located between two entities, for example: "<u>We</u> trust in <u>God</u>". The method does not provide relations that are located differently in the text, for example: "In God we trust". Given an input sentence s, ReVerb follows the next algorithm:

- Step 1. For each verb v in the sentence s, find the longest sequence of words r_v such that
  - r_v starts at the verb v,
  - r_v satisfies the syntactic constraint,
  - r_v satisfies the lexical constraint.
  - r_v satisfies the lexical constraint. If any pair of verbal sequences r_{v1} and r_{v2} are adjacent or overlap in the sentence s, they are merged into a single sequence. Therefore, the relation phrase must be a contiguous span of words in the sentence.
- Step 2. For each relation phrase r identified in Step 1,
  - find the nearest noun phrase x to the left of r in a sentence s such that x is not a relative pronoun,
  - find the nearest noun phrase y to the right of r in a sentence s. If such an (x, y) pair could be found, return (x, r, y) as an extraction.

The syntactic constraint requires for English relation phrases to match POS-tag patterns such as V (a verb, e.g., *write*), VP (a verb followed by a preposition, e.g., *written by*), VN?P (a verb followed by a noun and ended with a preposition, e.g., *is a part of*), and so forth. The lexical constraint separates valid relation phrases from over-specified ones using an external relation database.

3.2 Experimental work

We realize ReVerb method with the help of R language and UDPipe text processing models. UDPipe it is a pipeline which is based on Universal Dependencies 2.4 Models and provides pre-trained language models for various languages [51]. Experiments on English texts have been carried out using the joint model "english-ewt-ud-2.4-190531". Tokenizer, POS tagger, lemmatizer and parser models have been applied to input texts (see Figure 2). The output was a preprocessed corpus (see Figure 3). Then POS-tagging patterns have been applied to corpus tokens in order to extract relations (verb phrases) and entities (noun phrases). For example, the pattern <VB>?<IN> has been applied to extract relations like "flows" or "flows into", and the pattern <DT>?<PRP>?<JJ>*<NN> has been applied to extract entities like "The Baltic Sea". Matched relations and entities phrases have been sorted in the order they appeared in the sentences, and if they formed a sequence "noun phrase – verb phrase – noun phrase" they have been extracted as a triplet. Finally, a concept map has been constructed from found triplets (see Figure 4).

```
library(udpipe)
library(stringr)
library(visNetwork)

#-----------------------------------
# Input texts
#-----------------------------------
txt <- c(d1 = "Thames flows through London",
         d2 = "The source of the Thames lies in Gloucestershire",
         d3 = "Thames flows into the North Sea",
         d4 = "The Baltic Sea is connected to the North Sea via the Skagerrak and Kattega

#-----------------------------------
# Parsing
#-----------------------------------
u.model = udpipe_load_model("english-ewt-ud-2.4-190531.udpipe")
corpus  = udpipe(txt, object=u.model)
```

Figure 2: Applying UDPipe to English texts

doc_id	paragraph_id	sentence_id	sentence	start	end	term_id	token_id	token	lemma	upos	xpos
d1	1	1	Thames flows through London	1	6	1	1	Thames	Thames	PROPN	NNP
d1	1	1	Thames flows through London	8	12	2	2	flows	flow	VERB	VBZ
d1	1	1	Thames flows through London	14	20	3	3	through	through	ADP	IN
d1	1	1	Thames flows through London	22	27	4	4	London	London	PROPN	NNP
d2	1	1	The source of the Thames lies in Gloucestershire	1	3	1	1	The	the	DET	DT
d2	1	1	The source of the Thames lies in Gloucestershire	5	10	2	2	source	source	NOUN	NN
d2	1	1	The source of the Thames lies in Gloucestershire	12	13	3	3	of	of	ADP	IN
d2	1	1	The source of the Thames lies in Gloucestershire	15	17	4	4	the	the	DET	DT
d2	1	1	The source of the Thames lies in Gloucestershire	19	24	5	5	Thames	Thames	PROPN	NNP
d2	1	1	The source of the Thames lies in Gloucestershire	26	29	6	6	lies	lie	VERB	VBZ
d2	1	1	The source of the Thames lies in Gloucestershire	31	32	7	7	in	in	ADP	IN
d2	1	1	The source of the Thames lies in Gloucestershire	34	48	8	8	Gloucestershire	Gloucestershire	PROPN	NNP
d3	1	1	Thames flows into the North Sea	1	6	1	1	Thames	Thames	PROPN	NNP

Figure 3: The output of UDPipe for English texts

Experiments on Russian texts have been carried out using the joint model "russian-gsd-ud-2.4-190531" from Universal Dependencies 2.4 Models. Tokenizer, POS tagger, lemmatizer and parser models have been applied to input texts (see Figure 5). The output was a preprocessed corpus (see Figures 6-7). The POS-tag patterns, as is the case with English, have been applied to tokens, and a concept map has been constructed from found triplets (see Figure 8). Despite the fact that there are some parsing errors in the corpus, these errors generally do not affect

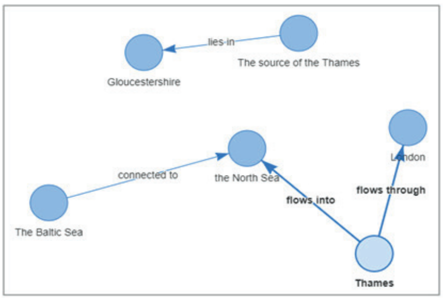

Figure 4: The final concept map based on English texts

the result of extracting relations and entities. Logical errors of the ReVerb algorithm are more serious. In particular, from the sentence "Иртыш несет свои воды в Северный Ледовитый океан"(“Irtysh carries its water to the Arctic Ocean”) three entities, viz. “Иртыш” (“Irtysh”), “свои воды” (“its water”) and “Северный Ледовитый океан” (“the Arctic Ocean”), and one relation “несет” (“carries”) are extracted, so the resulting triplet is constructed as “Иртыш – несет – свои воды” (“Irtysh – carries – its water”). However, the correct version should be “Иртыш – несет свои воды в – Северный Ледовитый океан” (“Irtysh – carries its water to – the Arctic Ocean”).

```
setwd("/home/anugumanova/spaceR/")

library(udpipe)
library(stringr)
library(visNetwork)

#-----------------------------------
# Load models
#-----------------------------------
u.model = udpipe_load_model(file = "russian-gsd-ud-2.4-190531.udpipe")


#-----------------------------------
# Input texts
#-----------------------------------
txt <- c(d1 = "Иртыш несет свои воды в Северный Ледовитый океан",
         d2 = "Иртыш вливается в проточное озеро Зайсан",
         d3 = "Иртыш протекает через город Усть-Каменогорск",
         d4 = "Иртыш вливается в реку Обь",
         d5 = "Красивейшая река Сибири Обь впадает в Карское море",
         d6 = "Иртыш берет свое начало в Китае на границе с Монгольским Алтаем")

corpus <- udpipe(txt, object=u.model)
```

Figure 5: Applying UDPipe to Russian texts

token	lemma	upos	xpos	feats
Иртыш	Иртыш	PROPN	NNP	Animacy=Anim|Case=Nom|Gender=Masc|Numbe
несет	несть	VERB	VBC	Aspect=Imp|Mood=Ind|Number=Sing|Person=3|T
свои	свой	DET	PRP$	Animacy=Inan|Case=Acc|Number=Plur
воды	вода	NOUN	NN	Animacy=Inan|Case=Acc|Gender=Fem|Number=F
в	в	ADP	IN	NA
Северный	северный	ADJ	JJL	Animacy=Inan|Case=Acc|Degree=Pos|Gender=M
Ледовитый	Ледовитый	ADJ	JJL	Animacy=Inan|Case=Acc|Degree=Pos|Gender=M
океан	океан	NOUN	NN	Animacy=Inan|Case=Acc|Gender=Masc|Number=
Иртыш	Иртыш	PROPN	NNP	Animacy=Anim|Case=Nom|Gender=Masc|Numbe
вливается	вливаться	VERB	VBC	Aspect=Imp|Mood=Ind|Number=Sing|Person=3|T
в	в	ADP	IN	NA
проточное	проточный	ADJ	JJL	Case=Acc|Degree=Pos|Gender=Neut|Number=Si
озеро	озеро	NOUN	NN	Animacy=Inan|Case=Acc|Gender=Neut|Number=
Зайсан	Зайсан	NOUN	NN	Animacy=Inan|Case=Nom|Gender=Masc|Number

Figure 6: The output of UDPipe for Russian texts

	token	type
1	Иртыш	noun_phrase
4	свои воды	noun_phrase
8	Северный Ледовитый океан	noun_phrase

Figure 7: Entities (noun phrases) extracted from the sentence "Иртыш несет свои воды в Северный Ледовитый океан"

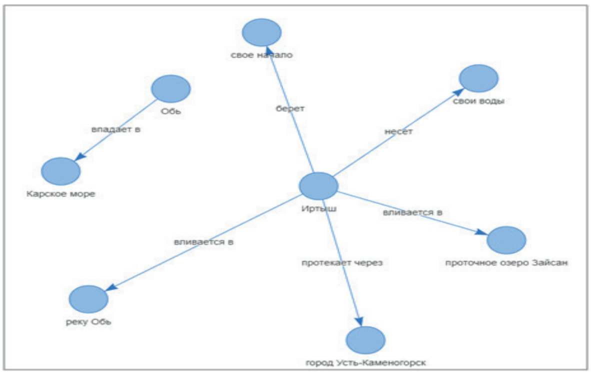

Figure 8: The final concept map based on Russian texts

Experiments on Kazakh texts have been carried out using the joint model “kazakh-ud-2.0-170801” from Universal Dependencies 2.0 Models (there is no models for Kazakh language in the Universal Dependencies 2.4 Treebank). Tokenizer, POS tagger, lemmatizer and parser models have been applied to input texts (see Figure 9). The output was a preprocessed corpus, and as it shown in Figure 10, there are a number of serious POS tagging errors which leads to a very low performance in triple extraction (see Figure 11). At the same time, manually correcting POS tags results in a more believable concept map (see Figure 12).

```
#-----------------------------------
# Load models
#-----------------------------------
u.model = udpipe_download_model(language="kazakh",
                                udpipe_model_repo='jwijffels/udpipe.models.ud.2.0')


#-----------------------------------
# Input texts
#-----------------------------------
txt <- c(d1 = "Ертіс Обь өзеніне құяды",
         d2 = "Қара Ертіс өзені Қытайдан басталады",
         d3 = "Ертіс өзені Өскемен арқылы өтеді",
         d4 = "Әдемі Павлодар қаласы Ертіс өзенінің жанында орналасқан")

corpus <- udpipe(txt, object=u.model)
corpus[corpus$token=="Ертіс", "upos"] = "PROPN"
corpus[corpus$token=="өзені", "upos"] = "NOUN"
```

Figure 9: Applying UDPipe to Kazakh texts

token	lemma	upos
Ертіс	Ертіс	NUM
Обь	Обь	NOUN
өзеніне	өзеніне	PRON
құяды	құя	VERB
Қара	Қара	NOUN
Ертіс	ертіс	NUM
өзені	өзені	ADJ
Қытайдан	Қытайдан	NOUN

Figure 10: The output of UDPipe for Kazakh texts

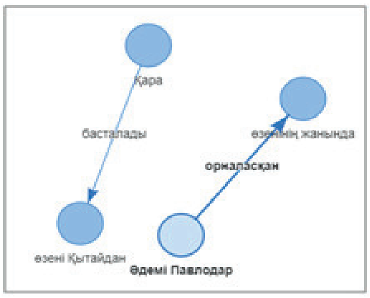

Figure 11: The output of UDPipe for Kazakh texts. An incorrect version of a concept map based on wrong preprocessing of Kazakh texts

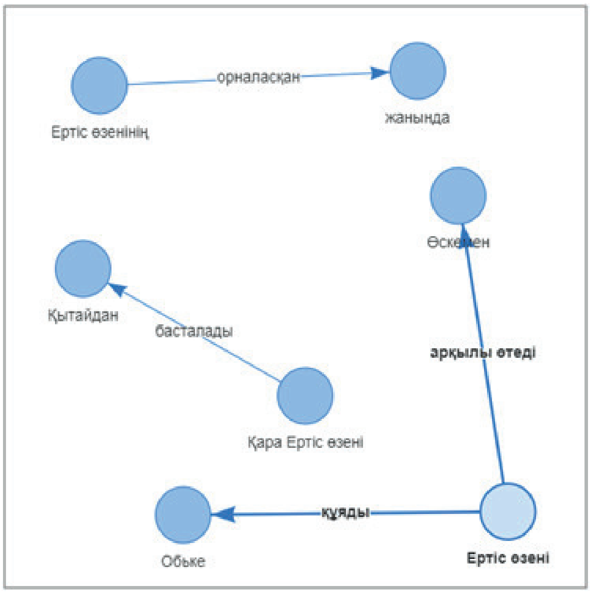

Figure 12: A version of a concept map based on manual preprocessing of Kazakh texts

4 Conclusion

In this paper, we considered an algorithm for extracting triplets of the "entity - relation - entity"type from texts in English, Kazakh and Russian. The algorithm is based on the use of syntax patterns and depends on the availability of annotated linguistic resources. It demonstrates acceptable results for Russian and English. However, experiments carried out for Kazakh texts have shown that the algorithm demonstrates very low quality in the absence of such linguistic resources. In our future work, we plan to explore alternative ways of constructing concept maps for low-resource languages such as Kazakh.

References

[1] Novak J. D., Canas A. J., "Theoretical origins of concept maps, how to construct them, and uses in education" , *Reflecting Education 3-1* (2007): 29-42.

[2] Falke, T., "Automatic Structured Text Summarization with Concept Maps" , Doctoral dissertation, Technische Universitet (2019).

[3] Kudryavtsev, D., & Gavrilova, T., "From anarchy to system: A novel classification of visual knowledge codification techniques" , *Knowledge and Process Management 24,* (2017): 3-13.

[4] Baron N., "Do students lose depth in digital reading"– The Conversation. Available at: https://theconversation. com/do-students-lose-depth-in-digital-reading-61897 (2016).

[5] Fader, A., Soderland S., Etzioni O., "Identifying relations for open information extraction" , Proceedings of the 2011 Conference on empirical methods in Natural language processing (2011): 1535-1545.

[6] Oliveira A., Pereira F. C., Cardoso A.,"Automatic reading and learning from text" , Proceedings of the International Symposium on Artificial Intelligence (2001).

[7] Rajaraman K., Tan A. H., "Knowledge discovery from texts: a concept frame graph approach" , Proceedings of the eleventh international conference on Information and knowledge management (2002): 669-671.

[8] Shen R., Richardson R., Fox E. A., "Concept maps as visual interfaces to digital libraries: summarization, collaboration, and automatic generation" , Joint Conference on Digital Libraries (2003).

[9] Sue P. C. et al., "A new approach for constructing the concept map" , IEEE International Conference on Advanced Learning Technologies, Proceedings. – IEEE, (2004):76-80.

[10] E. A. Fox, R. Richardson., "Using concept maps in digital libraries as a cross-language resource discovery tool" , Proceedings of the 5th ACM/IEEE-CS Joint Conference on Digital Libraries (JCDL '05) – (2005):256-257, doi: 10.1145/1065385.1065443.

[11] Leake, A. Valerio D., "Jump-starting concept map construction with knowledge extracted from documents" , In Proceedings of the Second International Conference on Concept Mapping, (2006).

[12] Bichindaritz I., Akkineni S., "Concept mining for indexing medical literature" , *Engineering Applications of Artificial Intelligence,* V 19, No 4 (2006):411-417.

[13] S. Tseng, P. C. Sue, J. M. Su, J. F. Weng, & W. N. Tsai, "A new approach for constructing the concept map" , *Computers & Education,* 49(3) (2007):691-707. DOI: 10.1016/j.compedu.2005.11.020.

[14] Bai S. M., Chen S. M., "Automatically constructing concept maps based on fuzzy rules for adapting learning systems" , *Expert systems with Applications,* V. 35(1-2) (2008):41-49.

[15] Zouaq A., Nkambou R., "Building domain ontologies from text for educational purposes" , IEEE Transactions on learning technologies, V. 1:49-62.

[16] Chen N. S. et al., "Mining e-Learning domain concept map from academic articles" , *Computers & Education,* V. 50, No 3 (2008):1009-1021.

[17] Villalon J. J., Calvo R. A., "A definition and a framework for its evaluation" , 2008 IEEE/WIC/ACM International Conference on Web Intelligence and Intelligent Agent Technology, IEEE, V. 3, (2008):357-360.

[18] Wang W. M. et al., "Mining knowledge from natural language texts using fuzzy associated concept mapping" , *Information Processing & Management,* V. 44 No 5 (2008):1707-1719.

[19] Villalon J., Calvo R. A., "Concept extraction from student essays, towards concept map mining" , 2009 Ninth IEEE International Conference on Advanced Learning Technologies. – IEEE, (2009):221-225.

[20] Lau R., Song D., et al., "Toward a fuzzy domain ontology extraction method for adaptive E-learning" , IEEE Transactions on Knowledge and Data Engineering, V. 21 No 6 (2009):800-813. DOI: 10.1109/TKDE.2008.137.

[21] Dalmolin L. et al., "A concept map extractor tool for teaching and learning" , 2009 Ninth IEEE International Conference on Advanced Learning Technologies. – IEEE, (2009):18-20.

[22] Kowata, J. H., Cury, D., and Silva Boeres, M. C., "Concept Maps core elements condidates recongnition from text" , In Concept Maps: Making Learning Meaningful. Proceedings of the 4th International Conference on Concept Mapping, (2010):120–127.

[23] Tseng Y. et al, "Mining concept maps from news stories for measuring civic scientific literacy in media" , *Computers & Education,* V.55, No 1 (2010):165–177.

[24] Villalon J., Calvo R. A., Montenegro R., "Analysis of a gold standard for Concept Map Mining–How humans summarize text using concept maps" , Proceedings of the Fourth International Conference on Concept Mapping, (2010):14-22.

[25] Olney A., Cade W. L., Williams C., "Generating concept map exercises from textbooks" , Proceedings of the Sixth Workshop on Innovative Use of NLP for Building Educational Applications, (2011):111-119.

[26] Zubrinic K., Kalpic D., Milicevic M., "The automatic creation of concept maps from documents written using morphologically rich languages" , *Expert systems with applications,* V. 39, No 16 (2012):12709-12718.

[27] S. M. Chen, & P. J. Sue, "English2mindmap: An automated system for mindmap generation from english text" , 2012 IEEE International Symposium on Multimedia, IEEE, (2012):326-331.

[28] Zubrinic K., Kalpic D., Milicevic M., "Constructing concept maps for adaptive learning systems based on data mining techniques" , *Expert systems with applications,* 40(7) (2013):2746-2755. DOI: 10.1016/j.eswa.2012.11.018.

[29] Karannagoda E. L. et al., "Document analysis based automatic concept map generation for enterprises" , 2013 International Conference on Advances in ICT for Emerging Regions (ICTer), IEEE (2013):154-159.

[30] Qasim, I., Jeong, J.-W., Heu, J.-U., and Lee, D.-H., "Concept Map Construction From Text Documents Using Affinity Propagation" , *Journal of Information Science,* 39(6) (2013):719–736.

[31] Yi N., Li H., "A practical approach for automatically constructing concept map in E-learning environments" , 2014 IEEE International Conference on Progress in Informatics and Computing, IEEE (2014):582-586.

[32] Pipitone A., Cannella V., Pirrone R., "Automatic concept maps generation in support of educational processes" , *Journal of e-Learning and Knowledge Society,* V.10 No 1 (2014).

[33] Yoon W.C., Lee S., Lee S., "Burst analysis of text document for automatic concept map creation" , International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems, Springer, Cham (2014):407-416.

[34] Atapattu T., Falkner K., Falkner N., "Evaluation of concept importance in concept maps mined from lecture notes" , Proceedings of the 6th International conference on computer supported education, (2014):75-84.

[35] S. Lee, Y. Park, & W. C Yoon., "Burst analysis for automatic concept map creation with a single document" , *Expert Systems With Applications,* 42(22) (2015):8817-8829.

[36] Zubrinic K., Obradovic I., Sjekavica T., "Implementation of method for generating concept map from unstructured text in the Croatian language" , 2015 23rd International Conference on Software, Telecommunications and Computer Networks (SoftCOM), IEEE (2015):220-223.

[37] Nugumanova A. et al., "An Automatic Construction of Concept Maps Based on Statistical Text Mining" , International Conference on Data Management Technologies and Applications, Springer, Cham (2015):29-38.

[38] Panopoulou et al., "Exploiting Concept Map Mining Process For E-Content Development" , 7th International Conference on Education and New Learning Technologies (2015).

[39] Wang S. et al., "Using prerequisites to extract concept maps from textbooks" , Proceedings of the 25th acm international on conference on information and knowledge management (2016):317-326.

[40] Aguiar, C. Z., Cury, D., and Zouaq, A., "Automatic Construction of Concept Maps from Texts" , In Proceedings of the 7th International Conference on Concept Mapping (2016):20–30.

[41] Falke T., Gurevych I., "Bringing structure into summaries: Crowdsourcing a benchmark corpus of concept maps" , arXiv preprint arXiv:1704.04452 (2017).

[42] Falke T., Gurevych I., "Utilizing automatic predicate-argument analysis for concept map mining" , IWCS 2017—12th International Conference on Computational Semantics (2017).

[43] Shao Z. et al., "Research on a New Automatic Generation Algorithm of Concept Map Based on Text Clustering and Association Rules Mining" , International Conference on Intelligent Computing. – Springer, Cham (2018):479-490.

[44] Aguiar C. Z., Cury D., Zouaq A., "Towards Technological Approaches for Concept Maps Mining from Text" , CLEI Electronic Journal, V.21, No 1 (2018).

[45] Alomari S., Abdullah S., "Improving an AI-Based Algorithm to Automatically Generate Concept Maps" , *Computer and Information Science,* V.12, No 4 (2019):72–83.

[46] Nacheva R. et al., "Concept Map Mining Approach Based on the Mental Models Retrieval" , *TEM Journal,* V.8, No 4 (2019):1484.

[47] Ahmed R., Ahmad T., "Fuzzy Concept Map Generation from Academic Data Sources" , *Applications of Artificial Intelligence Techniques in Engineering,* Springer, Singapore (2019):415-424.

[48] Mirbagheri G., Hakimian M., Kardan A. A., "Using a recommender system to suggest educational resources and drawing a semi-automated concept map to enhance the learning progress" , Proceedings of The International Conference on E-Learning in the Workplace 2019 (ICELW), V.6 (2019).

[49] Falke, T., "Automatic Structured Text Summarization with Concept Maps" , Doctoral dissertation, Darmstadt Technische Universitet (2019):240.

[50] Shao Z. et al., "Research on a new automatic generation algorithm of concept map based on text analysis and association rules mining" , *Journal of Ambient Intelligence and Humanized Computing,* V.11 No 2 (2020):539-551.

[51] Straka M. and Strakov J., "Universal Dependencies 2.4 Models for UDPipe (2019-05-31), LINDAT/CLARIAH-CZ digital library at the Institute of Formal and Applied Linguistics (FAL)" , Faculty of Mathematics and Physics, Charles University, http://hdl.handle.net/11234/1-2998.

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CONVOLUTIONAL DEEP LEARNING NEURAL NETWORK FOR STROKE IMAGE RECOGNITION: REVIEW

Deep learning is one of the developing area of artificial intelligence research. It includes artificial neural network-based machine learning approaches. One method that has been widely used and researched in recent years is convolution neural networks (CNN). Convolutional neural networks have different research issues and medicine is one of the main ones. Today, the predominant global problem is acute cerebral blood flow disorder - stroke. The most important diagnostic tests for stroke are computerized tomography (CT) imaging and magnetic resonance imaging (MRI). However, late recognition and diagnosis by a specialist can affect the lives of many patients. For such cases, the role and help of convolutional neural networks are extraordinary. In-depth clustering neural networks apply non-linear transformations and abstractions of high-level models in large databases. Year after year, advances in the field of deep learning architecture, namely crate neural networks for the recognition of stroke, are making an important contribution to medicine's evolution. In this paper, a review of the achievements of deep learning neural networks in the recognition of stroke from brain images is considered. This review chronologically presents the main neural network block diagram and open databases providing MRI and CT images. In addition, a comparative analysis of convolutional neural networks are used in this study of stroke detection is exhibited, in addition achieved indicators of the methodologies used.
Key words:: artificial intelligence, deep learning, stroke, convolutional neural network, MRI, CT.

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Инсультті тануға арналған конволюциялық нейрондық терең оқыту желісі: шолу

Терең оқыту-жасанды интеллектті зерттеудің дамып келе жатқан салаларының бірі. Ол жасанды нейрондық желілерге негізделген машиналық оқыту әдістерін қамтиды. Соңғы жылдары кеңінен қолданылатын және зерттелетін әдістердің бірі – конволюциялық нейрондық желілер (CNN). Конволюциялық нейрондық желілер әртүрлі зерттеу міндеттеріне ие, ал медицина саласы – басты міндеттердің бірі болып табылады. Бүгінгі таңда бас миының қанмен қамтамасыз етілуінің күрт бұзылуы – инсульт, басым жаһандық проблема болып саналады. Инсульттің ең маңызды диагностикалық зерттеулері – компьютерлік томография (КТ) және магниттік-резонанстық томография (МРТ). Алайда, маманның уақтылы диагноз қойып, көмектесе алмай қалуы көптеген пациенттердің өміріне әсер етуі мүмкін. Мұндай жағдайларда конволюциялық нейрондық желілердің рөлі мен көмегі зор. Терең оқытудың конволюциялык нейрондық желілері үлкен мәліметтер базасында жоғары деңгейлі модельдердің сызықты емес түрлендірулері мен абстракцияларын қолданады. Жылдан жылға терең білім беру архитектурасындағы жетістіктер, атап айтқанда инсультты тануға, конволюциялық нейрондық желілер медицинаның дамуына айтарлықтай үлес қосады. Бұл мақалада инсультты мидың суреттерінен анықтауға нейрондық терең оқыту желілериниң жетістіктері туралы шолу берілген. Берілген шолуда нейрондық желінің негізгі схемасы хронологиялық түрде ұсынылған және МРТ мен КТ суреттерін ашық түрде қолданысқа беретін мәліметтер базасы келтірілген. Сонымен қатар, инсультты анықтауда конволюциялық нейрондық желілерді қолдануға салыстырмалы талдау, сондай-ақ қолданылатын әдіснамалардың жетістік көрсеткіштері ұсынылған.
Түйін сөздер: жасанды интеллект, терең оқыту, конволюциялық нейрондық желі, инсульт, МРТ, КТ.

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Сверточная нейронная сеть глубокого обучения для распознавания изображений инсульта: обзор

Глубокое обучение является одним из развивающихся области исследования искусственного интеллекта. Она включает в себя методы машинного обучения, которые основаны на искусственных нейронных сетях. Одним из методов, который широко применяется и исследуется в последние годы, это - сверточные нейронные сети (CNN). Они имеют разный спектр задач исследований, и медицина одна из главных. На сегодняшний день, превалирующей глобальной проблемой считается острое нарушение кровоснабжения головного мозга - инсульт. Наиболее важными диагностическими исследованиями при инсульте считаются компьютерная томография (КТ), а также магнитно-резонансная томография (МРТ). Однако, несвоевременное распознавание и диагностирование со стороны специалиста могут повлиять на жизни многих пациентов. Для подобных случаев, роль и помощь сверточных нейронных сетей велика. Сверточные нейронные сети глубокого обучения использует нелинейные реформирования и абстракции моделей высокого уровня в больших базах данных. Год за годом, достижения в области архитектуры глубокого обучения, а именно сверточных нейронных сетей, для распознавания инсульта вносят значительный вклад в развитие медицины. В этой статье представлен обзор достижения нейронных сетей глубокого обучения в распознавания инсульта по изображениям мозга. В следующем обзоре хронологически представлено, основная блок-схема нейронной сети и открытые базы данных, предоставляющие изображения МРТ и КТ. Кроме того, представлен сравнительный анализ использования сверточных нейронных сетей при выявлении инсульта, а также достигнутые показатели используемой методологий.
Ключевые слова: искусственный интеллект, глубокое обучение, сверточная нейронная сеть, инсульт, МРТ, КТ.

1 Introduction

Data mining and deep learning have recently become the most popular topics among the scientific community, with surprising results in robotics, image recognition and artificial intelligence (AI). For many years, vast amounts of data have been collected for statistical purposes. Statistical curves can be used to forecast future behavior by describing the past and present. For decades, however, only traditional methods and algorithms have been employed, and refining these algorithms can result in successful self-learning [1]. Based on existing numbers, diverse criteria, and advanced statistical methodologies, better decision-making can be achieved. As a result, one of the most important applications of this optimization is in medicine, where vast datasets of symptoms, causes, and medical judgments can be utilized to anticipate the best approaches for quick mapping and treatment of stroke [2].

Stroke, also known as impaired cerebral circulation, is a central nervous system injury that is one of the major causes of death in developed countries. The average global lifetime risk of stroke grew from 22.8 percent in 1990 to 24.9 percent in 2016, according to The Global Burden of Disease 2016 Lifetime Risk of Stroke Collaborators, a relative rise of 8.9 after accounting for the competing risk of death from any cause other than stroke [3]. According to a research from the Centers for Disease Control and Prevention, stroke has risen from third position in 2007 to fifth position in 2017 due to mortality in the United States[4-6]. Nevertheless, there is the possibility of recovery in the injured area if treatment is given immediately. In the past, neuroimaging was performed mainly to exclude haemorrhagic, tumour diseases and infectious etiology from the ischaemic cause of stroke, and the role of diagnostic and therapeutic efficacy of neuroimaging and interventional neuroradiology in the acute treatment of stroke has been limited. With recent advances, neuroimaging is now an important part of stroke treatment. Information from CT scans can be useful to show the differences and similarities between the objects that make it up. The medical examiner performs visual segmentation during the examination in order to highlight specific areas capable of displaying pathological changes. However, the changes that appear on MRI and CT scans are usually subtle and in some cases imperceptible to simple visual analysis [7]. These digital images contain approximately 4,000 different shades of grey, whereas the human visual system can distinguish approximately 64 shades of grey in the same light condition. Thus, computational analysis can reveal a large number of features that are not easily detectable to the human observer. In this context, it is crucial to develop computational tools to help the physician to make a diagnosis in order to prevent, detect and monitor this neurological disease. These tools are based on research of medical images, as they can provide a lot of information about a patient's health. Such computational analysis often reduces the subjectivity of diagnosis while providing higher accuracy of invasive treatments [8]. The reported research suggests that an automated diagnostic system trained to use large amounts of patients' physiological signal and image data, based on artificial integration of advanced signal processing and deep learning in an automated fashion, can help neurologists, neurosurgeons, radiologists and other medical professionals make better clinical decisions. So far, convolutional neural networks, one of the deep learning techniques, have given good practical results in medical imaging research. In view of this, it is an urgent task to study convolutional neural network artificial intelligence for stroke diagnosis from medical-images.

The purpose of this paper is to provide an overview of deep learning convolutional neural networks and image processing approaches for detecting strokes in MRI and CT images.

2 Material and methods

Deep learning was initially coined in 2006 as a new discipline of machine learning research. It was first described in a similar way to hierarchical learning in [9], and it usually encompassed a wide range of pattern recognition-related disciplines of study. Deep learning takes into account a few crucial elements, including non-linear processing in layers or stages, as well as supervised or unsupervised learning. [10].The input algorithm in nonlinear multilayer processing of the active layer is the preceding layer. Between the layers, a hierarchy is formed to sort the value of the data that is regarded helpful. A similar time, supervised or unsupervised learning is linked to the class target label, with its presence indicating a supervised system and its absence indicating an unsupervised system.

2.1 Convolutional neural networks

Artificial neural networks have risen to prominence in the processing of unstructured data in recent years, including pictures, text, audio, and voice. Convolutional neural networks (CNNs) are giving very good results for such unstructured data. Whenever there is a topology associated with the data, convolutional neural networks are doing a good job on features extraction from given set of data. Architecturally, CNNs are inspired by multilayer perceptron neural network (MLP-NN). By setting local connectivity constraints between neurons of neighboring layers, CNNs make use of local spatial correlation. The core element of CNN is the processing of data through a convolution operation. The convolution of any signal with another signal creates a third signal that can reveal more information about the signal than the original signal itself. Convolutional neural networks (CNNs) are based on image convolution and detect features based on filters that are learned by CNNs through training. For example, we do not apply any known filters to detect disease or to remove Gaussian noise, but using a convolutional neural network to train, the algorithm learns the image processing filters itself, which may be very different from conventional image processing filters. For supervised learning, the filters are learned so that the overall cost function is reduced as much as possible. Typically, the first layer of convolution learns to detect the presence of stroke, while the second layer can learn to detect more complex forms that can be formed by the formation of different types of stroke, such as ischaemic, haemorrhagic, and so on. The third layer and beyond learns much more complex objects based on those created in the previous layer. One of the fundamental things about convolutional neural networks is the sparse connectivity that arises from the weight of sharing, which decreases the number of parameters to learn by a significant amount. The same filter can learn to detect the same edge in any given part of the image because of equivariance property, which is an excellent convolution property useful for object detection.

2.2 Convolutional neural network components

A convolutional neural network's typical components are listed below:

The image's pixel intensity will be stored in the input layer.. For the red, green, and blue (RGB) color channels, an input picture with width 64, height 64, and depth 3 would have an input size of 64x64x3.

The convolution layer takes the pictures from the preceding layers and convolves them with a specified number of filters to generate output object mappings. The supplied number of filters equals the number of output object mappings. TensorFlow's or PyTorch's CNNs have largely employed 2D filters up until now, however 3D convolution filters have just been added.

Theactivation functions for CNN are usually ReLUs. After traveling through the ReLU activation layers, the output measurement is the same as the input value. The ReLU layer gives the network non-linearity while also providing unsaturated gradients for the positive network inputs.

In terms of height and breadth, the merging layer will lower the dimensionality of the 2D activation cards. The depth or number of activation maps is unaffected and remains constant.

Traditional neurons in fully linked layers get distinct sets of weights from preceding layers; unlike convolution processes, there is no weight sharing between them. Through independent weights, each neuron in this layer will be linked to all neurons in the previous layer or all coordinate outputs in the output maps. Class output neurons are fed inputs from finite fully connected layers for classification.

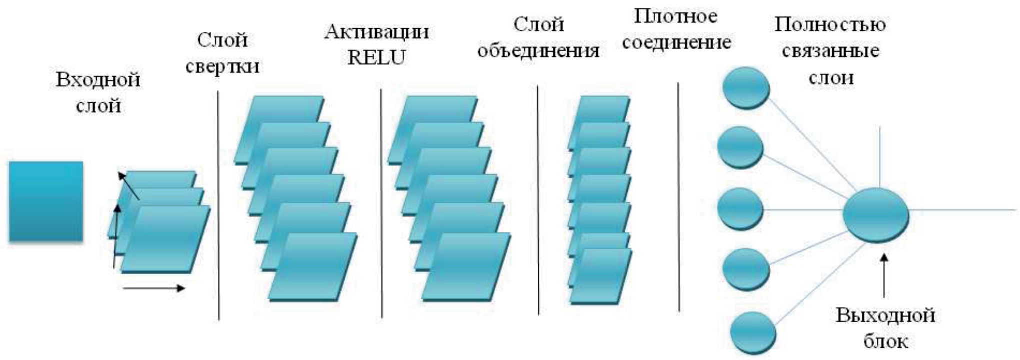

Figure 1: Basic block diagram of a convolutional neural network

Figure 1 shows a basic convolutional neural network that uses one convolutional layer, one ReLU layer and one association layer, followed by a fully connected layer and finally an output classification layer. The network tries to distinguish images of the brain with a stroke from images of a healthy brain. The output device can be thought of as a sigmoid activation function because it is a binary image classification problem. Typically, for most CNN architectures, multiple layer-ReLU convolutional layer combinations are stacked one after the other before the completely unified layers.

2.3 Data sets Algorithms that use deep learning of convolutional neural networks are promising, but they require large training data sets to optimize the performance of large productions. For research and software development in stroke detection, vast datasets have been created and are publicly available, here are some of them:

- ISLES - data collection on patients with stroke and related expert segmentations.
- ATLAS Open Data provides open access to proton-proton collision data on the tank for educational purposes. Developed in collaboration with students and educators, ATLAS Open Data resources are suitable for high school and college students as well as researchers.
- BraTS, which uses multi-institutional preoperative MRI scanning and focuses on the segmentation of internally heterogeneous (in appearance, shape and histology) brain tumours, namely gliomas.
- BioGPS is a free, extensible and customizable gene annotation portal, a complete resource for studying gene and protein function, with datasets.
- The International Stroke Database is designed to provide the international stroke research community with access to clinical and research data to accelerate the development and application of advanced neuroinformatics in clinical settings to improve patient management and ultimately patient outcomes.

3 Results and discussion

Convolutional neural networks have proven to be a leader in image recognition and segmentation for large datasets among deep learning methods. Year by year, researchers in the field of brain lesions, namely stroke, get good results in identifying convolutional neural networks were used to study the illness in compatibility with various machine learning algorithms. Table 1 below shows the results of researchers in the field of stroke using convolutional neural networks over the last 4 years and their results.

Table 1: Comparative analysis of the use of convolutional neural networks for stroke detection

Link to article	Year of research	Neural network	Image type	Dice or Accuracy
[11]	2020	DCNN	MRI	0.86
[12]	2020	CNN	CT	0.88
[13]	2020	DCNN	MPT	74%
[14]	2020	3DRCNN	MRI	0.64
[15]	2020	CNN	CT	79%
[16]	2020	CNN	2DMRI	95.33%
[17]	2019	CNN (LeNet, SegNet)	MRI	96%, 85%
[18]	2019	3DCNN	KT	93%
[19]	2018	CNN	CT	90%
[20]	2018	CNN	CT	88%
[21]	2017	CNN	CT	0,89
[22]	2017	3DCNN	CT	0.996
[23]	2017	CNN	MRI	0.83

Convolutional neural networks applied to 3D and 2D MRI and CT images in combination with different algorithms and machine learning techniques have been found to give the best results.

4 Conclusion

This article reviewed the application of convolutional deep learning neural networks for the recognition of brain stroke from MRI and CT images. Neuroimaging has favorably aided in the diagnosis of brain disease and has also advanced the world of research in medicine. Due to the development of artificial intelligence, the collaborative use of deep learning techniques, namely CNN has shown good results in many works of brain scientists. Convolutional neural networks are a new era in the diagnosis of disease varieties, so it will be a good aid in detecting brain stroke for the specialist, enabling timely treatment and prolonging the patient's life.

References

[1] Deep Learning: review, accessed july 12, 2019, https://habr.com/ru/company/ /otus/blog/459785/

[2] Kim In-Wha, Oh Jung Mi, "Deep learning: from chemoinformatics to precision medicine Journal of Pharmaceutical Investigation. 2017

[3] Salim Virani Alvaro Alonso, Emelia Benjamin, et al. "Heart disease and stroke Statistics 2020 update: A report from the American Heart Association". Circulation. 2020.

[4] Emelia Benjamin, et al. "Heart disease and stroke statistics 2017 update: A report from the American Heart Association". Circulation. 2017.

[5] Jiaquan Xu, et al. "Mortality in the United States, 2012". US Department of Health and Human Services. 2014.

[6] Jiaquan Xu, et al. "Deaths: final data for 2007". National vital statistics reports: from the Centers for Disease Control and Prevention. 2010.

[7] Temesguen Messay, Russell Hardie, Steven Rogers. "A new computationally efficient CAD system for pulmonary nodule detection in CT imagery". Medical image analysis. 2010.

[8] Rodrigues da Silva E. " Ambiente virtual colaborativo de diagnostico a distancia integrado a ferramentas de manipulacao de imagens. Universidade Federal de Pernambuco 2010.

[9] Amir Mosavi, Annamaria Varkonyi-Koczy. " Integration of machine learning and optimization for robot learning". Springer, 2017.

[10] Y. Bengio. " Learning deep architectures for AI". Now Publishers Inc, 2009.

[11] Naiqin Feng, Xiuqin Geng, Lijuan Qin. " Study on MRI Medical Image Segmentation Technology Based on CNN-CRF model". IEEE Access, 2020.

[12] P. Calvachi. " European Stroke 2020: Stroke detection and subtype classification using Convolutional Neural Networks (CNNs)". Alzheimer's and dementia. 2020.

[13]Liangliang Liu. " Deep convolutional neural network for automatically segmenting acute ischemic stroke lesion in multi-modality MRI". Neural Computing and Applications. 2019.

[14] Naofumi Tomita, et al. " Automatic post stroke lesion segmentation on MR images using 3D residual convolutional neural network". NeuroImage: clinical. 2020.

[15] Manon Tolhuisen, et al. " A convolutional neural network for anterior intra-arterial thrombus detection and segmentation on non-contrast computed tomography of patients with acute ischemic stroke". Applied Sciences. 2020.

[16] Integrating uncertainty in deep neural networks for MRI based stroke analysis accessed august 13, 2020, https://arxiv.org/abs/2008.06332

[17] Bhagyashree Gaidhani, et al. " Brain stroke detection using convolutional neural network and deep learning models". IEEE, 2019.

[18] Oli Oman, et al. " 3D convolutional neural networks applied to CT angiography in the detection of acute ischemic stroke". European radiology experimental. 2019.

[19] J. Marbun, et al. " Classification of stroke disease using convolutional neural network". Journal of physics. 2018.

[20] Danillo Pereira, et al. " Stroke lesion detection using convolutional neural networks". IEEE, 2018.

[21] Aneta Lisowska, et al. " Context-aware convolutional neural networks for stroke sign detection in non-contrast CT scans". Springer. 2017.

[22] Aneta Lisowska, et al. " Thrombus detection in ct brain scans using a convolutional neural network". Scitepress. 2017.

[23] Liang Chen, et al. " Fully automatic acute ischemic lesion segmentation in DWI using convolutional neural networks". NeuroImage: Clinical. 2017.

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DEVELOPMENT OF A DECISION SUPPORT SYSTEM FOR EVALUATING INVESTMENT PROJECTS TAKING INTO ACCOUNT MULTI-FACTORITY BASED ON THE METHOD OF HIERARCHY ANALYSIS AND GAME THEORY

The paper describes the developed software product-the decision support system (DSS) "DSS Invect 2020"and its module "IT INVESTMENT which can be used both independently and as part of the DSS "DSS Invect 2020". The DSS is designed to make recommendations to the DSS during the selection of rational financial strategies by investors. When solving problems in poorly structured subject areas, the task of choosing rational investor strategies can be explained by increasing the relevance of the subject model to increase the reliability of proposals developed with the help of the DSS. The computing core of "DSS Invect 2020"is based on the method of hierarchy analysis, as well as on the mathematical solution obtained for the first time, which is based on the tools of bilinear multistep quality games with multiple terminal surfaces. "DSS Invect 2020"allows you to evaluate the attractiveness of investment projects in the field of digitalization of enterprises. The DSS "DSS Invect 2020"is implemented on a modular basis. The modular paradigm used during development makes it possible to supplement the DSS with other modules as necessary to expand the functionality of the DSS.

Key words: decision support system, investment project, software product, module.

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Иерархиялар мен ойын теориясын талдау әдісі негізінде көп факторлы ескере отырып, инвестициялық жобаларды бағалаудағы шешім қабылдауды қолдау жүйесін әзірлеу

Жұмыста әзірленген бағдарламалық өнім - "DSS Invect 2020"шешімдерді қабылдауды қолдау жүйесі (ШҚҚЖ) және оның "IT INVESTMENT"модулі сипатталған, оны өз бетінше де, "DSS Invect 2020"ШҚҚЖ құрамында да пайдалануға болады. ШҚҚЖ инвесторлардың ұтымды қаржылық стратегияларын таңдау кезінде ШҚТ ұсыныстарын жасауға арналған. Нашар құрылымдалған пәндік салалардағы мәселелерді шешу кезінде инвестордың ұтымды стратегияларын таңдау міндетін ШҚҚЖ көмегімен жасалған ұсыныстардың сенімділігін арттыру үшін пәндік модельдің өзектілігін арттыру арқылы түсіндіруге болады. "DSS Invect 2020"есептеу ядросы иерархияны талдау әдісіне, сондай-ақ бірнеше терминалды беттері бар көп сатылы сапалы ойындардың құралдарына негізделген алғаш рет алынған математикалық шешімге негізделген. "DSS Invect 2020"кәсіпорындарды цифрландыру саласындағы инвестициялық жобалардың тартымдылығын бағалауды іске асыруға мүмкіндік береді. "DSS Invect 2020"ШҚҚЖ модульдік қағидат бойынша орындалған. Даму барысында қолданылатын модульдік парадигма ШҚҚЖ функционалдығын кеңейту қажеттілігіне қарай ШҚҚЖ-ді басқа модульдермен толықтыруға мүмкіндік береді.

Түйін сөздер: шешім қабылдауды қолдау жүйесі, инвестициялық жоба, бағдарламалық өнім, модуль.

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Разработка системы поддержки принятия решений для оценки инвестиционных проектов с учетом многофакторности на основе метода анализа иерархий и теории игр

В работе описаны разработанный программный продукт - система поддержки принятия решений (СППР) "DSS Invect 2020"и его модуль "IT INVESTMENT который может использоваться как самостоятельно, так и в составе СППР "DSS Invect 2020". СППР предназначена для выработки рекомендаций ЛПР в ходе выбора рациональных финансовых стратегий инвесторами. При решении проблем в слабо структурированных предметных областях задача выбора рациональных стратегий инвестора может быть объяснена повышением актуальности предметной модели для повышения надежности предложений, разработанных с помощью СППР. Вычислительное ядро "DSS Invect 2020"базируется на методе анализа иерархий, а также на впервые полученном математическом решении, которое основано на инструментарии билинейных многошаговых игр качества с несколькими терминальными поверхностями. "DSS Invect 2020"позволяет реализовывать оценку привлекательности инвестиционных проектов в сфере цифровизации предприятий. СППР "DSS Invect 2020"выполнена по модульному принципу. Модульная парадигма, используемая в ходе разработки, дает возможность дополнять СППР другими модулями по мере необходимости расширения функционала СППР.

Ключевые слова: система поддержки принятия решений, инвестиционный проект, программный продукт, модуль.

1 Introduction

The problem of effective financial investment in advanced information technologies is one of the most important in the field of digitalization of industrialized countries. Such investment projects are usually characterized by a high degree of uncertainty and risk. Many researchers note that in order to increase the effectiveness and efficiency of evaluating such large investment projects related to the digitalization of enterprises, it is advisable to use the potential of various computerized decision support systems (DSS). This is especially true for the analysis of different variant strategies of investors. That is, the number of options for investment strategies can be quite large.

The "DSS Invect 2020"program allows you to:

1. balance in accordance with the method of hierarchy analysis, set expert estimates of the comparison of options in the form of a matrix of paired comparisons. Estimates can be set not only in numerical form, but also in graphical representation, which makes it easier for experts to work. The program implements the receipt of final estimates and the construction of the resulting diagrams;

2. find rational strategies for investors with visualization of calculation data in tabular and graphical form.

2 Main material

The use of the hierarchy analysis method in "DSS Invect 2020"allows you to include in the hierarchy all the knowledge and intuition available to the expert group on the problem under consideration. This method is simple and gives a good correspondence to intuitive ideas [1]. The first stage in solving the problem of decision-making is the decomposition of the problem through the definition of its components and the relationships between them (Fig. 1).

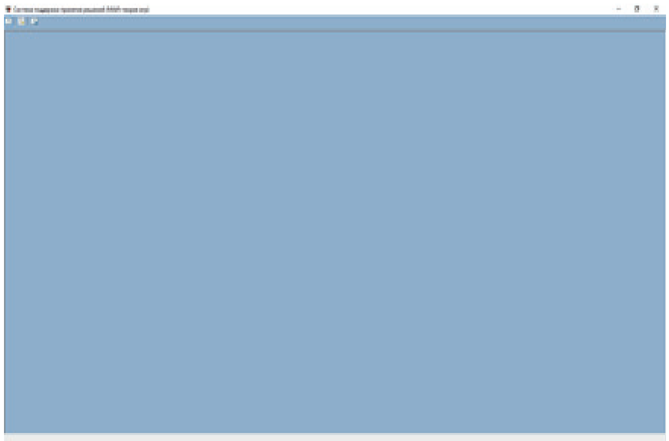


Figure 1: General view of the DSS "DSS Invect 2020"

If the computing resources of the computer are insufficient, for example, the number of criteria and alternatives exceeds 10, then the DSS issues a corresponding warning, see Fig. 2

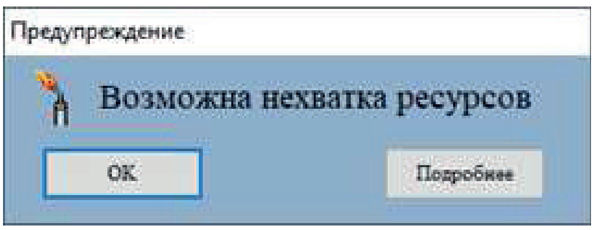
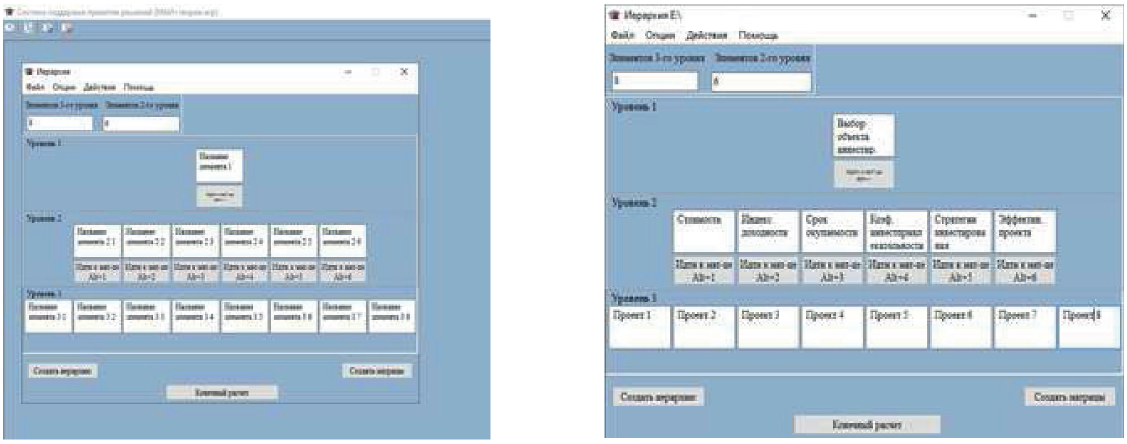


Figure 2: DSS alert "DSS Invect 2020"

When working with the DSS "DSS Invect 2020", the expert must fill out the forms proposed by the system in the interactive mode. After that, the corresponding hierarchies will be automatically formed for the subsequent evaluation of the investment project, see Fig. 3

The DSS "DSS Invect 2020" system has a well-developed menu, which makes working with it very convenient even for an unprepared user.

The "DSS Invect 2020" includes a module that allows you to take into account the degree of disparity of the compared alternatives when making decisions during the process of investing in digitalization objects, taking into account the multi-factor nature. Fig. 5 describes



a) b)

Figure 3: Example of creating hierarchies in the DSS "Invect 2020 DSS"

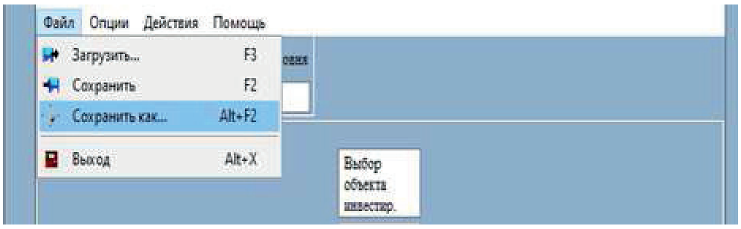


Figure 4: The main menu of the DSS "DSS Invect 2020"

in more detail the stages of filling in the matrices by experts in the context of the problem being solved. Fig. 5 (a) shows all the matrices in the child forms of the application, and the criteria were set at the last step of the project evaluation. Fig. 5 (b) shows an example of the criterion matrix already filled in by the expert – "Selection of the investment object".

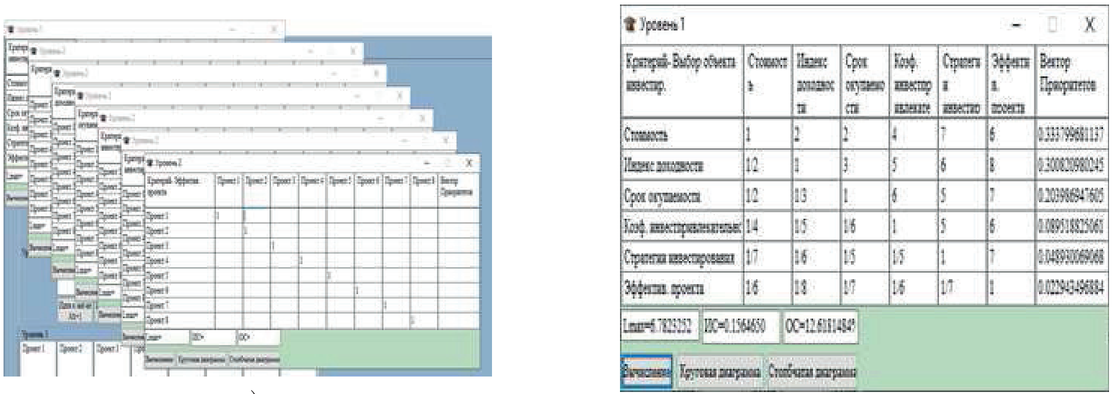
In the lower part of the form, each of the criteria is automatically calculated and controlled by such indicators as: $\lambda_{\max}$ (or $L_{\max}$ – the eigenvalue of the matrix); ИС (the consistency index); ОС (the consistency ratio).

For clarity, the results can be visualized, for example, in the form of a bar chart, see Fig. 6.

Fig. 7 shows the resulting matrix for the initial selection stage of the project that can be selected for investment.

Also, for convenience, you can keep a history of evaluation procedures, which is convenient when you need to perform several evaluations by different experts or groups of experts. The rating history file is stored in its own format (see Fig. 8) of the "DSS Invect 2020" project, so it cannot be opened without the corresponding software. The latter circumstance can be useful to prevent unauthorized access to the calculation data, for example, by unscrupulous competitors.

The "IT INVESTMENT" module is described below, which can be used both independently and as part of the DSS "DSS Invest 2020".



a) b)

Figure 5: Examples of matrices filled in by experts

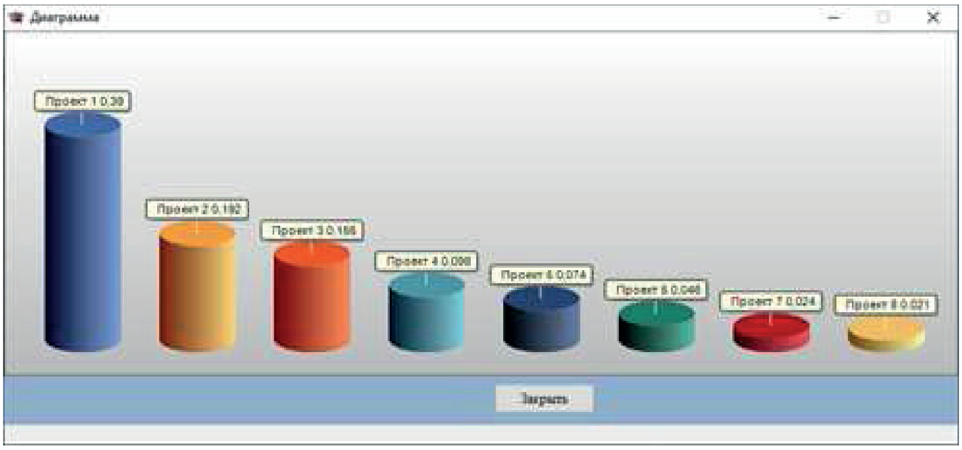
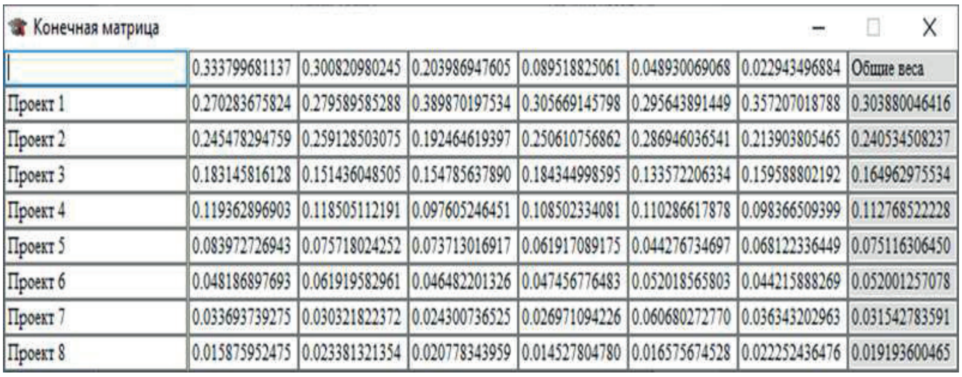


Figure 6: Bar chart obtained during the selection of strategies for investing in digitalization objects, taking into account the multi-factor nature



Конечная матрица

	0.333799681137	0.300820980245	0.203986947605	0.089518825061	0.048930069068	0.022943496884	Общие веса
Проект 1	0.270283675824	0.279589585288	0.389870197534	0.305669145798	0.295643891449	0.357207018788	0.303880046416
Проект 2	0.245478294759	0.259128503075	0.192464619397	0.250610756862	0.286946036541	0.213903805465	0.240534508237
Проект 3	0.183145816128	0.151436048505	0.154785637890	0.184344998595	0.133572206334	0.159588802192	0.164962975534
Проект 4	0.119362896903	0.118505112191	0.097605246451	0.108502334081	0.110286617878	0.098366509399	0.112768522228
Проект 5	0.083972726943	0.075718024252	0.073713016917	0.061917089175	0.044276734697	0.068122336449	0.075116306450
Проект 6	0.048186897693	0.061919582961	0.046482201326	0.047456776483	0.052018565803	0.044215888269	0.052001257078
Проект 7	0.033693739275	0.030321822372	0.024300736525	0.026971094226	0.060680272770	0.036343202963	0.031542783591
Проект 8	0.015875952475	0.023381321354	0.020778343959	0.014527804780	0.016575674528	0.022252436476	0.019193600465

Figure 7: The resulting matrix for the initial evaluation of investment projects

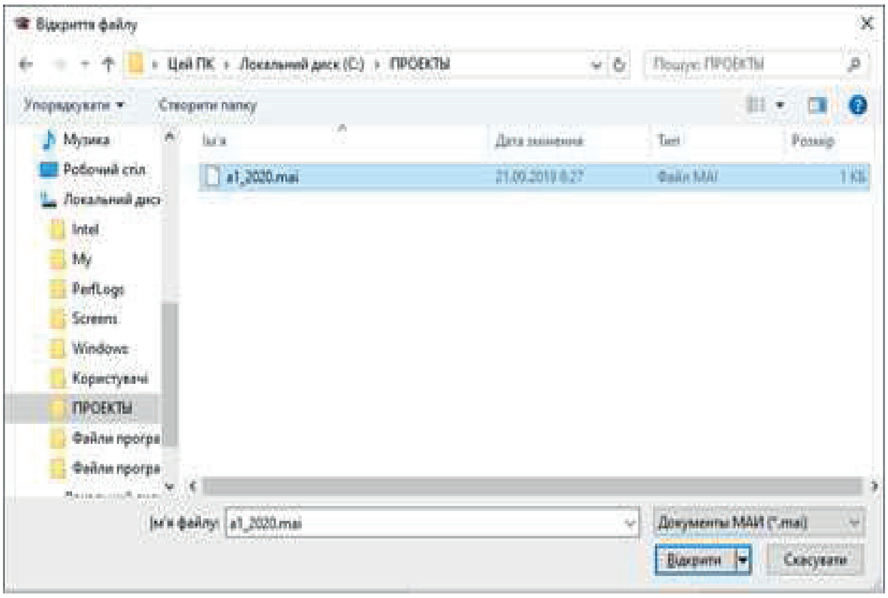


Figure 8: File with the history of expert assessments of investment projects

A general view of the form for setting the source data in the "IT INVESTMENT" module is shown in Fig. 9

The "IT INVESTMENT" DSS module is implemented in the Visual Studio 2019 programming environment.

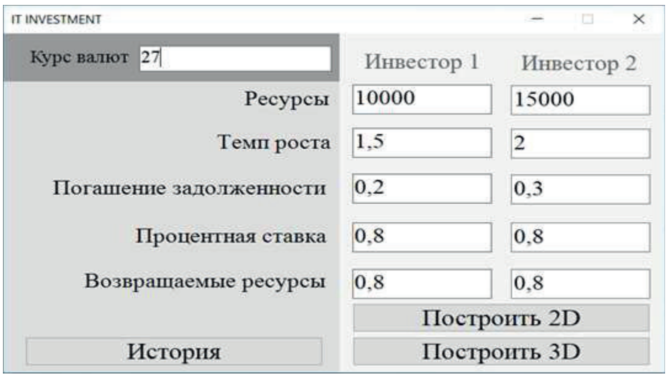
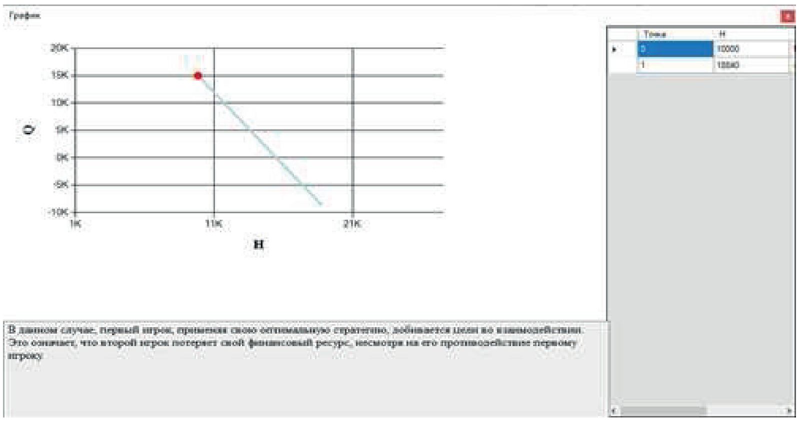


Figure 9: General view of the "IT INVESTMENT" module

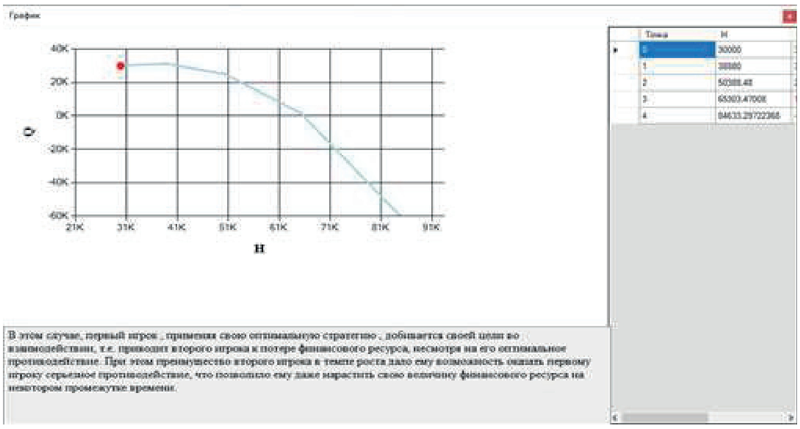
All text fields on the main form are used for setting initial values and coefficients. By default, an empty field contains 0.

3 Results

The "Build 2D" and "Build 3D" buttons allow you to perform calculations.



a)



b)

Figure 10: 2D graph output and solution interpretation in the "IT INVESTMENT" module

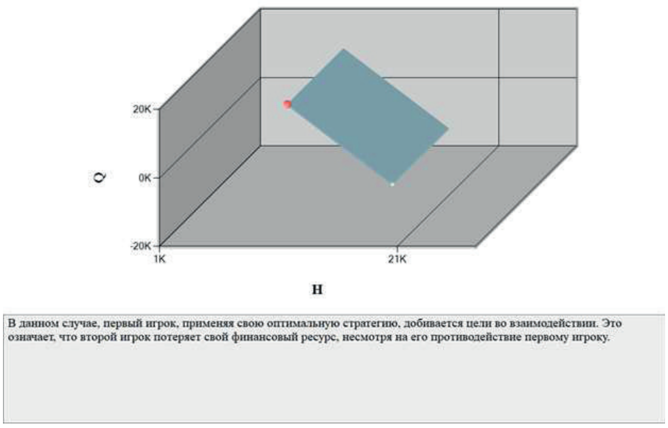


Figure 11: Example of a three-dimensional graph for two calculation points and interpretation of the solution in the "IT INVESTMENT" module

The cases when the players' strategies lead them to the corresponding terminal surfaces were modeled. On the plane, the abscissa axis is the financial resources of the first investor. The ordinate axis is the financial resources of the second investor.

In the lower part of the window, Fig. 10, shows the output generated in the "IT INVESTMENT" module during the search for rational strategies by investors, as well as the intersection area of the first investor's preference set. The rational strategy of the investor in Figure 10 a), b) is shown by a gray-blue line for a 2D chart or a plane for a 3D chart. The starting point is shown in red. The right part of the visualization form shows a table with the calculated values of the points that make up the trajectory of the optimal strategy of the investor.

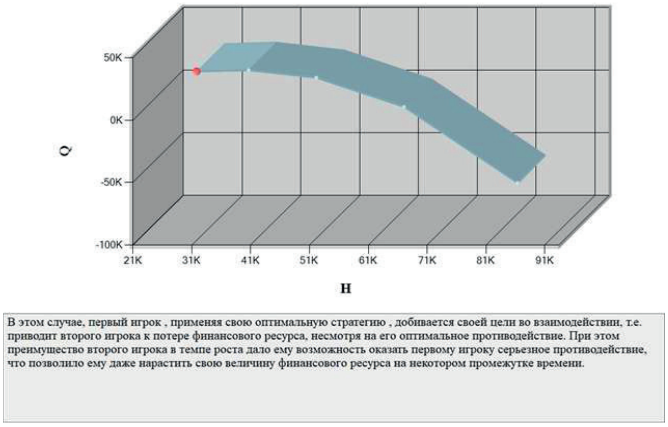


Figure 12: Example of a three-dimensional graph for five calculation points and interpretation of the solution in the "IT INVESTMENT" module

On the central part of the form there is a graph. It can be two-dimensional or three-dimensional, depending on the user's choice. The starting point of the graph is always marked in red. On the right is a table with the coordinates of each point. A text description of the result obtained during the calculations is placed at the bottom. The "History"button displays a table containing information about all previous calculations.

During the testing of the model and the "IT INVESTMENT" module, the correctness of the results was established.

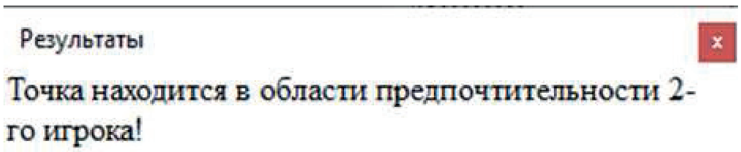


Figure 13: Example of generating the DSS output if the point on the graph is in the preference area of the 2nd player

In order to facilitate the work of experts or PLRs, the main form of the "IT INVESTMENT" module has a "History" button, which allows you to view the history of evaluation procedures.

DSS "DSS Invect 2020" allows you to collect expert information, and after processing with the help of integrated modules, present the results in a convenient format, either graphic or text. In addition, the DSS "DSS Invect 2020" allows you to find the optimal solution not only on the basis of the traditional and well-proven MAI, but also on the basis of new approaches based on bilinear quality games. The evaluation of the investment project performed with the help of "DSS Invect 2020" allows not only to determine the most attractive projects for the investor from the set of possible ones, but also to analyze and predict investment strategies more deeply, taking into account the multi-factor approach, using the apparatus of bilinear differential equations.

4 Conclusion

A software product has been developed – the decision support system "DSS Invect 2020". The DSS is designed to make recommendations to the DSS during the selection of rational financial strategies by investors. "DSS Invect 2020" allows you to evaluate the attractiveness of investment projects in the field of digitalization of enterprises. The "DSS Invect 2020" is implemented on a modular basis. This makes it possible to supplement the DSS with other modules. In particular, the DSS module "IT INVESTMENT" has been implemented, which allows to reduce the discrepancies between forecasting data and the real return on investment in enterprises in the field of digitalization.

References

[1] Karacı A., "Performance Comparison of Managed C# and Delphi Prism in Visual Studio and Unmanaged Delphi 2009 and C++ Builder 2009 Languages", *International Journal of Computer Applications* 26 (1) (2011): 9-15.

[2] Romashkina G.F., Tatarova G.G., "Koefficient konkordacii v analize sociologicheskih dannyh [Concordance coefficient in the analysis of sociological data]", *Sociologiya: metodologiya, metody, matematicheskoe modelirovanie* 20 (4M) (2005): 131-158.

[3] Kache F., & Seuring S. "Challenges and opportunities of digital information at the intersection of Big Data Analytics and supply chain management", *International Journal of Operations & Production Management* 37 (1) (2017): 10-36.

[4] Smit H.T., & Trigeorgis L. *Flexibility and games in strategic investment* (2015).

[5] Arasteh A. "Considering the investment decisions with real options games approach", *Renewable and Sustainable Energy Reviews* 72 (2017): 1282-1294.

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RECOGNITION OF A PSYCHOEMOTIONAL STATE BASED ON VIDEO SURVEILLANCE: REVIEW

Recognition of human activity based on video is currently one of the most active areas of research in the field of computer vision. Various studies show that the effectiveness of recognizing actions depends on the type of functions performed and how these actions are expressed. At the same time, determining the psychoemotional state of controlled persons, including students and schoolchildren, is an urgent socially significant problem. The pace of technology development and the growing interest of foreign and domestic specialists indicate that automation of the detection of psychoemotional reactions is an urgent and popular area of research. The main purpose of this study is to review various literary sources and study methods of image recognition, artificial intelligence technology as a means of determining the psychoemotional state observed based on video surveillance. The article discusses modern types of emotional artificial intelligence that allow a computer to recognize and interpret human emotions and respond to them. The camera reads a person's state, and the neural network processes data to detect emotions. The pattern recognition methods discussed in this article can solve many problems, and you can find a method that matches the programming language used.
Key words: Computer vision, image processing, cluster analysis, signal processing, neural network, filtering.

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Бейнебақылау негізінде психоэмоционалды жағдайды тану: шолу

Бейне негізінде адамның іс-әрекетін тану қазіргі уақытта компьютерлік көру саласындағы зерттеулердің ең белсенді бағыттарының бірі болып табылады. Әр түрлі зерттеулер іс-әрекеттерді танудың тиімділігі шығарылатын функциялардың түріне және осы әрекеттердің қалай көрсетілетініне байланысты екенін көрсетеді. Сонымен қатар бақылаудағы адамдардың, соның ішінде студенттер мен оқушылардың психоэмоционалды жағдайын анықтау әлеуметтік маңызы бар өзекті мәселе болып табылады. Технологияның даму қарқыны, шетелдік және отандық мамандардың қызығушылығының артуы психоэмоционалды реакцияларды анықтауды автоматтандыру зерттеудің өзекті және сұранысқа ие бағыты екенін көрсетеді. Бұл зерттеудің негізгі мақсаты бейне бақылау негізінде байқалатын психоэмоционалды жағдайды анықтау құралы ретінде әртүрлі әдеби дереккөздерге шолу және бейнелерді тану әдістерін, жасанды интеллект технологиясын зерттеу болып табылады. Мақалада компьютерге адамның эмоциясын тануға және түсіндіруге және оларға жауап беруге мүмкіндік беретін эмоционалды жасанды интеллекттің қазіргі түрлері қарастырылған. Камера адамның жағдайын оқиды, ал нейрондық желі эмоцияны анықтау үшін деректерді өңдейді. Мақалада қарастырылған үлгіні тану әдістері көптеген мәселелерді шешуге қабілетті және сіз қолданылған бағдарламалау тіліне сәйкес әдісті таба аласыз.
Түйін сөздер: Компьютерлік көру, суретті өңдеу, кластерлік талдау, сигналдарды өңдеу, нейрондық желі, сүзгілеу.

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Распознавание психоэмоционального состояния на основе видеонаблюдения: обзор

Распознавание действий человека на основе видео в настоящее время является одним из самых активных направлений исследований в области компьютерного зрения. Различные исследования показывают, что эффективность распознавания действий в значительной степени зависит от типов извлекаемых функций и способа выражения этих действий. В то же время определение психоэмоционального состояния наблюдаемых по видео, в том числе студентов и учеников является актуальной задачей, имеющая социальную значимость. Темпы развития технологий и повышенный интерес зарубежных и отечественных специалистов показывают, что автоматизация определения психоэмоциональных реакций – актуальное и востребованное направление исследований. Основная цель данного исследования заключается в обзоре литературы и методов распознавания образов, технологии искусственного интеллекта, как средства определения психоэмоционального состояния наблюдаемых на основе видео наблюдений. В статье рассматриваются существующие виды эмоционального искусственного интеллекта, позволяющие компьютеру распознавать и интерпретировать человеческие эмоции и реагировать на них. Камера, считывают состояние человека, а нейросеть обрабатывает данные, чтобы определить эмоцию. Рассмотренные в статье методики распознавания образов способны решать широчайший спектр задач, и можно найти подходящий метод под используемый язык программирования.

Ключевые слова: Компьютерное зрение, обработка изображения, кластерный анализ, обработка сигналов, нейросеть, фильтрация.

1 Introduction

The task of automatic recognition of a psychoemotional state is interdisciplinary and constantly attracts researchers of different specialties-mathematicians, programmers, psychologists, and physiologists. The progress of modern automated control systems, security systems, emergency notification systems, etc. depends on its solution. The solution of this problem is of great scientific importance for all areas of basic human research and information technology. In recent years, interest in the analysis of video surveillance with the use of artificial intelligence technology, considered as the most convenient objective way to identify emotions, the emotional state of a person, has clearly increased. first, you need to control the child's emotions every second of learning. Recognition of human behavior and generalization of the image are complex tasks of computer vision.

Tragedies happen at school, so you need to use video surveillance technology to detect emotions at school. To identify young people who may pose a potential danger, it is not enough to catch them when they use weapons at school in a critical situation. Thus, it is impossible to avoid possible harm. Therefore, to influence young people and prevent this critical situation, it is necessary to identify them in advance.

Let us define the concept of "Emotion". Through emotions, we experience a person's attitude to something at a certain moment. An emotion is a higher level of emotional response than an emotional tone in evolutionary development. This is a reaction of adaptation to a specific situation, and not a reaction to a specific stimulus [1]. An emotional tone can evoke emotions, but emotions can arise when evaluating a situation. Differentiated assessment of emotions in different situations. The emotional tone gives a broader assessment, and the emotion more subtly reflects the meaning of a particular situation. This is not only a method of assessing the upcoming situation, but also a mechanism for early and adequate preparation through the mobilization of mental and physical energy. Like an emotional tone, it serves as a mechanism for predicting the significance of a particular situation for a person and a mechanism for consolidating positive and negative experiences (attempts at positive or negative reinforcement).

There are two main ways to analyze emotions:

1. Contact method. When a person is put on a device that reads his pulse, the electrical impulses of the body. These technologies allow you to determine emotions, stress levels.

2. Contactless. Emotion analysis is based on video and audio recordings. The computer learns facial expressions, gestures, eye movement, voice, and speech.

To train the neural network, they collect a sample of data, manually mark the change in the emotional state of a person. The program learns patterns and understands which signs relate to which emotions.

There is a database of images. Such images can be people's faces, images, different emotional states, different objects of the three-dimensional world. The task is to search for the desired image in the database. Moreover, the task can be formulated both for finding an accurate image, and as close as possible to the specified one. An important task is the selection of methods and algorithms for image processing that can provide a high-quality solution to the problem. Processing technologies in this case depend on many parameters: the size of the database, the size of the image, the image quality, the brightness and contrast parameters of the images, the presence of the background, the angles of the objects' location [1-2].

The purpose of this work is to review the literature and methods of image recognition, artificial intelligence technology, as a means of determining the psychoemotional state of the observed based on video observations.

2 Review of literature and methods

Many scientists from near and far abroad are engaged in this task. Russian scientists Zaboleeva-Zotova A.V., Orlova Yu.A., Fedorov O.S. are engaged in determining the emotional state of a person by his movements using neural networks. In their work, they wrote that a review of the developments of Ugobe, Machine Perception, NeuroSky, VibraImage, Sound Intelligence, TruMedia, FaceReader, Federal Express, ERIC laboratories, Affective Computing Research, the Massachusetts Institute of Technology (MIT), the Fraunhofer Institute, the Universities of Geneva and Tokyo, Microsoft, Apple, Sony shows that now there is no system that fully implements the analysis of all means of transmitting human emotional reactions [2]. The authors analyzed the theories of emotions, considered the fundamental and modern works of scientists. Rosaliev V.L. and Zaboleeva-Zolotova A.V. [7] consider information about human body movements presented in the bvh format as time series. In their opinion, to analyze information about body movements, it is necessary to formalize the activity of human body movements [7]. Activity is expressed in the number of body movements of a person: the fewer body movements, and as a result, the fewer changes in the file channels, the lower the activity value.

In the human body, there are certain vibrations by which you can judge the psychoemotional state of a person and get information from them. To assess the psychoemotional state of a person as a form of the mental state of a person, the most effective methods are those that do not depend on the opinion of the subject. This method is a vibration imaging system developed by scientists.

The system, according to Nguyen D.K. and Yuzhakov M.M., is designed to register, analyze, and study the psychological and emotional state of a person, quantify the emotional level, polygraph, psychophysiological diagnostics, and remote identification of potential dangers. This system allows you to intuitively and automatically assess the psychophysiological state of a person based on the vestibular-emotional reflex, using software visualization of the vibration halo obtained by processing the components of the amplitude-frequency vibration image.

Belarusian scientists Brilyuk, D.I., Starovoitov, V.V. [8] in the article "Neural network methods of image recognition" shows the architecture of a multi-layer neural network (MNS) consisting of sequentially connected layers, where the neuron of each layer relates to all the neurons of the previous layer by its inputs, and the outputs are connected to the neurons of the next layer. From this article, we can learn that a neural network with two decision layers can approximate any multidimensional function with any accuracy. Neural networks that have a single layer of solutions can form a linear distribution surface, which significantly reduces the range of problems they solve. A neural network with a nonlinear activation function and two layers of solutions allows you to create any convex region in the solution space and has three layers of solutions of any complexity-regions, including non-convex regions. In addition, the multi-layer neural network does not lose its ability to generalize. According to the author, the multilayer neural network is prepared using the inverse error distribution algorithm, which is a method of gradient descent in the weight space to reduce the overall error of the network. In this case, the error (more precisely, the adjusted weight value) propagates from the input to the output through the weight of the neuron associated with the opposite. [8]. Also, in their article [8] Brilyuk D.I, Starovoitov V.V. talk about the use of a multi-layer neural network for direct classification of images - the input is either the image itself in some form, or a set of previously extracted key characteristics of the image, the output neuron with maximum activity indicates belonging to the recognized class (Figure 1).

Rosaliev V. L., Bobkov A. S., Fedorov O. S. [9] in the article "The use of neural networks and granulation in the construction of an automated system for determining the emotional reaction of a person", developed an approach to delineating the human body. To increase the effectiveness of identifying the emotional state of a person, information about the body is divided into two zones: upper and lower. The upper zone contains the nodes of the body related to the arms, head, neck, and back. The lower zone includes nodes related to the legs and pelvis [9]. According to the authors, the initially recognized motion is fed to the input of the data preprocessing subsystem. Here, the data is filtered, and data blocks are allocated that describe the static and dynamic zones of the body. After the data preprocessing subsystem, the separated blocks are processed separately by the corresponding subsystems: the analysis of poses and body movements. The information obtained after the analysis is combined and a common result is formed in the results comparison subsystem. Then the data

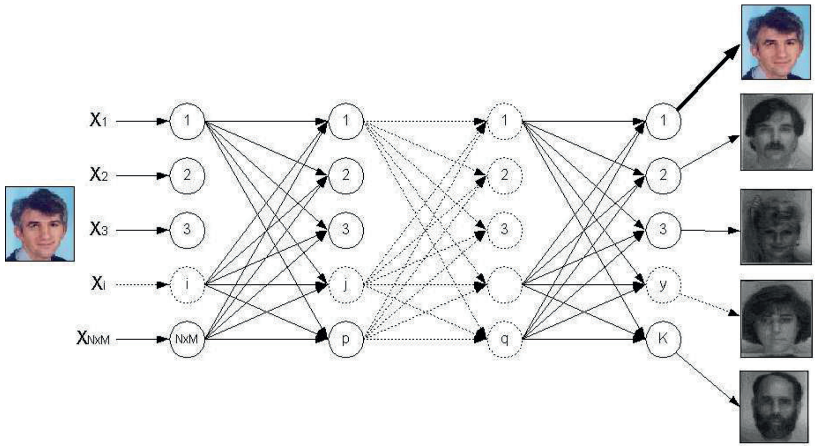

Figure 1: Multi-layer neural network for image classification. The neuron with the highest activity (here the first one) indicates that it belongs to the recognized class.

is supplemented with expert knowledge, which is stored in databases of characteristic poses and body movements and is sent to the user.

Gorokhovatsky V. A. [5] in the article "Studying the properties of clustering methods in relation to sets of characteristic features of images" , he proposed recognition methods based on the transformation of the space of structural features by clustering and applying the cluster characteristics of the base of reference images. The advantages, according to the authors of structural methods in image analysis, are the representation of visual objects in the form of a set of independent structural elements, which allows the recognition process to make effective decisions on subsets of elements and provide the necessary resistance to interference in the analyzed image. Gorokhovatsky V. A. believes that the cluster transformation of the space of structural features reduces the amount of computational costs, and hundreds of times improves the speed of recognition while maintaining the desired efficiency.

The problem of aggression, including children's aggression, has long attracted the attention of psychologists. One of the oldest psychological theories of aggressiveness – psychoanalytic-explains aggression as one of the types of natural instincts in a person who, during socialization, finds acceptable channels for exit and ways of expression in society. In this case, the increased aggressiveness of the child is explained by the insufficient strength of the "I" , which controls the behavior, as well as the insufficient development of psychological mechanisms for using aggressive energy "for peaceful purposes" . Increases the aggressiveness of the child's acquired knowledge of himself as "unmanageable" , "angry" , "brawler" , etc. Until now, the psychoanalytic interpretation of aggressiveness remains one of the most popular.

3 Recognition of emotions by actions

The system for monitoring the psychoemotional state of a person (VibraImage) is designed to detect aggressive and potentially dangerous people, using contactless remote scanning to ensure security at airports, schools, and other protected facilities [1]. The system allows you to automatically calculate and visually evaluate the psychoemotional state of a person using software processing of a television signal and its conversion into a vibration image. The psychoemotional state of a person is characterized based on patented algorithms for analyzing the vestibular-emotional reflex and macro-movements. The Natal project from Microsoft. The software provides full 3-dimensional recognition of body movements, facial expressions, and voice. Natal can recognize emotions by voice and face.

In the literature [10], many methods of classifying human actions by visual observation have been proposed. The recognition of human actions based on video surveillance data can be viewed from different angles. Two-dimensional analysis of the recognition of actions when a person interacts with a computer requires a good exposure of the human body to obtain video. The proposed methods, according to Omar Elharrouss et al. [10], can be divided into three categories: motion-based methods, appearance-based methods, and space-time-based methods. Motion-based methods consist of calculating parametric and general optical fluxes before comparing the results with motion patterns. For appearance-based methods, the motion history of the images is extracted for comparison with the active shape models. In spatiotemporal approaches, spatiotemporal characteristics with learning outcomes are used in the spatiotemporal domain.

Dasari R, Chen CW [11] classified human actions by tracking the trajectories of the human body CDVS. El-Masry et al. in their work [12] begin by selecting areas (patches) in a video that can be described as actions.They then generate blocks containing the detected movements, and each of these blocks is assigned a discrimination score. To recognize actions, the authors applied a clustering method to each block to identify different actions.

In [3], Tejero-de-Pablos et al. propose a new video generalization method that uses player actions as a hint to determine the main points of the original video. The deep neural network approach is used to isolate two types of action-related functions and to classify video segments into interesting and uninteresting parts. The proposed method is applicable to any sports in which games consist of a sequence of actions. Elharrouss O et al. [4] proposed a video summation method based on motion detection. Sensor noise (capture and digitization noise) and changes in scene illumination are the biggest limitations of background subtraction methods. To solve these problems, this paper presents an approach based on a combination of background subtraction and structure-texture-noise decomposition.

In the article Kamiński Ł, Maćkowiak S, Domański M (2017) [6], a new method for recognizing human activity is presented. The proposed solution uses a single stationary camera to detect common human actions, such as: waving hands, walking, running, etc. Unlike other methods that use different types of characteristic point descriptors to describe human postures, the proposed solution uses a CDVS descriptor that is part of the MPEG-7 standard. This allows you to efficiently compute a compact handle in the camera.

In an article by authors Xu C, He J, and Zhang X (2017) [18], the authors believe that recognizing human movement-related actions using wearable sensors could potentially enable the use of various useful everyday applications. So far, most studies consider this as a separate problem of mathematical classification without considering the physical nature of human movements. Consequently, they suffer from data dependencies and face a dimensionality problem and a mismatch problem, and their models never become readable. Wang L, Huynh DQ, Koniusz P (2019) [19] in their paper analyze and compare 10 recent Kinect-based algorithms for both cross-action recognition and cross-action recognition using six control data sets.

El-Henawy et al. [13] proposed a method for recognizing human actions using fast HOG3D and Smith-Waterman partial matching of the shape of each frame. First, the foreground of the video subsequence is extracted from the input stream. The keyframes of the current subsequence are then mixed before extracting the contour of the resulting frame. To classify the HOG3D functions, the author uses a nonlinear SVM decision tree. For video surveillance systems, the human body is in some cases incomplete, which is a problem for recognizing human actions [10]. This method recognizes multiple actions of multiple actors.

JIN CB et al [14] in their paper presented a sub-action descriptor for detailed action detection. The auxiliary action descriptor consists of three levels: pose, movement, and gesture level. The three levels give three categories of sub-actions for a single action aimed at solving the problem of representation. The proposed action detection model simultaneously localizes and recognizes the actions of multiple people in video surveillance using time-based appearance functions with multiple CNNs.

Another paper by AKULA A et al [15] demonstrates the use of IR cameras in the AAL region and discusses their effectiveness in recognizing human actions (HAR). Special attention is paid to one of the most responsible actions – falling. In the work, a set of IR image data was generated, consisting of 6 classes of actions - walking, standing, sitting on a chair, sitting on a chair with a table in front, falling on a table in front and falling / lying on the ground. To achieve reliable recognition of actions, the authors developed a controlled convolutional neural network (CNN) architecture.

4 Results and their comparison

If we consider using these methods to detect and recognize multiple human actions, we can compare the accuracy of the results of these methods. In this case, for comparison, I took the three methods described in [14,15,16].

The degree of accuracy for modern methods related to recognizing multiple human actions that use the same category of representation datasets:

Methods	Accuracy %
Jin CB and drl. 2017 [14]	83. 5% ICVL
Akula A et al. 2018 [15]	87.44% (infrared video)
CS-HOG-TD map [13]	97.5%

Bloisi DD et al. [16] presented the results of the multimodal background modeling method, which allows us to create a reliable initial background model, even without a clear framework. They used background subtraction as a method of detecting moving objects in image sequences.

In Elharrouss O et al [17], the authors presented a result on block background initialization using the sum of absolute differences (SAD), as well as simulations using block entropy estimation with low computational costs, which makes them suitable for an embedded platform. The author's method is effective for background generation and detection of moving objects.

To consolidate the visualized results, I used various metrics:

– total number of erroneous pixels (OKOP);

– signal-to-noise ratio (SSSH);

– multilevel structural similarity index (MISS);

– color image quality indicator (PKCI).

These indicators for the methods [13, 16, 17] are presented in Table 1.

Table 1: Performance results of the compared background modeling methods

Метод	OKOP	SSSH	MISS	PKCI
[16] IMBS-MT	9.8507	22.7278	0.9090	34.0028
[17] SAD	2.5313	27.7099	0.9892	39.6381
[13] HOG	1.1586	36.3866	0.9964	43.2006

5 Conclusion and discussion

The proposed methods suffer from some limitations, especially in the case of occlusion and very crowded scenes. Indeed, it is quite difficult to detect and recognize multiple human bodies in a crowded environment. One solution to overcome these limitations is to apply some pre-processing and learning process to deal with various occlusions and crowded scenarios. Also, it's worth mentioning that, to the best of our knowledge, there are no publicly available datasets for detecting people in very busy scenes that could be used in the training process.

Neural networks allow you to develop and modify the image recognition system due to the possibility of combining and interconnecting different networks. The effectiveness of cluster recognition significantly depends on the formed system of clusters in the application database of classes set by standards. The transition to the vector-cluster view significantly increases the speed of recognition by simplifying processing [5].

Life is rapidly being informatized, and the introduction of systems for monitoring psychoemotionality by video order transmitted over 3G networks or via the Internet (i.e., without significant delays and in good quality) will allow the introduction of systems that can monitor the life of society, revealing aggression, anger [1].

The results of the action recognition may be affected by changes in the lighting. Detecting any lighting changes in the scene can be useful to improve the video captured before the action is recognized. The proposed methods for detecting changes in illumination allow you to apply an improvement in video quality only if there is any change.

Building a complex recognition system requires a preliminary analysis of all available information about the objects being studied. The variety and complexity of the tasks of recognizing the psychoemotional state do not make it possible to implement one universal approach to the solution. Thus, existing image recognition techniques are able to solve a wide range of tasks, and depending on the limiting factors (development budget, speed and accuracy of image recognition), you can find a suitable method for the programming language used.

References

[1] Orlova Yu.A., *Review of modern automated systems for recognizing human emotional reactions* (Volgograd, Russia: VolgSTU, 2011).

[2] Zaboleeva-Zotova A.V., Orlova Yu. A., Rozaliev V. L., Fedorov O. S., "Determination of the emotional state of a person by his movements using neural networks", *Volgograd, Russia: Bulletin of the RSUPS* 2 (2011).

[3] Tejero-de-Pablos A., Nakashima Y., Sato T., Yokoya N., Linna M., Rahtu E., "Summarization of user-generated sports video by using deep action recognition features", *IEEE Trans Multimed* 20 (8) (2018): 2000-2011.

[4] Elharrouss O., Al-Maadeed N., Al-Maadeed S., "Video Summarization based on Motion Detection for Surveillance Systems", *In 2019 15th International Wireless Communications & Mobile Computing Conference (IWCMC), IEEE* (2019): 366-371.

[5] Gorokhovatsky V.A., *Studying the properties of clustering methods in relation to sets of characteristic features of images* (Kiev, Ukraine, 2016).

[6] Kamiński Ł., Maćkowiak S., Domański M., "Human activity recognition using standard descriptors of MPEG CDVS", *International Conference on Multimedia & Expo Workshops. IEEE* (2017).

[7] Rozaliev V.L., Zaboleeva-Zolotova A.V., "Modeling of the emotional state of a person based on hybrid methods", *Volgograd, Russia: VOLGGU Izv.* (2008).

[8] Brilyuk D.I., Starovoitov V.V., "Neural network methods of image recognition", *Minsk, Belarus: ITK NANB* (2013).

[9] Rozaliev V.L., Bobkov A.S., Fedorov O.S., "The use of neural networks and granulation in the construction of an automated system for determining the emotional reaction of a person", *Volgograd, Russia: Izvestia* (2010).

[10] Omar Elharrouss, Noor Almaadeed, Somaya Al-Maadeed, Ahmed Bouridane, Azeddine Beghdadi, "A combined multiple action recognition and summarization for surveillance video sequences", *Appl Intell* 51 (2020): 690-712. https://doi.org/10.1007/s10489-020-01823-z

[11] Dasari R., Chen CW., "MPEG CDVS Feature Trajectories for Action Recognition in Videos", *Conference on Multimedia Information Processing and Retrieval. IEEE* (2018).

[12] El-Masry M., Fakhr M.W., Salem M.A.-M., "Action recognition by discriminative EdgeBoxes", *IET Comput. Vis.* 12 (4) (2017): 443-452.

[13] El-Henawy I., Ahmed K., Mahmoud H., "Action recognition using fast HOG3D of integral videos and Smith-Waterman partial matching", *IET Imag. Process* 12 (6) (2018): 896-908.

[14] Jin C-B., Shengzhe L.I., Hakil et KIM, "Real-Time Action Detection in Video Surveillance using Sub-Action Descriptor with Multi-CNN", *arXiv*: 1710.03383. (2017)

[15] Akula A., Shah AxK., et Ghosh R., "Deep learning approach for human action recognition in infrared images", *Cogn. Syst. Res.* 50 (2018): 146-154.

[16] Bloisi D.D., Pennisi A., Iocchi L., "Parallel multi-modal background modeling", *Pattern Recogn. Lett.* 96 (2017): 45-54.

[17] Elharrouss O., Abbad A., Moujahid D., et al., "Moving object detection zone using a block-based background model", *IET Comput. Vis.* 12 (1) (2017): 86-94.

[18] Xu C., He J., Zhang X., "DFSA: A classification capability quantification method for human action recognition", *IEEE SmartWorld, Ubiquitous Intelligence & Computing, Advanced & Trusted Computed, Scalable Computing & Communications, Cloud & Big Data Computing, Internet of People and Smart City Innovation, SmartWorld/ SCALCOM/ UIC/ ATC/ CBDCom/ IOP/ SCI* (2017)

[19] Wang L., Huynh D.Q., Koniusz P., "A comparative review of recent Kinect-based action recognition algorithms", *IEEE Trans Image Process* 29 (2019): 15-28.

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REQUIREMENTS FOR SYMMETRIC BLOCK ENCRYPTION ALGORITHMS DEVELOPED FOR SOFTWARE AND HARDWARE IMPLEMENTATION

The hardware and software cryptographic information protection facility is one of the most important components of comprehensive information security in information and communication systems and computer networks. This article outlines and systematizes the basic requirements for modern cryptographic information protection facilities (CIPFs), and describes the stages of developing a symmetric block encryption algorithm. Based on the basic requirements for CIPFs, criteria for evaluating the developed cryptographic encryption algorithms were determined. The possibilities and necessary limitations of the types of cryptographic transformations (primitives) in the software and hardware implementation of the developed symmetric block encryption algorithm are considered and defined. On the basis of the developed encryption algorithm, SDTB Granit plans to implement a model of a hardware-software complex for off-line (linear) data encryption, taking into account all the listed requirements and characteristics. The article presents a new version of the "AL01" encryption algorithm, which is guided by the basic requirements for creating symmetric block ciphers.
Key words: symmetric block algorithm, cryptographic information protection facility, software, hardware-software, and hardware implementation of encryption algorithms, cryptographic primitives.

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Бағдарламалық-аппараттық іске асыру үшін әзірленетін симметриялық блоктық шифрлау алгоритмдеріне қойылатын талаптар

Ақпараттық-коммуникациялық жүйелері мен компьютерлік желілерде ақпараттық қауіпсіздігін кешенді түрде қамтамасыз етуде маңызды құрамдас бөліктің бірі болып ақпаратты криптографиялық қорғаудың бағдарламалық-аппараттық құралдары саналады. Бұл еңбекте ақпаратты криптографиялық тұрғыда қорғаудың заманауи құралдарына (АКҚҚ) қойылатын негізгі талаптар баяндалған және жүйелендірілен, симметриялық блоктық шифрлау алгоритмін әзірлеу кезеңдері көрсетілген. АКҚҚ-ға қойылатын негізгі талаптарын ескере отырып, әзірленетін криптографиялық шифрлау алгоритмдерін бағалау критерийлері талқыланған. Әзірленетін симметриялы блоктық шифрлау алгоритмін бағдарламалық-аппараттық іске асыруда криптографиялық түрлендірулердің (примитивтердің) түрлерінің ерекшеліктері, мүмкіндіктері мен қажетті шектеулері қарастырылып, нақтыланды.
Әзірленіп жатқан шифрлау алгоритмі "Гранит" АКТБ базасында көрсетілген талаптар мен сипаттамаларды ескере отырып, деректерді алдын ала (желілік) шифрлауға арналған бағдарламалық-аппараттық кешен жасап шығару жоспарлануда. Симметриялы блокты шифрлар жасаудың негізгі талаптарын ескере отырып, мақаланың мақалада "AL01" шифрлау алгоритмінің жаңа нұсқасы көрсетілген.
Түйін сөздер: симметриялы блоктық алгоритмі, ақпаратты криптографиялық қорғаудың құралдары, бағдарламалық, бағдарламалы-аппараттық және аппараттық іске асырудың шифрлау алгоритмдері, криптографиялық примитивтер.

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Требования к симметричным блочным алгоритмам шифрования, разрабатываемым для программно-аппаратной реализации

Программно-аппаратное средство криптографической защиты информации является одним из важнейших составляющих комплексного обеспечения информационной безопасности в инфокоммуникационных системах и компьютерных сетях. В данной статье изложены и систематизированы основные требования к современным средствам криптографической защиты информации (СКЗИ), приведены этапы разработки симметричного блочного алгоритма шифрования. Исходя из основных требований, предъявляемых к СКЗИ, определены критерии оценки разрабатываемых криптографических алгоритмов шифрования. Рассмотрены и определены возможности и необходимые ограничения видов криптографических преобразований (примитивов) при программно-аппаратной реализации разрабатываемого симметричного блочного алгоритма шифрования. На основе разрабатываемого алгоритма шифрования в СКТБ "Гранит" планируется реализовать модель программно-аппаратного комплекса для предварительного (линейного) шифрования данных с учетом всех перечисленных требований и характеристик. В статье представлена новая версия алгоритма шифрования "AL01" учитывающая все основные требования, предъявляемые при создании симметричных блочных шифров.
Ключевые слова: симметричный блочный алгоритм, средство криптографической защиты информации, программная, программно-аппаратная и аппаратная реализации алгоритмов шифрования, криптографические примитивы.

1 Introduction

Today cryptography is a relatively new research area at the intersection of mathematics and computer science and is related to information security. Its role and application to ensure confidentiality and integrity when processing information in modern information and telecommunication systems have become an integral part of the life of modern society [1-3].

To ensure the inaccessibility of the semantic part of confidential data, three types of encryption are used: hardware, hardware-software, and software encryptions. The main differences between them, in addition to their cost and maintenance costs, are the encryption methods and the level of data security. In practice, these three criteria are decisive when choosing the type of encryption for users. Currently, the most affordable of them are considered to be software following by hardware-software, and the most expensive are hardware cryptographic information protection facilities (CIPFs). Although the cost of hardware CIPFs is significantly higher than the cost of the other CIPFs, the difference in monetary terms is incomparable with a significant increase in the level of data security. The advantage of hardware CIPFs is the guaranteed invariability of the algorithm itself, whereas, with software or hardware-software implementation, the encryption algorithm can be deliberately changed. Also, a hardware encryptor, through technical protection aimed at increasing the security of technological data, excludes any interference in the encryption process.

Hardware CIPFs have the ability to load encryption keys into the cryptoprocessor, bypassing the PC's RAM, while in software CIPFs the cryptographic keys are stored in the PC's RAM. Based on hardware CIPFs, it is possible to implement control and restriction of access to a PC. Also, hardware encryption takes the load off the PC's central processor [3-5]. Therefore, it is relevant to develop and implement an encryption algorithm in the form of a hardware-software complex for off-line encryption. Off-line encryption includes a procedure for preliminary cryptographic transformation of the transmitted information, after which the method of its transmission is determined. This method of ensuring the confidentiality of information is used in the case of an e-mail when information is encrypted in advance and then transmitted through established communication channels.

2 Materials and methods

2.1 Determination of basic requirements for cryptographic information protection facilities

The use of cryptographic systems to ensure data security is determined by technological capabilities, depending on the conditions and scope of the tasks being solved. This often leads to a revision (tightening) of requirements for cryptographic strength, flexibility, and performance of encryption, as well as cost acceptability for hardware implementation. New areas of information security, in particular the development of modern types of encryption in the USA (NIST competition), Europe (NESSIE competition), and other developed countries, show the recognition of the scientific and technological value and the growing role of encryption. When developing ciphers, the use of new cryptographic primitives is promising for technological applications. The cryptographic primitives traditionally used in the building symmetric block cryptographic systems are substitutions and permutations, arithmetic and algebraic transformations, and also other additional operations that well implement data diffusion.

2.1.1 Principles and stages of development of a symmetric block encryption algorithm

At present, along with asymmetric systems, symmetric block encryption algorithms are considered to be one of the main cryptographic means of ensuring the necessary level of secrecy when storing, processing, and transmitting information in modern information and communication systems. When developing this type of ciphers, the following general requirements are imposed, such as in [6-8]:

1. Providing the required level of cryptographic strength;
2. Simplicity, availability, and cost of software, hardware-software, or hardware implementation.
3. Providing high performance and flexibility in software, software-hardware, or hardware implementation;

However, the above requirements are rather controversial and contradictory, since increasing the cryptographic strength requires additional rounds of encryption, and this, as a rule, affects the decrease in the encryption speed. Nevertheless, international algorithms such as DES, AES [9], NESSIE [10-12], CryptRec [13], and others indicate the possibility of achieving the most suitable indicators for practical application.

In practice, the length of a message encrypted with a symmetric block algorithm is significantly larger than the length of the encryption key (the entropy of messages exceeds the entropy of the key). In this case, practical criteria of strength are considered, that is, the inadmissibility of the successful implementation of a crypto attack on algorithms in the conditions of modern computing potential (taking into account the upcoming optimistic development of computer technology and nanotechnology) in a certain period of time.

As known, the strength of an algorithm depends on the complexity of the implementation of a crypto attack on a symmetric block algorithm. Proceeding from this, as its indicators, as a rule, the following are used [6]:

1. Time complexity is the mathematical expectation of time (guaranteed security time) required to carry out a cryptographic attack using available and expected in the short term computing means;

2. Space complexity is the dependence of the amount of memory occupied on the size of the input data when performing cryptographic analysis of the algorithm;

3. The minimum number of encrypted and corresponding plaintexts (the minimum number of pairs) required to implement the attack.

The primary assessment of cryptographic strength is made in relation to known brute-force attacks, such as complete key enumeration, dictionary attacks, and others. Provided that the required level of strength to such types of attacks is ensured, an assessment of strength to analytical attacks and statistical methods of cryptanalysis is carried out.

For modern symmetric block encryption algorithms, it is proposed to apply the following conclusions as a criterion for evaluating cryptographic strength to analytical attacks [8]:

1. The total number of cipher/plaintext pairs required to perform a cryptanalysis attack exceeds the number of all possible cipher/plaintext pairs;

2. The complexity of a brute force attack should be less than the complexity of any other analytical attack.

To assess the complexity of an analytical attack according to the second criterion - the complexity of a brute-force attack - the following quantitative indicators are considered:

- Time criterion is the required minimum number of encryption operations (actions) for the implementation of an analytical attack (no fewer than with a complete key enumeration);

- The minimum memory size required to store intermediate and additional results during an analytical attack (not fewer than when implementing a dictionary attack on a full cipher).

A cipher is protected from a cryptanalysis attack if the above brute force attack indicators are lower than the cryptanalysis attack complexity.

Thus, when designing modern symmetric block encryption algorithms, the following basic requirements should be taken into account:

1. Cryptographic strength against brute-force attacks (by the minimum memory size required to store intermediate and additional results and by the time criterion);

2. Lack of ways to find and solve a system of algebraic (Boolean) equations describing the relationship between the plaintext, ciphertext, and encryption key;

3. Practical inaccessibility of implementing well-known analytical crypto attacks on the algorithm or their high computational complexity;

4. Availability of the optimal "margin of cryptographic strength" of the algorithm, taking into consideration the dynamics of innovative technological development of the industry;

5. The cryptographic strength of a simplified version of the algorithm, where some operations of the basic version were replaced or simplified by more elementary ones;

6. Statistical properties of the output ciphertext (cryptographic bitstream, cryptograms) should be close to the properties of a truly random sequence.

Considering the above, during the design and analysis of block symmetric algorithms, it is proposed to take into account the following approaches [6]:

- "Conservative design". This means that you should only use repeatedly verified, that is, reliable, designs, cryptographic primitives, and methods that provide guaranteed security;
- Strength against all known cryptanalysis attacks;
- Accessible and simple structure and design principles of the algorithm;
- Formation of the optimal "security margin" of the algorithm, the possibility of further secure use of the algorithm, given the emergence of potential cryptanalytic attacks and/or the development of computer technology;
- Protecting against all known vulnerabilities of the algorithm;
- The lack of a set of "bad" keys;
- The prevalence of strength over the performance of computer technology;
- Ensuring high performance close to the best world indicators.

When designing hardware-software and hardware CIPFs, it is also necessary to take into account the possibility of technical attacks based on changes in the temperature regime of the device, the appearance of ionizing radiation, measuring the consumed currents, side electromagnetic radiation, etc.

2.1.2 Design steps for block ciphers

The block cipher development process includes the following steps [8]:

1. Determination of the application area. At this step, the class of the cryptosystem is pre-selected, then the necessary list of requirements for its main initial parameters is drawn up.
2. Selection of key length. Currently, the actual key length is at least 128 bits, but in lightweight cryptography, a key with a length of 64 bits can be used, where timing is of particular interest.
3. Key generation scheme, consisting of a key installation system and a key management system. The key setting system includes algorithms and methods of generation, as well as a key verification rule. In turn, the key management system determines the further use, the order of transfer, storage, change, copying, and recovery, as well as the guaranteed destruction of keys.
4. Selection of basic cryptographic primitives and development of a cryptographic scheme. Clarification of the main approaches in the development of the algorithm, types and classes of symmetric block cryptosystems, the cost of their hardware, and hardware-software implementations. Optimal timing parameters are determined.
5. Assessment of technological resources for software, hardware-software, and hardware implementation of the encryption algorithm. Development work on encryptors and their software designs.
6. Estimation of the encryption speed. Various approaches are evaluated to obtain the required encryption speed for software, hardware-software, and hardware implementations. If the estimates derived at steps 5 and 6 do not correspond to the values obtained at step 1, then, taking into account the results obtained at the current step, it is proposed to return to step 4.

7. Evaluation of the cryptographic strength of the cipher using cryptanalysis and other methods.

8. Changing the algorithm with due account for the intermediate cryptanalysis of the cipher. Considering the results obtained at step 7, the main nodes of the cryptosystem are optimized to increase the efficiency of countering various types of crypto attacks. Until an acceptable degree of strength is obtained, this step can be repeated several times.

9. Conducting a basic, more detailed analysis of the cipher. If significant weaknesses are found, it is necessary to repeat step 8, if it is impossible to obtain a positive result, return to step 4.

10. At this step, the development of the encryption algorithm code or its implementation on the FPGA is carried out. The encrypted data is checked using statistical tests and special experiments to verify the completeness and reliability of the theoretical analysis. It is recommended to use well-known sets of statistical tests. All these tests are considered within the framework of mathematical statistics.

2.2 Requirements for symmetric block encryption algorithms developed for software and hardware implementation

The use of CIPFs is regulated by various normative and normative-methodological documents. The applied CIPFs should perform the following main functions:

- Key generation and key information management;
- Encryption of information in accordance with the standard ST RK 1073-2007;
- Identification and authentication of user connection and access to work (use of additional means of protection of CIPFs, such as tokens, iButton, etc.)

The developed symmetric block encryption algorithm, first of all, must comply with the standard ST RK 1073-2007 "Cryptographic information protection facilities. General technical requirements".

Given the experience gained during the previous R&D, it is advisable to use symmetric block ciphers previously developed in the information security laboratory, such as Qamal, EM, AL01, and others [14-17], which have been comprehensively studied and repeatedly tested against generally accepted requirements for symmetric block encryption algorithms.

The preliminary structure of the developed algorithm includes a variant of a substitution-permutation network (SP-network). This network uses an iterative transformation consisting of a substitution layer (nonlinear elements), a linear (mixing) layer, and a key adding layer. This design, due to the transformation of the entire data block at each iteration, provides a much faster mixing of the input vector in comparison with the Feistel network [18-19].

a) Building substitution boxes (nonlinear transformation nodes)

The use of pseudo-randomly generated lookup tables (S-boxes), selected according to the criteria of resistance to various cryptanalysis methods and the degree of nonlinearity of Boolean equations describing transformations, is considered relevant. These tables do not have an explicit mathematical structure inherent in known ciphers, allowing to reveal any algebraic dependencies between input, key, and output. This approach provides resistance to algebraic attacks.

To reduce the possibility of building and the probability of finding the correct solution to a system of algebraic equations, it is practiced to apply several substitutions at once.

When forming S-boxes of the developed cipher, the following basic requirements are imposed [5, 6]:

- Pseudo-random generation (ensuring a low probability of obtaining strict mathematical relationships between input and output data);
- Minimization of the maximum value of the probability of passing the difference through the substitution;
- Minimization of the maximum value of the probability of linear approximation of a substitution;
- Non-linear substitution order.

Next, consider the cryptographic primitives used in the design of the algorithm.

b) Linear transformation block

Currently, a comprehensively studied MDS transform is used as a linear transformation unit to obtain the best diffusion characteristics. It is recommended to use a 64-bit MDS code, which provides the best "avalanche effect" after the second round of encryption, which gives the best performance.

c) Subkey generation scheme

At this stage, it is necessary to exclude as much as possible the following disadvantages of known encryption algorithms:

- The ability to fully or partially recover the master key based on a known one or more subkeys;
- A fairly simple mathematical link connecting the subkeys. This flaw may allow for a "related-key attack";
- Acceptance of the master key itself as the first subkey;
- Using a design other than the round function in the generation circuit;
- Lack of the "avalanche effect" property on the subkeys, i.e. low effect of changes to the master key or subkey on other subkeys;
- differences in the computational complexity of generating subkeys for encryption and decryption.

Requirements key generating schemes of the developed algorithm:

- There should be a non-linear relationship between each bit of a subkey and the master key. In other words, there should be an "avalanche effect" when a change in one input bit of the master key should lead to changes in about 50
- High resistance - the ability to resist cryptanalysis methods and all known types of crypto attacks;
- Lack of a class of weak keys that make the round key generation scheme vulnerable to some attacks aimed at detecting any weaknesses in cryptographic properties;
- Recovery of a master key based on one or several subkeys should require high computational complexity;
- The simplicity of the software, hardware-software, and hardware implementation through the use of cyclic transformation;
- The computational complexity of generating subkeys should not exceed the computational complexity of the encryption algorithm itself;
- Generation of subkeys should be carried out in any direction of encryption.

Summing up, based on the results of the analysis of the requirements for encryption algorithms given in Section 1, for the designed encryption algorithm, the following conclusions and requirements can be formulated:

- The encryption algorithm should be symmetric and block based;
- The algorithm should provide a high level of cryptographic strength;
- The algorithm should be efficiently modified for all security levels and meet the relevant requirements as much as possible;
- The strength of the algorithm should not be based on the unavailability of the algorithm, i.e. the algorithm should be publicly available;
- The algorithm should provide for implementation on different platforms. It should be possible to effectively implement it in software on modern universal microprocessors and work on integrated microcontrollers and other small and medium-sized processors while maintaining the optimal ratio between size, cost, and performance;
- The algorithm should be easily adapted on specially designed encryption equipment, and its implementation in the form of a CIPF should be energy-efficient;
- The algorithm should be simple and easy to write program code to prevent errors that allow engineering analysis;
- The algorithm should use simple and low-resource operations that are effective on microcontrollers and microprocessors (XOR, addition, substitution boxes, cyclic shift);
- The algorithm should not have a set of "weak" keys, which facilitates its cryptanalysis, i.e. accept any random string of a certain length as the master key.

3 Results and discussions

The conditions and requirements for any symmetric block cipher algorithms are listed above. Let's take a look at a new encryption algorithm that has been designed with these requirements in mind.

The encryption algorithm AL03 is one of the modifications of the symmetric block encryption algorithm AL01 [8 - 9]. The developed algorithm uses blocks and keys with a length of 128 bits. The structure of the cipher is a variant of a substitution-permutation network (SP-network), and encryption is performed in R1=24 rounds. The encryption process includes key addition using the exclusive or (XOR) operation, substitution S-boxes, and the bitwise shift operation (Figure 1).

Each round of the encryption process consists of 3 transformations, called Step-1, Step-2, and Step-3, and ends with the addition of the round keys modulo 2 with the results obtained.

In the **Step-1** transformation, an input block of 16 bytes (128 bits) is divided into 4 subblocks equal to 4 bytes (32 bits): $a_0^0, a_1^0, a_2^0, a_3^0, a_0^1, a_1^1, a_2^1, a_3^1, a_0^2, a_1^2, a_2^2, a_3^2, a_0^3, a_1^3, a_2^3, a_3^3$ where the superscript represents the subblock number and the subscript represents the byte number in the subblock. The internal transformation of subblocks is performed as follows: 1st and 2nd bytes, 2nd and 3rd bytes, and 3rd and 4th bytes are summed modulo 2 and form new 1st, 2nd, and 3rd bytes, respectively. Further, the new 1st byte, after passing through the nonlinear transformation using the first S-box, is added modulo 2 to the 4th byte and produces a new 4th byte. The same transformation is performed for the remaining 3 subblocks, that is $b_i^j = a_i^j \oplus a_{i+1}^j, b_3^j = S_1({b_1}^j) \oplus a_3^j, i = 0, 1, 2; j = 0, 1, 2, 3.$

Thus, all the results of the addition operation pass through the substitution S-box S_1.

$$c_i^j = S_1(b_i^j), i = 0, 1, 2, 3; j = 0, 1, 2, 3.$$

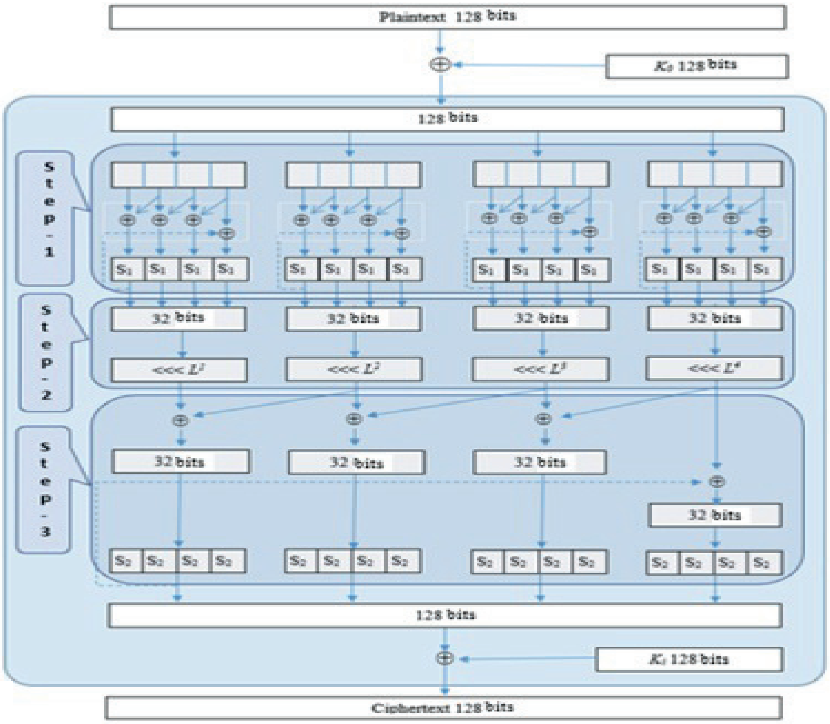

Figure 1: Scheme of the encryption algorithm AL03

Step-2. In each subblock, a concatenation operation is performed over its 4 bytes, followed by a left cyclic shift by a predetermined number of bits $L^j (L^j = 1, 3, 5, 3$ bits) for each subblock, respectively:

$$c_0^j \parallel c_1^j \parallel c_2^j \parallel c_3^j = (c_0^j \parallel c_1^j \parallel c_2^j || c_3^j) \lll L^j, j = 0, 1, 2, 3.$$

The **Step-3** transformation is carried out similarly to **Step-1**, except that the operations are performed not on adjacent bytes, but on neighboring subblocks. The corresponding values of the 1st and 2nd subblocks, the 2nd and 3rd subblocks, and the 3rd and 4th subblocks are summed modulo 2 and give the new values of the 1st, 2nd, and 3rd subblocks. The new values of the 4th subblock are determined as follows: the newly obtained values of the 1st subblock after the second S-box transformation are summed modulo 2 with the corresponding values of the 4th subblock:

$$d_i^j = c_i^j \oplus c_i^{j+1}, d_i^3 = S_2(d_i^0) \oplus c_i^3, i = 0, 1, 2, 3; j = 0, 1, 2.$$

Thus, all the obtained values of the 4 subblocks pass through the substitution S-box S_2.

$$e_i^j = S_2(d_i^j), i = 0, 1, 2, 3; j = 0, 1, 2, 3.$$

Each i-th round of encryption ends by adding modulo 2 values e_i^j to the round key K_i.

$S_1(256) = \{98, 233, 16, 142, 0, 40, 127, 30, 25, 76, 169, 130, 19, 72, 58, 93, 77, 235, 148, 162,$
$196, 150, 38, 232, 82, 152, 177, 164, 211, 101, 188, 245, 165, 145, 115, 13, 121, 9, 234, 214, 180,$

117, 26, 138, 147, 247, 33, 189, 183, 179, 32, 255, 161, 2, 172, 83, 218, 167, 21, 95, 201, 199, 80, 28, 157, 96, 109, 60, 74, 190, 113, 137, 85, 205, 84, 143, 66, 62, 206, 146, 181, 55, 12, 59, 31, 91, 90, 24, 14, 191, 51, 159, 228, 65, 248, 244, 231, 135, 194, 129, 213, 114, 79, 111, 184, 102, 122, 207, 208, 163, 134, 128, 151, 100, 254, 227, 20, 29, 223, 118, 220, 176, 230, 197, 212, 48, 89, 222, 52, 202, 1, 106, 105, 175, 149, 224, 210, 39, 170, 68, 41, 61, 8, 168, 192, 5, 249, 182, 241, 54, 78, 45, 174, 215, 246, 99, 171, 6, 63, 140, 132, 4, 187, 160, 64, 186, 226, 86, 11, 110, 250, 112, 203, 67, 124, 216, 35, 57, 155, 119, 158, 166, 87, 73, 120, 75, 252, 103, 56, 217, 97, 47, 42, 251, 50, 221, 242, 240, 116, 88, 156, 34, 123, 141, 198, 18, 10, 44, 236, 193, 71, 239, 7, 125, 154, 53, 136, 23, 219, 229, 3, 238, 69, 107, 43, 185, 36, 108, 126, 92, 15, 253, 37, 46, 104, 237, 131, 81, 94, 139, 209, 225, 144, 49, 27, 200, 133, 70, 173, 17, 204, 22, 153, 243, 178, 195};

$S_2(256)$ = {168, 176, 8, 107, 204, 99, 10, 223, 243, 160, 118, 180, 146, 179, 230, 30, 59, 244, 212, 219, 105, 120, 92, 201, 163, 152, 193, 101, 36, 56, 26, 161, 44, 254, 166, 88, 83, 123, 178, 188, 84, 15, 31, 97, 190, 224, 48, 220, 29, 213, 129, 35, 76, 183, 124, 198, 17, 137, 229, 240, 78, 3, 67, 5, 109, 226, 132, 227, 222, 169, 234, 66, 95, 39, 214, 111, 86, 187, 32, 162, 194, 139, 81, 207, 141, 127, 40, 100, 126, 114, 46, 74, 153, 155, 47, 16, 211, 175, 196, 165, 182, 38, 140, 37, 98, 80, 149, 199, 75, 195, 209, 27, 251, 102, 202, 77, 237, 54, 242, 250, 60, 128, 121, 6, 131, 151, 115, 116, 0, 173, 112, 154, 117, 90, 231, 68, 113, 192, 63, 110, 12, 241, 20, 73, 119, 238, 216, 135, 13, 171, 94, 125, 156, 91, 189, 14, 69, 43, 138, 11, 133, 185, 148, 157, 1, 247, 58, 174, 158, 239, 130, 205, 57, 64, 235, 24, 21, 19, 61, 218, 51, 50, 104, 34, 22, 253, 72, 45, 164, 236, 108, 184, 206, 215, 53, 28, 143, 203, 23, 167, 2, 25, 79, 89, 9, 49, 134, 87, 225, 106, 150, 41, 62, 52, 70, 197, 255, 252, 18, 103, 144, 7, 217, 221, 71, 159, 181, 200, 249, 210, 245, 142, 145, 93, 147, 233, 172, 42, 208, 65, 177, 82, 33, 248, 136, 191, 186, 122, 232, 246, 96, 228, 170, 4, 55, 85}.

The algorithm for generating round keys. An algorithm for generating round keys from the master key $K(k_0, k_1, k_2, \ldots, k_{15})$ with a length of 16 bytes is considered. We will use the master key K as the round key K_0. The total number of round keys (excluding K_0) is the same as the number of rounds R_2. The values of the round key $K_0(k_0, k_1, k_2, \ldots, k_15)$ are written as a 4x4 square matrix A:

$$A = \begin{pmatrix} k_0 & k_1 & k_2 & k_3 \\ k_4 & k_2 & k_6 & k_7 \\ k_8 & k_9 & k_{10} & k_{11} \\ k_{12} & k_{13} & k_{14} & k_{15} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix};$$

The algorithm for generating round keys, schematically shown in Figure 2, consists of **Stage-1**, **Stage-2**, and **Stage-3** transformations, which are described below. **Stage-1 transformation**

At this stage of the transformation, which consists of two steps, from a given matrix A, we will obtain a new matrix A of the same size.

Step 1. The intermediate values c_{ij} of the matrix A are determined from left to right, from top to bottom by adding modulo 2 all four elements of the ith row and three elements of the jth column, except for the c_{ij} itself.

Step 2. At this step, the obtained value c_{ij} passes through the substitution S-box and is written to the same place as the new value of the matrix A.

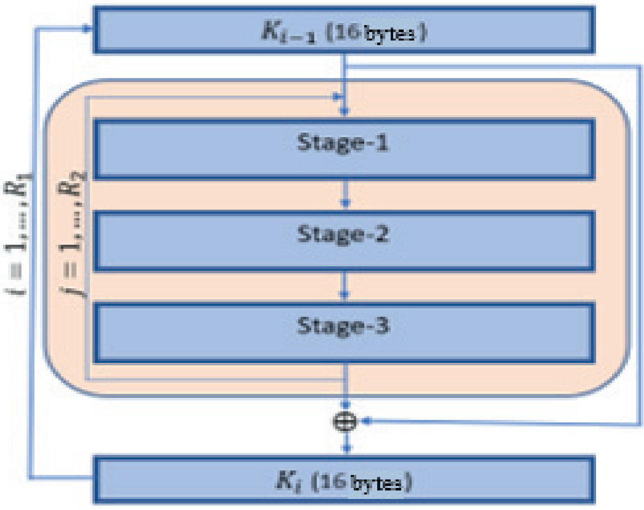

Figure 2: Scheme of the algorithm generating round keys

The Stage-1 transformation, consisting of the 1st and 2nd steps, can be written as:

$$m_{ij} = \oplus\sum_{k=0}^{3} a_{ik} \oplus (\oplus\sum_{k=0}^{3} a_{kj});$$

$$a_{ij} = S_1(m_{ij}); i = 0,..,3; j = 0,..,3;$$

where c_{ij} is the intermediate value of the matrix A, S is the substitution using the S-box S_1.

Stage-2 transformation

This transformation consists of only one operation. The elements of matrix A, obtained in Stage-1, are written in the form of a one-dimensional array $(a_{00}, a_{01}, a_{02}, a_{03}, a_{10}, a_{11}, a_{12}, a_{13}, a_{20}, a_{21}, a_{22}, a_{23}, a_{30}, a_{31}, a_{32}, a_{33})$. Further, all elements of the array are treated as bytes and are concatenated using the concatenation operator: $a_{00} \parallel a_{01} \parallel a_{02} \parallel a_{03} \parallel a_{10} \parallel a_{11} \parallel a_{12} \parallel a_{13} \parallel a_{20} \parallel a_{21} \parallel a_{22} \parallel a_{23} \parallel a_{30} \parallel a_{31} \parallel a_{32} \parallel a_{33}$.

Next, a cyclic left shift is performed by 1 bit. The obtained 16-byte result is taken as the new values of the 4x4 matrix A whose elements are located from left to right, from top to bottom.

Stage-3 transformation

This transformation is similar to the Stage-1 transformation. Here, too, the matrix Aundergoes a two-step transformation. The only difference is that the elements of the matrix are selected from right to left, from bottom to top. After transforming Stage-3, we get a new matrix A of the same size. We write this transformation, consisting of the 1st and 2nd steps, similar to the previous one:

$$m_{ij} = \oplus\sum_{k=0}^{3} a_{ik} \oplus (\oplus\sum_{k=0}^{3} a_{kj});$$

$$a_{ij} = S_2(m_{ij}); i = 0,..,3; j = 0,..,3;$$

where c_{ij} is the intermediate value of the matrix A, S is the substitution using the S-box S_2. Finally, the result of one round is the values obtained after the Stage-3 transformation. The Stage-1, Stage-2б and Stage-3 transformations are repeated $R_2 = 8$ times, and then the resulting 16-byte value is added to the previous round key $K_{i-1} modulo 2(XOR)$ and finally we get the next round key K_i, where $i = 1, \ldots, R_1$.

We have modified the AL01 symmetric encryption algorithm. The structure of the algorithm uses XOR operations and substitution S-boxes. In one round of encryption, only 56 operations are performed, therefore the complete algorithm is performed in 1,344 operations. Thus, with an effective software implementation, the encryption rate of this algorithm will be close to or even exceed the encryption rate of other known algorithms.

One of the tasks of the research work is to create a hardware prototype of the encryption algorithm developed in the LIS and adapted for this purpose. The creation of such a device is possible at the Special Design and Technology Bureau (SDTB) Granit LLP. The SDTB is engaged in the development and production of radar stations and single-board minicomputers. It has developed a device that is designed to emulate the operating mode of the CIPF. On its basis, SKTB Granit, within the framework of this project, plans to implement a software and hardware complex for preliminary encryption of information with the characteristics required for the project. The developed encryption algorithm will be implemented on a device that is a single-board minicomputer for CIPFs. Taking into account the peculiarities of the operation of the file encryption software, the device will be made in a truncated hardware configuration without connecting a keyboard and monitor.

4 Conclusion

The paper provides an overview of the basic requirements for modern CIPFs, studies the stages of developing a symmetric block encryption algorithm, systematizes the basic requirements for the developed cryptographic information security facilities, on the basis of which criteria for evaluating the developed cryptographic information protection systems are determined. The technical task for the creation of a software and hardware complex that implements the developed encryption algorithm has been clarified and agreed upon.

The research being carried out is new. The development and study of the encryption algorithm designed to ensure the protection of information, including classified information constituting a state secret, by the cryptographic transformation of data presented in the form of files, its implementation in the form of a hardware-software complex is aimed at the creation and development of domestic cryptographic means to ensure information security.

In addition, this article presents a new structure of the symmetric block cipher algorithm, its key generation algorithm that meets the basic requirements and recommendations of block cipher algorithms. Currently, work is underway to analyze the cryptographic strength of the algorithm using statistical and algebraic approaches.

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References

[1] Ivanov M.A., Chugunkov I.V.,"Teoriya, primenenie i ocenka kachestva generatorov psevdosluchajnyh posledovatelnostej [Theory, application, and quality assessment of pseudorandom sequence generators]", Moscow, K-OBRAZ (2003), 136, [in Russian].

[2] Babenko L.K., Ischukova E.A.,"Sovremennye algoritmy blochnogo shifrovaniya i metody ih analiza [Modern block cipher algorithms and methods for their analysis]", Moscow, Helios ARV (2006), p. 376, [in Russian].

[3] Schneier B., "Applied Cryptography,: Protocols, Algorithms, and Source Code in C". 2-nd ed.; John Wiley & Sons, Inc. (1996): 118.

[4] Ivanov M.A., "Kriptograficheskie metody zashchity informacii v komp'yuternyh sistemah i setyah [Cryptographic methods of information security in computer systems and networks]", Moscow, K-OBRAZ (2001), 368, [in Russian].

[5] Gorbenko I. D., Dolgov V., Oleynikov R. V., Ruzhentsev V. I., Mikhaylenko, M. S., Gorbenko, Y. I.,"Razrabotka trebovanij i princip proektirovaniya perspektivnogo simmetrichnogo blochnogo algoritma shifrovaniya [Development of requirements and design principle perspective symmetrical block encryption algorithm]", Izvestiya yuzhnogo federal'nogo universiteta. Tekhnicheskie nauki no. 1, V. 76 (2007), 183-189, [in Russian].

[6] Gorbenko I. D., Dolgov V., Oleynikov R. V., Ruzhentsev V. I., Mikhaylenko, M. S., Gorbenko, Y. I.,"Razrabotka trebovaniy i printsip proektirovaniya perspektivnogo simmetrichnogo blochnogo algoritma shifrovaniya [Development of requirements and design principle perspective symmetrical block encryption algorithm] Izvestiya YUFU. Tekhnicheskie nauki. no. 1. URL: https://cyberleninka.ru/article/n/razrabotka-trebovaniy-i-printsip- proektirovaniya- perspektivnogo-simmetrichnogo-blochnogo-algoritma-shifrovaniya (2007), (3.11.2020), [in Russian].

[7] Apparatnoe shifrovanie dlya PK [Hardware encryption for PC].Press center Company Active, 2013. URL: https://www.aktiv-company.ru/press-center/publication/2003-04-10.html (23.11.2020), [in Russian].

[8] Znaenko N.S., Kapitanchuk V.V., Petrishchev I.O., Shubovich V.G.,"Nekotorye kriterii ocenki kachestva algoritmov shifrovaniya [Some criteria for evaluating the quality of encryption algorithms]", NovaInfo.Ru. Tekhnicheskie nauki no. 59 (2017) URL: https://novainfo.ru/article/11211 (23.11.2020), [in Russian].

[9] AES discussion forum:http://aes.nist.gov.

[10] New European Schemes for Signatures, Integrity, and Encryption NESSIE:URL: http://cryptonessie.org.

[11] Final report of European project number IST-1999-12324, named New European Schemes for Signatures, Integrity, and Encryption. Springer-Verlag, Berlin Heidelberg NewYork, etc. (2004).

[12] NESSIE public report D20.NESSIE Security Report. URL: http://cryptonessie.org.

[13] URL: http://cryptrec.org/ Cryptography Research and Evaluation Committees.

[14] Report on research work "Development of software and firmware for cryptographic protection of information during its transmission and storage in info-communication systems and general-purpose networks 2018, State registration no. 0118PK01064..

[15] Biyashev R.G., Smolarsh A., Algazy K.T., Khompysh A., "Encryption algorithm "QAMAL NPNS"using non-positional polynomial notations Journal of Mathematics, Mechanics, and Computer Science, Bulletin of KazNU no. 1 (105), Almaty (2020), 198-207. .

[16] Nursulu Kapalova, Ardabek Khompysh, Muslum Arici, Kunbolat Algazy., A block encryption algorithm based on exponentiation transform // Cogent Engineering. - 2020. - No. 7 (1788292). - P. 1-12 // https://doi.org/10.1080/23311916.2020.1788292.

[17] Kapalova N.A., Haumen A.,"Simmetrichnyj blochnyj algoritm shifrovaniya dannyh "VS-2"[BC-2 symmetric block algorithm for data encryption]", Bezopasnye informacionnye tekhnologii". Sbornik trudov Desyatoj mezhdunarodnoj nauch-no-tekhnicheskoj konferencii, Moscow, Bauman MSTU, 2019, 161-166, [in Russian].

[18] Jonathan K., Yehuda L.,"Introduction to Modern Cryptography CRC PRESS, London-New York- Washington, (2007), 160.

[19] Panasenko S.P., "Algoritmy shifrovaniya [Encryption algorithms]", Special reference book, Saint Petersburg, BHV-Petersburg (2009), 576, [in Russian].

Список литературы

[1] Иванов М. А., Чугунков И. В.,Теория, применение и оценка качества генераторов псевдослучайных последовательностей. - М.: К-ОБРАЗ. - 2003. - 136 с.

[2] Бабенко Л.К., Ищукова Е.А., Современные алгоритмы блочного шифрования и методы их анализа. - М.: Гелиос АРВ, 2006. - 376 с.

[3] Schneier B., "Applied Cryptography.: Protocols, Algorithms, and Source Code in C". 2-nd ed.; John Wiley & Sons, Inc.(1996): 118.

[4] Иванов М.А., Криптографические методы защиты информации в компьютерных системах и сетях. - М.:К-ОБРАЗ, 2001. - 368 с.

[5] Горбенко И.Д., Долгов В., Олейников Р.В., Руженцев В.И., Михайленко М.С., Горбенко Ю.И., Разработка требований и принцип проектирования перспективного симметричного блочного алгоритма шифрования // Известия южного федерального университета. Технические науки. - 2007. - Т. 76, № 1. - С. 183-189.

[6] Горбенко И.Д., Долгов И.В., Олейников Р.В., Руженцев В.И., Михайленко М.С., Горбенко Ю.И., Разработка требований и принцип проектирования перспективного симметричного блочного алгоритма шифрования // Известия ЮФУ. Технические науки. 2007. №1. URL: https://cyberleninka.ru/article/n/razrabotka-trebovaniy-i-printsip-proektirovaniya-perspektivnogo-simmetrichnogo-blochnogo-algoritma-shifrovaniya (3.11.2020).

[7] Аппаратное шифрование для ПК // Пресс-центр "Компания "Актив". 2013. URL: https://www.aktiv-company.ru/press-center/publication/2003-04-10.html (23.11.2020).

[8] Знаенко Н.С., Капитанчук В.В., Петрищев И.О., Шубович В.Г., Некоторые критерии оценки качества алгоритмов шифрования // NovaInfo.Ru. Технические науки. 2017. №59. URL: https://novainfo.ru/article/11211 (23.11.2020).

[9] AES discussion forum: http://aes.nist.gov.

[10] New European Schemes for Signatures, Integrity, and Encryption NESSIE: URL: http://cryptonessie.org.

[11] Final report of European project number IST-1999-12324, named New European Schemes for Signatures, Integrity, and Encryption. Springer-Verlag, Berlin Heidelberg NewYork, etc. 2004.

[12] NESSIE public report D20. NESSIE Security Report. URL: http://cryptonessie.org.

[13] URL: http://cryptrec.org/ Cryptography Research and Evaluation Committees.

[14] Report on research work "Development of software and firmware for cryptographic protection of information during its transmission and storage in info-communication systems and general-purpose networks 2018, State registration no. 0118PK01064.

[15] Biyashev R.G., Smolarsh A., Algazy K.T., Khompysh A., "Encryption algorithm "QAMAL NPNS"using non-positional polynomial notations Journal of Mathematics, Mechanics, and Computer Science, Bulletin of KazNU no. 1 (105), Almaty (2020), pp. 198-207.

[16] Nursulu Kapalova, Ardabek Khompysh, Muslum Arici, Kunbolat Algazy., A block encryption algorithm based on exponentiation transform // Cogent Engineering. - 2020. - No. 7 (1788292). - P. 1-12 // https://doi.org/10.1080/23311916.2020.1788292.

[17] Капалова Н.А., Хаумен А., Симметричный блочный алгоритм шифрования данных "BC-2"// "Безопасные информационные технологии". Сборник трудов Десятой международной научно-технической конференции - М.: МГТУ им. Н.Э. Баумана, 2019. - С. 161-166.

[18] Jonathan K., Yehuda L., "Introduction to Modern Cryptography CRC PRESS, London-New York- Washington, (2007), 160.

[19] Панасенко С.П., Алгоритмы шифрования. Специальный справочник. - СПб.: БХВ-Петербург, 2009. - 576 с.

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DEVELOPMENT OF THE INFORMATION AND ANALYTICAL SYSTEM OF THE UNIVERSITY

Today, the introduction of analytical systems into the university management circuit has become a necessary and priority task of higher educational institutions. It was led by the desire of the university management to understand the nature of data to improve the quality of educational services and decision-making. This article discusses the analysis of information systems, comparison of business intelligence platforms, and approaches to designing information and analytical system within the university as one of the key elements of the university's information infrastructure.

The presented work describes the data architecture of the corporate information systems of al-Farabi Kazakh National University, designing and implementing an information and analytical system at the university and on the Microsoft Power BI cloud business analysis platform. This system integrates all the disparate data of the university's corporate information systems and transactional data sources. Furthermore, the logic of data extraction, transformation, implementation of visual reporting in Power BI, and the model of role-based access to them are also described.

Key words: data, information and analytical system, visualization, data management, university.

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Университеттің ақпараттық-талдау жүйесін әзірлеу

Бүгінгі таңда университетті басқару контурына аналитикалық жүйелерді енгізу жоғары оқу орындарының қажетті және басым міндеті болып табылады. Бұан университет басшылығының білім беру қызметтерінің сапасын жақсарту және шешім қабылдау мақсатында деректердің табиғатын түсінуге деген ұмтылысы себеп болды. Бұл мақалада ақпараттық жүйелерді талдау, бизнес-талдау платформаларын салыстыру және университеттің ақпараттық инфрақұрылымының негізгі элементтерінің бірі ретінде университет ішіндегі ақпараттық-аналитикалық жүйені жобалау тәсілдері қарастырылады.

Ұсынылған жұмыста Қазақ ұлттық университетінің корпоративтік ақпараттық жүйелерінің архитектурасы, ақпараттық-талдау жүйесін Microsoft Power BI бизнес-талдаудың бұлтты платформасында жобалау және іске асыру тәсілі сипатталған. Бұл жүйе университеттің корпоративтік ақпараттық жүйелері мен транзакциялық деректер көздерінің барлық шашыраңқы деректерін біріктіреді. Сондай-ақ, деректерді алу, түрлендіру, Power BI қолдана отырып, визуалды есеп беру нысанын енгізу логикасы және оларға рөлдік қол жетімділік моделі келтірілген.

Түйін сөздер:деректер, ақпараттық-аналитикалық жүйе, визуализация, деректерді басқару, университет.

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Разработка информационно-аналитической системы университета

Сегодня внедрение аналитических систем в контур управления университетом стало необходимой и приоритетной задачей высших учебных заведений. К этому привело желание руководства университета понимать природу данных, с целью совершенствования качества образовательных услуг и принятия решений. В данной статье рассматривается анализ информационных систем, сравнение платформ бизнес-аналитики и подходы проектирования информационно-аналитической системы внутри университета, как один из ключевых элементов информационной инфраструктуры ВУЗа.

В представленной работе описана архитектура данных корпоративных информационных систем Казахского Национального университета им. аль-Фараби, подход проектирования и реализации информационно-аналитической системы в ВУЗе, на облачной платформе бизнес-анализа Microsoft Power BI. Данная система интегрирует все разрозненные данные корпоративных информационных систем университета и транзакционных источников данных. Также изложена логика извлечения, преобразования данных, реализация формы визуальной отчетности в Power BI и модель ролевого доступа к ним.

Ключевые слова: данные, информационно-аналитическая система, визуализация, управление данными, университет.

1 Introduction

The manager needs to operate with all data flows in the information space of the organization to make managerial decisions. Sources of data for information flows of space are corporate information systems, each of which automates key business processes and, often, does not have consolidated data. There is a need for a qualitative improvement of information and analytical support in educational institutions, including educational processes at all levels.

The complex task of managing a university, according to the authors, is to improve the quality of the scientific and educational process, which requires a systematic and timely analysis of comprehensive and reliable information about the state of the university's activities for decision-making. The solution to this problem is possible by introducing modern information technologies into the management process of the university and constantly improving them. Therefore, higher educational institutions are constantly looking for effective ways to manage scientific and educa-tional activities (GCD) in connection with the university's information infrastructure being developed.

Information infrastructure is a set of solutions of its own and local developments and forms the information space of the university. The information and analytical system (IAS) is a modern, highly effective tool to support the adoption of strategic, tactical, and operational management decisions based on the visual and prompt provision of the entire necessary set of data to users respon-sible for analyzing the state of affairs and making management decisions. The main purpose of the IAS is the dynamic presentation and multidimensional analysis of historical and current data, trend analysis, modeling, and forecasting of the results of various management decisions.

The purpose of developing an IAS in the contour of a corporate information system (CIS) on the example of the al-Farabi Kazakh National University: to create an aggregating system for extracting data from various sources of CIS, transforming them, and uploading them to storage in order to build an operational and intelligent data analysis for their effective perception by consumers. For the implementation of IAS, methods and models were used, such as programming technologies, design of information systems, database theory, statistics, data mining.

The complexity of information and analytical systems affects the entire management vertical of the university: corporate reporting, financial and economic planning, and strategic planning. An information-analytical system is a platform in which databases (MS SQL, MySQL, etc.) of disparate information systems of an organization's information infrastructure and transactional data sources are integrated. This data integration accumulates at the cloud storage layer. The cloud architecture of the platform allows you to connect various intelligent data mining services like Microsoft Azure Learning, Analysis Services, and Google Analytics for the building purpose. Using a systematic approach and applying methods and models of economic and mathematical modeling and data mining, it is possible to build visualization services and predictive analytics. With the Power BI service, you can securely publish reports to your organization and set up automatic data refresh to keep all users up to date. The implementation of the IAS will help in the implementation of such tasks at the university as increasing the efficiency of university management and the processes of scientific and educational activities (GCD); improving the quality of educa-tion; assistance in the tasks of improving the qualifications of teaching staff and the effective use of pedagogical potential; identifying the reasons for academic failure; assessment of the effectiveness of educational and methodological complexes; control of the organization of the educational process.

2 Literature review

The analysis of Business Intelligence (BI) systems made it possible to conclude that the Microsoft Power BI platform is the most optimal for the education system since it quickly analyzes a large amount of data. It also allows you to visualize the results of processing an array of data with the ability to personalize; supports the possibility of joint work with data by placing data on an LDAP server; provides secure publication of dashboards and view them from any device with Internet access. In addition, a real-time reporting system enables group work with data and automatic data synchronization for all users.

Power BI is an online service developed by Microsoft for BI, which is a modern, highly effective tool for supporting the adoption of strategic, tactical, and operational management decisions based on the visual and prompt provision of the entire necessary set of data to users responsible for analyzing the state of the organization's activities and making management decisions [3]. Power BI allows you to create adhoc analytic reporting forms that are tailored to the needs of different categories of users and customized visualizations. There is also a mobile Power BI application available on various operating systems for continuous monitoring of the state of affairs and instant response to emergencies. It is also worth noting that Gartner has recognized Microsoft as a leader for fourteen years in a row in the Analysis and BI Platforms nomination in the Gartner Magic Quadrant for 2021 [4]. The implementation of an information and analytical system needs an environment that allows combining data from disparate systems, significantly reducing labor costs for preparing reports and improving the quality of information for strategic decision-making. This function was taken over by BI systems developed on the basis of cloud platforms, which are designed to receive operational information in real-time for making strategic decisions [1]. The following platforms are distinguished for processing and analyzing data and presenting them in the form of reports: Microsoft BI; Oracle BI; SAP Business Objects; QlikView; Qlik Sense; etc. Table 1 compares the functionality of Qlik, Tableau, and Power BI.

Table 1: *Comparison of Quik, Tableau and Power BI functionality.*

	Qlick	Tableau	Power BI
Availability	Qlik is a paid software that costs $30 per user per month, with different offers for corporate clients. You can also get a free trial period for using the program. For self-study, there is a free introductory course on their official website.	Tableau is also a $70 per user/month paid software, with different offers for corporate clients. You can also get a free trial period for using the program. For self-study, there are training videos on their official website. There is an adapta-tion of reports for different forms of presentation (you can download and print in .pdf format).	There are three versions of Power BI: Basic (free), Pro ($9.99 per user), and Premium ($20 per user and $4,995 per capacity (i.e., all employees in the organization)). Complete documentation of the product's features and functions is published on the official website. There is an adaptation of reports for different presentation forms (you can download and print in .pdf, .pptx formats, export tables/reports to Excel, CSV files).
ETL	Qlik has its data processor, which allows you to use "unprepared" data and transform it during loading.	Tableau allows you to combine several tables, get rid of empty or unwanted values, add calculated fields (amounts, number, and average).	Power BI fully implements internal ETL, which allows you to import, transform, and download data.
Relations be-tween tables	Qlik creates keys between tables based on common field names. The Data Model Viewer gives a complete understanding of the loaded tables and the relationship structure between them.	In Tableau, by default, the link is also built by fields with the same names, but this can also be edited in the section "Editing links."	In Power BI, relationships between tables are built by default by fields with the same names; you can also edit, de-lete, or create new relationships in the "Data Model" section.
Measurements and indicators	B In Qlik, the user initially sees a list of all loaded fields, can distribute all the fields into dimensions and indicators using the Master Items functionality.	Tableau automatically determines which fields are dimensions and indicators in all loaded tables (string in dimensions, numeric in indicators).	With Power BI Desktop, you can access SSAS multidimensional models, com-monly referred to as SSAS MD..

Visualization	Both platforms have a set of beautiful charts. Qlik also allows you to extend your visualization capabilities using extensions (ready-made extensions can be found on the Qlik Branch resource or using open standards such as HTML5.	Tableau offers a broader variety of out-of-the-box visualizations such as box-and-whiskers, Bullet-graphs, Gantt charts. Each chart is separated into a separate object, which makes it easier for the user to change the presentation of the diagram (just a table, a table-heatmap, a chart with one axis, a chart with two axes, etc.).	Power BI has many visualization libraries ranging from map points to nested graphs, histograms, and more. There is also a store of visual elements with modified visualization elements. It should be noted that the user can create his visual element and publish it in this store after receiving approval from the creators.
Drill-down	In Qlik, to fall into the lower level, it is necessary to select one (and only one) value of the upper level, after which the diagram is rebuilt in the context of the new dimension.	There is no Dill-down function in Tableau. Instead, hierarchies are created, the user, by clicking on "+" or "-" can collapse/expand the object, that is, work on the principle of a pivot table.	In Power BI, you navigated through data levels using up and down arrows. Moreover, a single arrow shows a transition to one level, and a double arrow - to the lowest level of hierarchies.
Data choice	In Qlik, the selection of a value on one tab / specific object will be valid for the entire document - on all tabs; the selection will be performed, you cannot bind a filter to a particular object.	In Tableau, the opposite is true. The choice is made locally - that is, only to a specific object. To transfer the selection between sheets, you need to configure Actions on each sheet, which will select the necessary values upon activation.	Power BI makes it easy to combine queries, create groups and data selections into a new or existing table.
Data relevance	Qlik has a configurable auto-update for datasets.	Tableau implements configurable auto-update of datasets.	There is a configurable auto-update of datasets in the cloud version of Power BI web according to the established schedule by configuring the gateway.

Data security and privacy	Qlik allows customization of access by individual data, users, sheets, streams, and applications.	Tableau allows you to customize access to specific reports, delineation of rights for actions in responses (viewing, edit-ing, updating, etc.).	Power BI can apply confidentiality labels to Power BI data using Microsoft Information Protection and customize roles, functionality, and access levels for workspaces, analytic reports, and datasets.
Integration	Qlik can import data from Data Warehouses, Relational Data-bases, Files, SAAS, SAP, etc.	In Tableau, data loading is implemented by connecting to Data Stores, Databases, files, etc.	Power BI is integrated with other products from Microsoft (Office 365, etc.), with R and Python. It is possible to connect almost any data source (streaming data, cloud services, Excel work-books, SQL Server databases, MySQL, MS Asure, and other third-party applications) and combine them. Power BI has a simple API for integrating into its own applications.
Convenience	Supports multiple platforms (web and mobile).	Work in Tableau is carried out in the web version.	Power BI is easy to use and has an intuitive interface. Supports multiple platforms (web, desktop, and mobile).

3 Materials and methods

Today, at al-Farabi Kazakh National University (further - KazNU), the information infrastructure uses such information systems as self-developed systems (IS "UNIVER" - a system for automating the educational process, IS "Science" - a system of accounting for research activities, system of indicative planning and rating based on IS "UNIVER" [5]); electronic document management system "Directum"; accounting and personnel accounting systems ("1C: Enterprise 8.2"); time tracking system "Perco 2.0"; the system of statistical reporting of the contact center "VoIPTime Contact Center." [6] These information systems were integrated with the cloud platform Power BI to build an IAS within the KIS KazNU (fig. 1).

The work of the information and analytical system (IAS) is based on the application of knowledge about the organization of business processes of the university, methods, and models of working with data, data analysis and monitoring, data interpretation, so that IAS users have the opportunity to offer objective solutions to emerging issues, and make more objective management solutions.

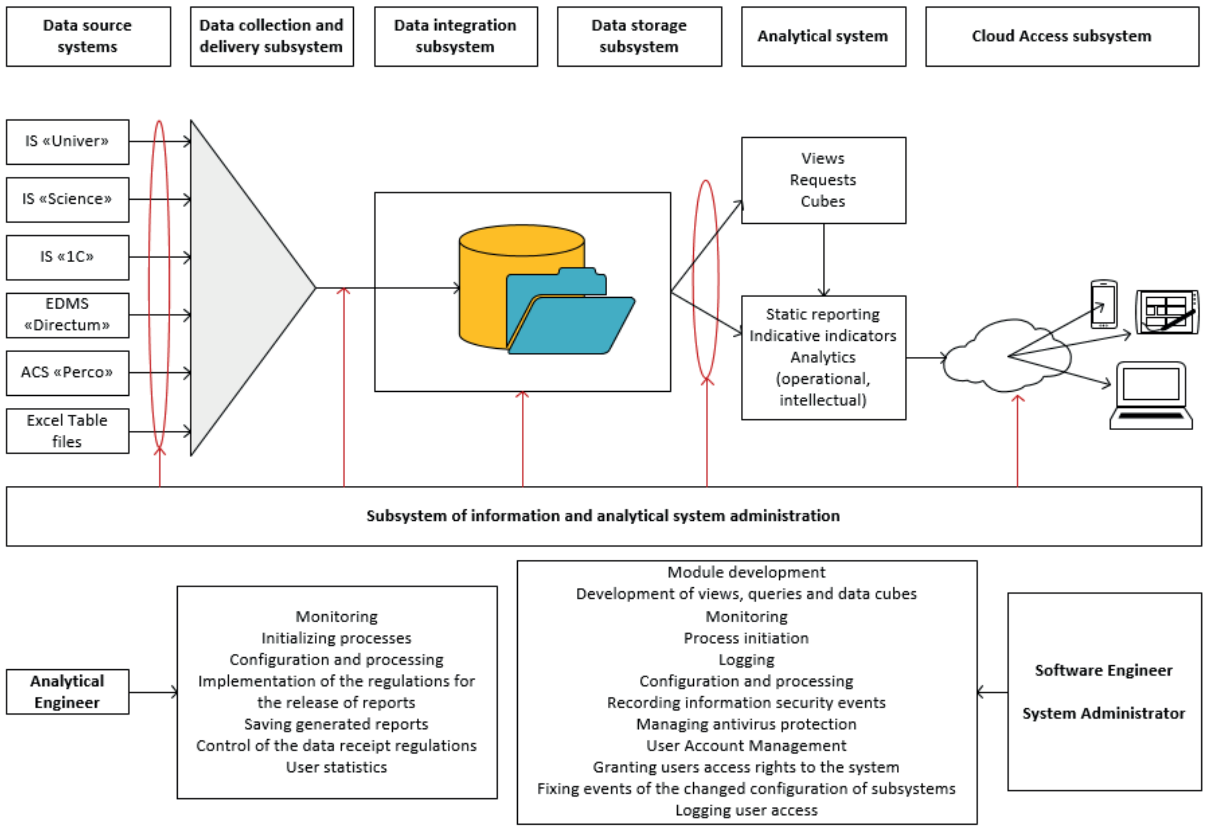

Figure 1: The architecture of IAS CIS of KazNU

The analysis of the statistical reporting of the corporate information system of KazNU showed that there are various categories of indicators at the university that are used in the formation of the reporting system. In this regard, the logic of the formation of analytical data was proposed, which easily inherits the hierarchical organization of data at the university (primary data, operational data, statistical indicators, statistical reports, indicative indicators, strategic indicators) and is easily programmed to build analytical indicators [5].

For the university leadership, there are clear benefits from the development and implementation of IAS in the university management loop, as a function: prompt response, operational monitoring, control of the University development strategy, reduction of labor costs for routine operations of automation of the university's statistical reporting. The result of using IAS tools is regulatory analytical reports focused on the needs of users of vari-ous categories, tools for interactive analysis of information, and rapid construction of reports by non-programmers using familiar concepts of the subject area.

4 The architecture and data for IAS

IAS implementation includes several essential stages: deployment of Microsoft SQL Server DBMS on a dedicated server/computer; building a data warehouse that receives information from various sources; Importing data into the Power BI business intelligence platform construction and visualization of reports; publishing a report to Power BI; installation of a gateway and automatic updates [7].

For the implementation of analytical reporting, it is necessary to provide for an appropriate data architecture of the University's CIS (fig. 2), which should consist of a data warehouse (DD), a data lake (OD), and a data mart (serving layer) [8, 9].

Data Warehouse is a system that collects data from various database sources within the university, developed and designed to prepare reports and business analysis to support management decisions. HD represents detailed data, aggregated data, and metadata (technical and business metadata).

Data Lake is a repository that stores a vast amount of raw data in its original form (RAW format). In the data lake: each unit of data has value now or in the future; data is stored for as long as needed; data is transformed when the need arises; data is interpreted following research goals. The data lake has a flat architecture.

Data lakes include structured data from relational databases (rows and columns), semi-structured data (CSV, log files, XML, JSON), unstructured data (emails, documents, pdf), and even binary data (video, audio, image files). The model of the data lake operation involves unloading data from a data source, preparing data, transforming and analyzing data, publishing data in data marts, and consuming data in the required reporting forms.

When working with the objects of the data architecture of the corporate information system of the university, the important processes are data reception, data storage, data quality, data audit, data exploration, data discovery [10].

Data sources of ML and CD KIS KazNU are:

- educational process management systems (data and log files of the IPK "UNIVER" and IS "Moodle");

- accounting systems (IS Science, 1C: Enterprise 8.2, Perco, online telephony systems);

- electronic document management systems and electronic archives: EDMS "Directum";

- local documents: file storage IPK "UNIVER" and IS "Science";

- external sources: excel files.

Fig. 2 shows the standard data architecture of KIS KazNU, represented by the following levels:

- level of data resources: DB IS, corporate IS, users, data of log files;

- ETL level: level of extraction, transformation, and loading of data;

- data level: o CD: cloud, which integrates data from the CIS database; o data marts: grouped thematic dataset; o OD: cloud, into which the data of log files, video, audio data are uploaded:

- the level of analytics - visual forms of statistical and analytical reporting (for educational, scientific, social, administrative, financial, etc.);

- access level for data mining: access at the level of analytical engineers and software engineers.

The collection of data into the university's information systems is carried out both in manual and automatic modes. IS "UNIVER" collects all categories of data on the educational process of the university: students, teachers, educational process.

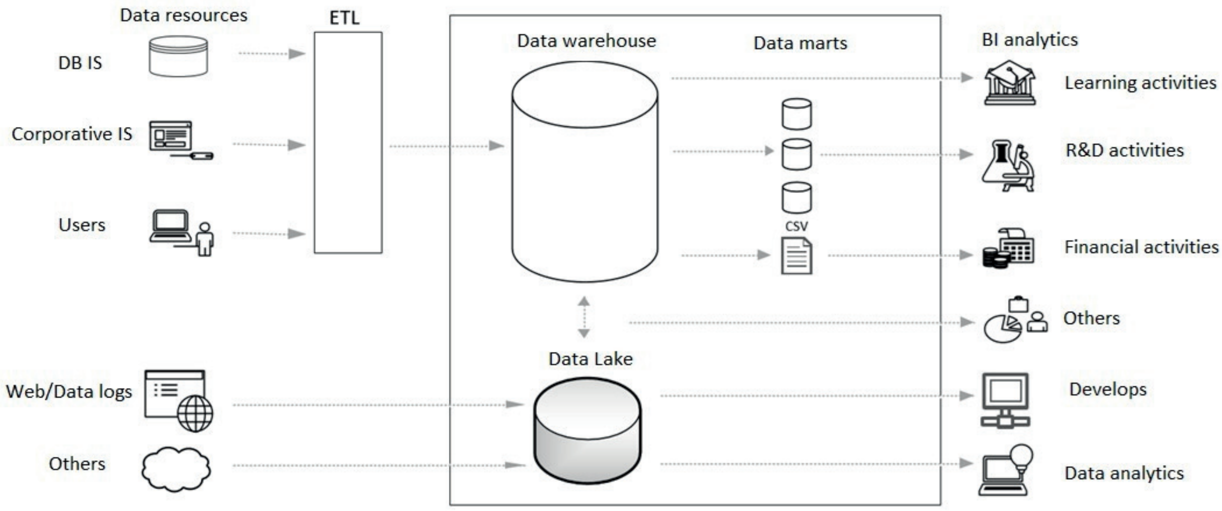

Figure 2: The standard architecture of CIS of KazNU

The following types of data are collected for students: personal data, citizenship, country, region of residence, faculty, specialty, department (ru/kz), stage of the study, language of instruction, course of study, a form of study (grant / paid), data on the results of UNT, etc.

On the educational process with the student: ID-group, the full name of the curator-adviser, name of the discipline by semester, the full name of the teacher of the discipline by semester, class schedule: semester, date, time, place, links to the online lesson; exam schedule: date, time, place, ID-admission to the exam, the format of the final exam, full name of those checking the student's examination data; discipline attendance data.

According to the student's progress: marks for foreign discipline controls, marks for MID-TERM by discipline, results of final exams in the discipline, GPA, the presence of arrears, data on retaking exams, data on non-attendance at the exam, the fact of violations during exams, etc.

According to the student's interaction with e-learning systems: the number of downloads of educational materials, the number of views and downloads of video recordings of online lectures, the number of visits online classes, the number of page views, etc.

According to the behavior of the student in the IPC "UNIVER": the number of visits to the system, the length of stay in the system, data on the visited web pages of the system, the most visited service/page, viewing the progress journal, viewing information about the teaching staff, the number of views/downloads of training materials, electronic resources, for a certain period, loading of completed tasks, data on passing tests and exams (duration of writing answers, grade, results of anti-plagiarism checks (in the case of a written exam)).

Personal data on the teaching staff of the university is extracted from the accounting and personnel accounting system “1C: Enterprise”, which is the primary source of data on university employees: personal data, citizenship, country, region of residence, address, work experience, the start date of employment, date of adoption, date dismissals, data on breaks (if any), diploma specialty, position, rate, academic degree, academic title, types of allowances, faculty, department.

Educational data on the teaching staff of the university in the IPK “UNIVER”: Name of the discipline, credits, language of instruction, specialty, semester, midterm points for the discipline, MIDTERM scores for the discipline, exam scores for the discipline, data based on the results of the questionnaire (teacher through the eyes of students, teacher through the eyes of colleagues, the quality of educational materials in the context of the semester, date), data on the loaded educational and methodological complexes of disciplines (name, type of EMCD, download date, update frequency, amount of material, data on the correspondence of the material to the discipline), ID-students in the discipline, schedule disciplines, data on the log of visits (date, discipline, date of entry of data on attendance, points for the lesson).

Ranking indicators for the teaching staff of the university in the IPK “UNIVER”: name, department of the employee, units, directions, year, indicators, code, name, result of the indicative indicator of this employee, questionnaire plan, fact of questionnaires, monitoring plan, goal achievement index.

The IS “Science” collects data on the research activities of the teaching staff and students of the university. Including scientific articles, publications, and books: author, title, the language of publication, year of publication, title, and rating of the scientific publication, type of publication, number of pages.

The data collection into the Perco time attendance system occurs automatically from the 1C: Enterprise IS name, faculty, department, division, identification code, student training period, and information received from devices (turnstiles) using Perco software: arrival time, departure time, date.

For storage in CD, OD, and display in the Data Showcase, it is necessary to carry out preprocessing procedures, which are carried out using SQL queries.

The primary processing procedures in the SQL query language are queries to retrieve data from the primary source of the data table; stored procedures for processing trigger operations for dynamic reports; presentation of data obtained from primary sources and stored procedures for obtaining visual reports in which related data from different databases are synchronized.

5 The program realization of IAS

After collecting data, the process of extracting, transforming, and loading data is carried out to generate analytical reports on all types of university activities.

Data for analytical research is retrieved by Power BI business intelligence software by connecting to the IAS data warehouse. After the data is retrieved, the relationship between the tables is configured, if necessary.

At the data transformation stage, the following operations are performed: the transformation of the data structure; aggregation of data; translation of values; combining data from multiple sources; changing the data type; creation of metrics; other.

The extracted and transformed data is a source of information for generating analytical re-ports in visual and tabular forms.

In addition to importing and transforming data, an equally important stage of work in Power BI is building analytical reports. Visualization allows you to compare data visually, make convenient accents on essential things and carry out a lot of research in different sections.

When building an interactive reporting form (data mart), you need to understand the answers to basic questions: why is this report?, for whom is this report?, what data can be visualized in this report ?.

Power BI offers an extensive range of visualization elements, allowing you to build reports of almost any complexity. In addition to the basic well-known histograms and graphs, there are a wide variety of charts (essential and funnel-shaped, waterfall and indicator, combined and bubble, and others), matrices, maps (base and cartograms), slices, plane trees, as well as images and simple cards with one number. The importance of a large number of elements in the analysis of learning activities cannot be overestimated.

An example of the report visualization is shown in fig. 3. The developed visual representations of the university's statistical reporting represent more than half of all forms of the university's statistical automation system IS "Univer 2.0".

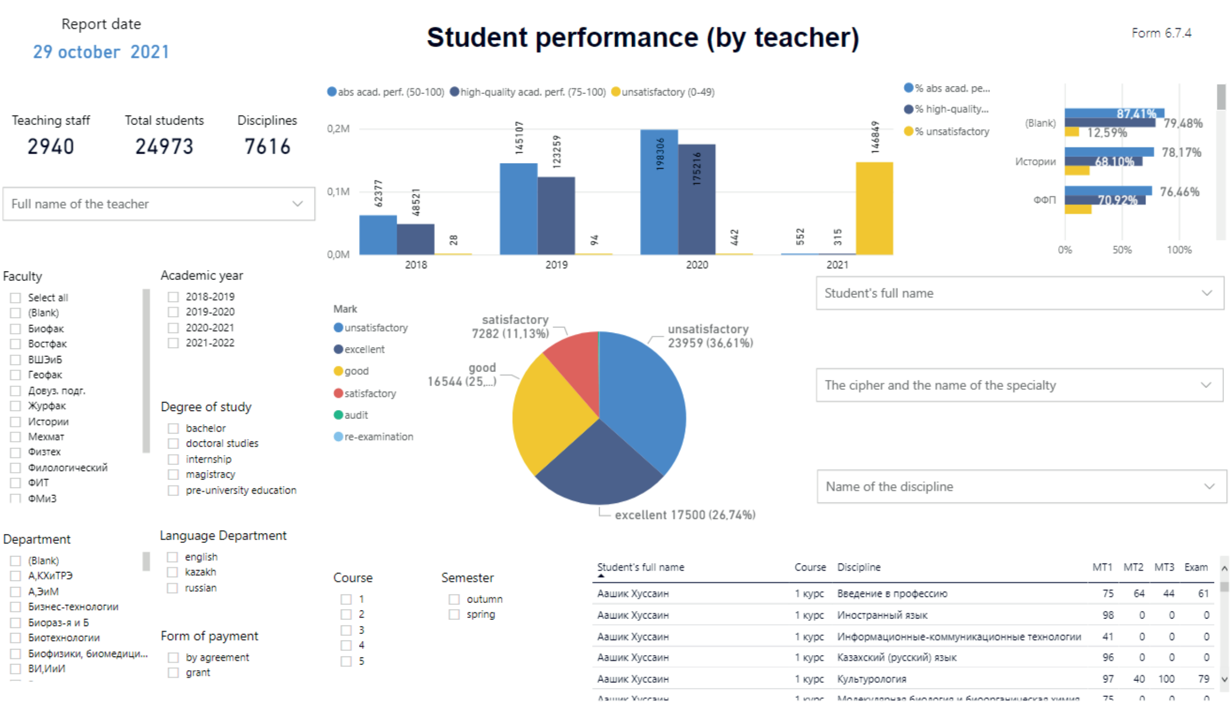

Figure 3: The visualization of reports on the educational activity of KazNU

The analytical report generated in the business intelligence software is loaded into the cloud version of the software, with the ability to configure access to reading and editing the report.

By the role-based policy of the KIS KazNU, the following roles of the IAS are provided, such as system administrator that has full rights to fully manage data, services, and system settings, users of CIS systems, such as dean, deputy dean, teacher, curator-adviser, DAV specialist, CSO employee, accountant, management. Students have rights with a limited scope of data visibility ac-cording to their role in the university's organizational structure.

Analytical engineers and Software Engineer have full rights to work with data at the data warehouse level, data lake, without the right to delete. The system administrator has full rights to configure and manage the DBMS, data warehouse, and data lake by system parameters.

IAS users, such as the rector, vice-rectors, department directors, deans, deputy deans, CSO employees, heads of structural divisions, have the right to view analytical reports following their type of activity in the organizational structure of the university.

Access to information systems and the information and analytical system can be provided temporarily and permanently.

The data from the IAS are used for generating statistical reports on the educational, scientific, social, administrative, and financial activities of the university; data analysis and visualization; identifying patterns in data using data mining methods; predicting student performance; providing feedback to support the work of teaching staff; providing guidance to students; identifying unwanted student behavior; optimization and modernization of the educational content of the training course; statistical analysis to investigate the nature of the data.

6 Results and discussion

Information and analytical system have been developed, which is integrated into the outline of the corporate information system of KazNU; instructions have been developed for programmers to create a system.

In the context of the digital transformation of the university, the development of digitalization processes, it is necessary to introduce a culture of data management: collection, analysis, communication, identification of bottlenecks, understanding, application, tools, management strategy, focus on results. This will allow you to assess the value of IT technologies and ensure effective decision-making. However, there is a problem here - data management is still under development in many institutions.

Therefore, it is necessary to start with simple tasks and make efforts to build the entire data management model of the educational process, automate the visual presentation of statistical reporting, and bring it into an analytical presentation.

Thus, at the Kazakh National University, work is underway to develop a data management strategy based on methods for determining how data is collected, stored, processed and used for various tasks of the development of the education system and decision-making, which makes it possible to build an information and analytical system that integrates and accumulates disparate corporate data.

Information systems provide an analysis of an educational organization's scientific and educational activities in the main areas of development, which are formalized as the quality of educational services, research potential, innovation orientation, information, and technical and technological infrastructure.

7 Conclusion

This article discusses an approach to the design and implementation of IAS on the cloud-based business analysis platform Microsoft Power BI, which integrates all the disparate data of corporate information systems of the university and transactional data sources. The analysis of data types used as a data source for analytical reporting is carried out, the architecture of the IAS is presented, the logic of data extraction and transformation is described, the role-based access model is presented, and the form of visual reporting is presented.

The approach used by the authors in the article to apply research with data in the education system on the example of the Kazakh National University named after al-Farabi can be scaled to any organization that has a corporate information system, consisting of its own and local developments of information systems and who want to conduct a deeper analysis of the organization's activities to make the right strategic decisions.

The economic effect of the introduction of such a system is early identification of "bottlenecks" in the organization of the education system, prompt decision-making; in a systematic and complex analysis of the main business processes of the organization of education, through building a data map and visualizing all statistical reports, using data mining algorithms, thereby contributing to the digital transformation of the main business processes of the education system.

References

[1] Publication "Comparison of the top 4 popular BI platforms"[Electronic resource], - Access mode: *https://habr.com/ru/company/newprolab/blog/349186/*.

[2] Publication "Comparison of BI systems"[Electronic resource], - Access mode: *https://habr.com/ru/post/438648/*.

[3] The official website of the Power BI software [Electronic resource], - Access mode: *https://powerbi.microsoft.com/*.

[4] Publication "2021 Gartner Magic Quadrant for Analytics and Business Intelligence Platforms"[Electronic resource], - Access mode: *https://info.microsoft.com/ww-Landing-2021-Gartner-MQ-for-Analytics-and-Business-Intelligence-Power-BI.html?LCID=EN-US*.

[5] Mamykova Zh.D., Mutanov G.M., Bobrov L.K., Gusev Y.B. *Indicative planning and rating assessments in strategic management of the university: information support system* // Bulletin of NGUEU: heading society and economy: problems of development. - Novosibirsk: Publishing house "NGUEU 2013. - № 1. - C. 10 - 21 (impact factor of the RSCI 0,254).

[6] Mamykova Zh.D., Mutanov G.M., Bobrov L.K. Electronic campus in a socially oriented model of Smart society // *Ideas and ideals: scientific journal.* - Novosibirsk: Publishing house "NGUEU 2013. - Volume 2.- № 2. – C. (impact factor of the RSCI 0,053).

[7] Kumargazhanova S., Uvaliyeva I., Baklanov A., Zhomartkyzy G., Mamykova Zh., Ipalakova M., Gyorgy G. Development of the Information and Analytical System in the Control of Management of University Scientific and Educational Activities // *Journal of Applied Sciences «Acta Polytechnica Nungarica»* // ISSN 1785-8860 (IF for 2018: 1.286)// DOI: 10.12700 / APH.15.4.2018.4.2 // *https://uni-obuda.hu/journal/Saule_Indira_Aleksander_Gulnaz_Zhanl_Madina_-Gyorok_83.pdf*.

[8] Bolatkhan M., Mamykova Zh.D., Kopnova O.L., Kistaubayev E.B., Karyukin V.I., Rabat Sh.Zh., Sundetova Zh. *Report "Applied research of data management in the education system for decision-making (on the example of a university)"* - 2020.

[9] Mamykova Zh.D., Mutanov G.M., Bobrov L.K. *IT infrastructure of the university as a platform for the development of information technologies* // Bulletin of the NSUEU. – 2013. – № 4. – C. 276-287.

[10] Abdrakhmanova, M., Mutanov, G., Mamykova, Z., Tukeyev, U. Agents interaction and queueing system model of real time control of students service center load balancing // *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics).* Volume 11055 LNAI, 2018, Pages 349-359. (CiteScore - 1.9, quartile 1 (percentile 91), link to the publication - *https://www.scopus.com/record/display.uri?eid=2-s2.0-85053192399&origin=resultslist&sort=plf-f&src=s&sid=6b052d7b5fbeaa90afbfab8dbbc6d3d9&sot=autdocs&sdt=autdocs&sl=17&s=AU-ID%286506682964%29&relpos=4&citeCnt=4&searchTerm=.*

4-бөлім	Раздел 4	Section 4
Қолданбалы математика	**Прикладная математика**	**Applied Mathematics**

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MATHEMATICAL MODEL OF THE EPIDEMIC PROPAGATION WITH LIMITED TIME SPENT IN EXPOSED AND INFECTED COMPARTMENTS

A discrete nonlinear mathematical model of the epidemic development is proposed. It involves dividing the population into eight compartments (susceptible, exposed, asymptomatic, easily sick, hospitalized, critically ill, recovered and deceased). At the same time, the time spent in compartments of exposed and all forms of patients is considered limited. Thus, any person who has been in contact with an infected person, after a while, either gets sick or does not, leaving the exposed compartment, and any patient, over time, for sure, either goes to the group of more severe patients, dies or recovers. This deterministic model is presented in a discrete form and simulates the quantitative change of various groups by day during the spread of the epidemic. It is a transformation of the SEIR model. The article also presents a numerical analysis of the proposed model. The development of the COVID epidemic in Kazakhstan is considered as an example. At the end, forecasts are given based on preliminary data from the first months of quarantine. Various parameters of the model when starting numerical experiments were found based on computational experiments. At the same time, for a given deterministic one, the effect of wavelike changes in the number of infected is observed.

Key words: epidemic, mathematical model, COVID.

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ЭПИДЕМИЯНЫҢ ДАМУЫНЫҢ КОНТАКТ ЖӘНЕ ЖҰҚТЫРҒАН ТОПТАРДА ШЕКТЕУЛІ УАҚЫТ БОЛАТЫН МАТЕМАТИКАЛЫҚ МОДЕЛІ

Эпидемиялық дамудың дискретті емес сызықтық математикалық моделі ұсынылған. Ол халықты сегіз топқа бөлуді қамтиды (сезімтал, жанаспалы, асимптоматикалық, жеңіл ауру, ауруханаға түскен, ауыр науқас, айыққан және қайтыс болған). Сонымен қатар, науқастардың барлық формаларында және байланыс түрлерінде өткізілген уақыт шектеулі болып саналады. Осылайша, кез-келген адам жұқтырған адаммен байланыста болғаннан кейін, біраз уақыттан кейін немесе ауырады, немесе байланыс тобынан шықпайды және кез-келген пациент уақыт өте келе негұрлым ауыр пациенттер тобына барады, қайтыс болады немесе қалпына келеді. Бұл детерминирленген модель дискретті түрде ұсынылған және эпидемияның таралуы кезінде әр түрлі топтардың сандық згеруін имитациялайды. Бұл SEIR моделін жаңарту. Мақалада ұсынылған модельдің сандық талдауы да келтірілген. Қазақстандағы COVID эпидемиясының дамуы мысал ретінде қарастырылады. Соңында, карантиннің алғашқы айларындағы алдын-ала мәліметтер негізінде болжамдар жасалады. Есептік тәжірибелер негізінде сандық эксперименттерді бастаған кезде модельдің әр түрлі параметрлері табылды, ол параметрлер негізінен әр үрлі топтар арасында адамдар алмасуды, науқастардың ауру жұқтыру жылдамдықтарын сипаттайды. Сонымен бірге, берілген детерминирленген үшін, жұқтырғандар санының толқын тәрізді өзгеруінің әсері байқалады.
Түйін сөздер: эпидемия, математикалық модель, COVID.

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МАТЕМАТИЧЕСКАЯ МОДЕЛЬ РАЗВИТИЯ ЭПИДЕМИИ С ОГРАНИЧЕННЫМ ВРЕМЕНЕМ ПРЕБЫВАНИЯ В ГРУППАХ КОНТАКТНЫХ И ИНФИЦИРОВАННЫХ

Предлагается дискретная нелинейная математическая модель развития эпидемии. Она предполагает разбиение популяции на восемь групп (восприимчивые, контактные, бессимптомные, легко больные, госпитализированные, критические больные, выздоровевшие и умершие). При этом время пребывания в группах контактных и всех форм больных считается ограниченным. Таким образом, любой человек, бывший в контакте с зараженным, через некоторое время либо заболевает, либо нет, покидая группу контактных, а любой больной со временем наверняка, либо переходит в группу более тяжелый больных, умирает или выздоравливает. Данная детерминистическая модель представлена в дискретном виде и моделирует количественное изменение различных групп по дням во время распространения эпидемии. Она является модернизацией SEIR модели. Так же в статье представлен проведенный численный анализ предложенной модели. В качестве примера рассматривается развитие эпидемии COVID в Казахстане. В конце даются прогнозы, полученные на основе предварительных данных первых месяцев карантина. Различные параметры модели при запусках численных экспериментов находились на основе вычислительных экспериментов. При этом для данной детерминированной наблюдается эффект волнообразных изменений количества инфицированных.
Ключевые слова: эпидемия, математическая модель, COVID.

1 Introduction

Modern mathematical models of epidemiology originate from the SIR model developed by W. Kermak and A. Mackendrick about a hundred years ago [1]. It involves dividing the entire population under consideration into susceptible, infected and recovered compartments. The mathematical model of the process is a system of differential or difference equations describing the change in the size of each of the indicated population groups. Its simplest modifications are the SIRD model, which adds a compartment of deceased (deceased) [2] and the SIS model of a disease to which immunity is not produced [3]. We also note the SIRS model, in which the recovered lose their immunity over time [4].

The disadvantage of the described models is the lack of an incubation period, i.e. the assumption that a person who had contact with a sick person immediately falls ill. As a result, the SEIR model was proposed, to which the exposed compartment was added, see, for example, [5]. Thus, in the process of infection, a person susceptible to the disease first becomes exposed and only then becomes infected.

2 Literature review

The overwhelming majority of mathematical models of the development of the epidemic that are currently used are modifications of the SEIR model. In particular, the SEIS model differs from it only in that immunity is not produced [6]. The SEIRD model additionally includes a group of deceased [7,8], the MSEIR model additionally includes a group of people who are maternally derived immunity [9, 10], and the SEIRHCD model includes a group of hospitalized and critical patients. [8, 11]. In a number of cases, models with a variable frequency of contacts are also considered [12], which take into account the spread of the epidemic over a certain territory and are described by partial differential equations [13], models that take into account vaccination [14], as well as stochastic models in which the transition from one group to another is a random event [3]. A significant number of models are also given in monographs [15–19] devoted to mathematical models of epidemiology.

In these models, one significant circumstance is not taken into account, namely, the limited stay in a compartment of exposed and different groups of patients. In particular, any person who has been in contact with a patient, after some time, will probably either get sick or not get sick, which means that he will certainly leave the exposed compartment. Anyone sick after some time will probably either recover or die, i.e. will definitely leave the group of infected.

In this paper, we propose a discrete dynamic model of the development of the epidemic, which assumes the division of the entire population into eight groups and considers the limited stay in groups of contact and various forms of patients. Based on this model, some calculations are made on the spread of the COVID-19 epidemic in Kazakhstan.

3 Method: Description of the model

A certain isolated population under the conditions of an epidemic is considered. The entire population is divided into the following compartments:

S: susceptible (healthy, but potentially sick);

E: exposed (healthy, in contact with sick);

A: asymptomatic (infected, asymptomatic);

I: easily sick (mild patients undergoing treatment at home);

H: hospitalized (seriously ill, hospitalized);

C: critically il (patients in critical condition);

R: recovered (recovered, who have no signs of illness);

D: deceased.

Further through S_k, E_k etc. denotes, respectively, the number of susceptible, exposed, etc. at time k. In this case k is understood as the k-th day from the moment of the start of the study. We do not take into account the natural fertility and mortality of the population, i.e. we consider the sum N of all the above mentioned compartments of the population constant.

As with the SEIR models, it is assumed that the susceptible person becomes infected by going through the exposed compartment stage. At the same time, only asymptomatic and easily sick people are sources of infection. In addition, it is assumed that all who have recovered are immunized, i.e. are not susceptible to disease.

When building a model, the following intergroup transitions are taken into account:

- es: the exposed person may not get sick and become susceptible again;

- ea: the exposed person may become asymptomatic;
- ei: the exposed person can become easily sick;
- eh: the exposed person may become hospitalized;
- ar: asymptomatic can become recovered;
- ae: asymptomatic can become easily sick;
- ih: easily sick patient can be hospitalized;
- ic: easily sick patient can become critically ill;
- ir: easily sick patient can become recovered;
- hc: the hospitalized person may become critically ill;
- hr: a hospitalized person can recover;
- cr: the critically ill can recover;
- cd: critically ill patient may die;
- se: susceptible can become exposed.

Moreover, let's indicate the proportions of exposed (e), passing over time into compartments of susceptible (s), asymptomatic patients (a), etc. δ_{es}, δ_{ea} etc. All these quantities lie between zero and one, and the obvious equalities $\delta_{es} + \delta_{ea} + \delta_{ei} + \delta_{eh} = 1$, etc. are fulfilled, i.e. any exposed will either not get sick at all, or get sick in one form or another.

The fundamental difference between the proposed model and the known ones is the assumption about the limited presence of a person in compartments of exposed and any form of patients. Further we indicate etc. the time (number of days) of being in the compartment of exposed (e), asymptomatic (a), etc. as n_e, n_a, and indicate the number of exposed on j-th day since contact, the number of asymptomatic j-th day since the start of the disease, etc. at time k as e_k^j, a_k^j, etc. Moreover, the following obvious equalities hold:

$$E_k = \sum_{j=1}^{n_e} e_k^j,\ A_k = \sum_{j=1}^{n_a} a_k^j,\ I_k = \sum_{j=1}^{n_i} i_k^j,\ H_k = \sum_{j=1}^{n_h} h_k^j,\ C_k = \sum_{j=1}^{n_c} c_k^j, \tag{1}$$

the number of exposed E_k at the moment of time k is the sum of the number of exposed of the first day, the second day, ..., n_e day at this moment of time, etc.

Note that the exposed of the j-th (previous) day at the previous moment of time becomes exposed of the $(j+1)$-th (subsequent) day at the subsequent moment of time, i.e.

$$e_{k+1}^{j+1} = e_k^j,\ j = 1, \ldots, n_e - 1. \tag{2}$$

Following equalities have a similar meaning

$$a_{k+1}^{j+1} = a_k^j,\ j = 1, \ldots, n_a - 1;\ i_{k+1}^{j+1} = i_k^j,\ j = 1, \ldots, n_i - 1; \tag{3}$$

$$h_{k+1}^{j+1} = h_k^j,\ j = 1, \ldots, n_h - 1;\ c_{k+1}^{j+1} = c_k^j,\ j = 1, \ldots, n_c - 1; \tag{4}$$

In accordance with the assumptions made earlier, the number of susceptible at the next moment in time is equal to the number of susceptible at the previous moment of time minus newly exposed at this moment of time plus the number of uninfected exposed of the last day at the previous moment in time:

$$S_{k+1} = S_k - e_{k+1}^1 + \delta_{es} e_k^{n_e}. \tag{5}$$

The number of exposed at the next moment in time is equal to the number of exposed at the previous moment in time plus the number of new exposed at this moment in time minus the number of exposed on the last day at the previous moment in time, i.e. who have left the exposed compartment at the moment:

$$E_{k+1} = E_k + e_{k+1}^1 - e_k^{n_e}. \tag{6}$$

Similarly, we have the following equalities

$$A_{k+1} = A_k + a_{k+1}^1 - a_k^{n_a}. \tag{7}$$

$$I_{k+1} = I_k + i_{k+1}^1 - i_k^{n_i}. \tag{8}$$

$$H_{k+1} = H_k + h_{k+1}^1 - h_k^{n_h}. \tag{9}$$

$$C_{k+1} = C_k + c_{k+1}^1 - c_k^{n_c}. \tag{10}$$

Further, the number of recovered at the next time point is equal to the number of recovered at the previous point in time plus all recovered patients of the last day of illness at this point in time:

$$R_{k+1} = R_k + \delta_{ar} a_k^{n_a} + \delta_{ir} i_k^{n_i} + \delta_{hr} h_k^{n_h} + \delta_{cr} c_k^{n_c}. \tag{11}$$

Finally, the number of deaths at the next time point is the sum of the number of deaths at the previous time point and the number of new deaths, i.e. critically ill of the last day of illness at this point in time who died:

$$D_{k+1} = D_k + \delta_{cd} c + k^{n_c}. \tag{12}$$

It remains to indicate the formulas for calculating the number of exposed and various patients on the first day at the next moment in time. In particular, the number of exposed on the first day, i.e. newly exposed at a later point in time is determined by the formula

$$e_{k+1}^1 = (\beta_a A_k + \beta_i I_k) \frac{S_k}{N}, \tag{13}$$

where $\beta_a,\ \beta_i$– positive constants characterizing the infectivity of asymptomatic and easily sick. The number of asymptomatic first day, i.e. newly ill at the next time point is equal to the number of those who left the exposed compartment at the previous time point who fell ill in an asymptomatic form, i.e.

$$a_{k+1}^1 = \delta_{ea} e_k^{n_e}. \tag{14}$$

The number of easily sick on the first day, i.e. newly ill at the next time point is equal to the number of those who left the exposed and asymptomatic compartments at the previous time point and fell ill in a mild form, i.e.

$$i_{k+1}^{1} = \delta_{ei} e_k^{n_e} + \delta_{ai} a_k^{n_a}. \tag{15}$$

The following formulas have a similar meaning

$$h_{k+1}^{1} = \delta_{eh} e_k^{n_e} + \delta_{ih} i_k^{n_i}. \tag{16}$$

$$c_{k+1}^{1} = \delta_{hc} h_k^{n_e} + \delta_{ic} i_k^{n_i}. \tag{17}$$

Relations (1) – (17) with the corresponding initial conditions constitute a mathematical model of the considered process.

4 Results and Analysis

As an example, we consider one variant of the COVID epidemic forecasting in Kazakhstan for the period from August 1, 2020. The model parameters were selected partly on the basis of official data from the Ministry of Health of the Republic of Kazakhstan, partly on the basis of expert assessments of epidemiologists. The following figures show graphs of the change over time in the total number of all cases, recovered and deaths, as well as daily increases in these characteristics.

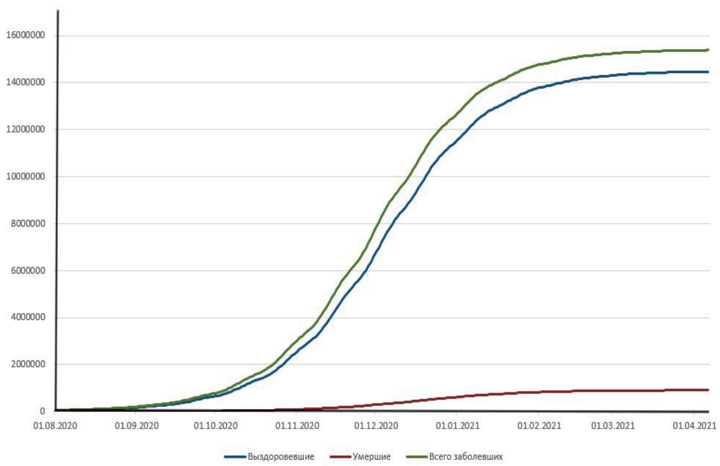

Figure 1: The total number of cases, recovered and deaths.

The obtained results of the numerical analysis are of a preliminary nature, since the information used to determine the parameters of the model is insufficiently complete and not accurate enough. Nevertheless, the established qualitative results indicate the sufficient effectiveness of the proposed model. In particular, we see that initially the epidemic develops exponentially. Then its growth slows down, reaches its maximum, declines and in the end the epidemic ends. It is characteristic that the development of the epidemic is not monotonous,

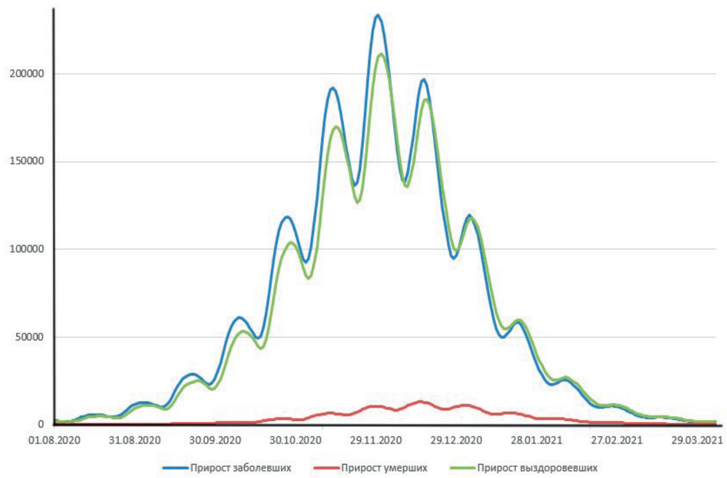

Figure 2: Daily increase in the total number of cases, recovered and deaths.

but wavy, which is consistent with the known data on the development of the epidemic both in Kazakhstan and in other countries. In addition, it was found that the introduction of a stricter quarantine, which is associated with a decrease in infection rates β_a and β_i leads to a shift in the time of the peak of the epidemic. Thus, the resulting model, subject to its more accurate identification, can be used for long-term forecasting of the development of epidemics.

5 Discussion and conclusion

The results of numerical analysis confirm the viability of the proposed model. In particular, comparing the results obtained with the actual development of COVID-19 in Kazakhstan in late summer and autumn 2020, a relatively high forecast accuracy can be noted. An important circumstance is the wave-like development of the epidemic obtained as a result of model calculations, which is not typical for most of the currently used mathematical models of epidemiology.

The model can be refined by taking into account the random nature of its individual parameters. This applies primarily to the time spent in the compartments. In addition, we can remove the restriction on the isolation of the population, considering the course of the epidemic in the system of regions, we can take into account the possibility of virus mutation, the emergence of a vaccine, the spread of the epidemic over a certain territory, etc.

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References

[1] Kermack W.O. and McKendrick, A.G. A Contribution to the Mathematical Theory of Epidemics. – Proc. Roy. Soc. Lond. A, 1927, 115, 700–721.

[2] Bailey N. The mathematical theory of infectious diseases and its applications (2nd ed.). London, Griffin, 1975.

[3] Bacaer N. Le Modele Stochastique SIS pour une Epiddmie dans un Environnement Aleatoire. – J. of Mathematical Biology, 2016, v. 73, 847–866.

[4] Wang X. An SIRS Epidemic Model with Vital Dynamics and a Ratio-Dependent Saturation Incidence Rate. – Discrete Dynamics in Nature and Society. 2015. https://doi.org/10.1155/2015/720682

[5] Mwalili S., Kimathi, M., Ojiambo, V. et al. SEIR model for COVID-19 dynamics incorporating the environment and social distancing. – BMC Res Notes 13, 352,2020. https://doi.org/10.1186/s13104-020-05192-1

[6] Wang J. Analysis of an SEIS Epidemic Model with a Changing Delitescence. – Abstract and Applied Analysis, 2012, 4. https://www.hindawi.com/journals/aaa/2012/318150/

[7] Sameni R. Mathematical Modeling of Epidemic Diseases; A Case Study of the COVID-19 Coronavirus. – arXiv:2003.11371. 2020.

[8] Krivorotko O.I., Kabanikhin S.I., Zyatkov N.Yu., Prikhodko A,Yu., Prokhoshin N.M., Shishlenin M.A. Matematicheskoye modelirovaniye i prognozirovaniye COVID-19 v Moskve i Novosibirskoy oblasti. – 2020 https://arxiv.org/pdf/2006.12619.pdf

[9] Hethcote H.W. The Mathematics of Infectious Diseases. – SIAM Review 42, 599–653, 2000.

[10] Almeida R., Cruz A., Martins N. and Monteiro N. An epidemiological MSEIR Model Described by the Caputo Fractional Derivative. – Int. J. of Dynamics and Control, 2019, 7, 776–784.

[11] Unlu E., Leger H. et al Epidemic Analysis of COVID-19 Outbreak and Counter-Measures in France. – 2020. medRxiv. 2020.04.27.20079962. DOI: 10.1101/2020.04.27.20079962.

[12] Greenhalgh, D. and Das R. Modeling Epidemics with Variable Contact Rates. – Theoretical Population Biology, 1995, 2 (47), 129-179.

[13] Huang H., Wang M. The Reaction-Diffusion System for an SIR Epidemic Model with a Free Boundary. – Discrete and Continuous Dynamical Systems. Series B, 2015, 20(7), 2039-2050.

[14] Gao S., Teng Z., Nieto J. and Torres A. Analysis of an SIR Epidemic Model with Pulse Vaccination and Distributed Time Delay. – Journal of Biomedicine and Biotechnology. 2007, 64870. doi:10.1155/2007/64870. PMC 2217597. PMID 18322563.

[15] May R., Anderson R. Infectious diseases of humans: dynamics and control. – Oxford: Oxford University Press, 1991.

[16] Brauer F. et al. Mathematical Epidemiology. Eds. – Springer, 2008.

[17] Capasso V. Mathematical Structures of Epidemic Systems. 2nd Printing. – Heidelberg, Springer, 2008.

[18] Vynnycky E., White, R.G., et al. An Introduction to Infectious Disease Modelling. – Oxford: Oxford University Press, 2010.

[19] Diekmann, O., Heesterbeek, H. and Britton, T. Mathematical Tools for Understanding Infectious Disease Dynamics. – Princeton Series in Theoretical and Computational Biology. Princeton University Press, Princeton, 2013.

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A BRIEF OVERVIEW OF MODERN RESEARCH OF THE PROCESSES DYNAMICS IN UNSTEADY WATER FLOWS USING THE SHALLOW WATER EQUATION

In hydrodynamics (hydraulics), there are numerous approaches to solving the problem of water flow dynamics control in river beds and channels, while the results of each methods differ, and estimates of their reliability do not always exist. The shallow water equation (or Saint-Venant's equations in one-dimensional form) is often used by hydraulic engineers in their practice. Its apparent simplicity and ability to describe well enough the behavior of rivers and flows make it a useful tool for many applications, such as the regulation of navigable rivers and irrigation networks in agriculture. The main direction of research in the field of numeric problems described by Saint-Venant equations is the development of numerical methods of computation implemented on super-powerful computers. Development of numerical models of surface water dynamics in the shallow water approximation is actively advancing during the recent years.
The article is devoted to a review of mathematical studies of the dynamics of processes in unsteady water flows using differential equations, as well as an assessment of these approaches from the point of view of the model's reflection of real processes.
The work is aimed at analyzing different approaches to modeling the dynamics of processes in non-stationary water flows. The objectives of the study include the analysis of scientific publications with different approaches to modeling the shallow water equation, taking into account factors, parameters, and modeling methods.
Key words: Saint-Venant equations, dynamics of unsteady river currents, numerical methods, a system of hyperbolic differential equations, high-performance computing.

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Таяз су теңдеуін қолдана отырып, тұрақсыз су ағындарындағы процестердің динамикасы туралы қазіргі заманғы зерттеулерге қысқаша шолу

Гидродинамикада (гидравликада) өзен арналары мен арналардағы су ағынының динамикасын басқару мәселесін шешудің көптеген тәсілдері бар, әр әдістің нәтижелері әр түрлі және олардың сенімділігін бағалау әрдайым бола бермейді. Таяз су теңдеуін (немесе Сен-Венан теңдеулерін бір өлшемді түрде) гидротехниктер өз тәжірибесінде жиі қолданады. Оның айқын қарапайымдылығы мен өзендер мен ағындардың мінез-құлқын жақсы сипаттау қабілеті оны көптеген қосымшалар үшін пайдалы құралға айналдырады, мысалы, өзендерді тасымалдау және ауыл шаруашылығындағы суару желілерін реттеу. Сен-Венан теңдеулерімен сипатталған Сандық есептер саласындағы зерттеулердің негізгі бағыты ауыр компьютерлерде жүзеге асырылатын сандық есептеу әдістерін жасау болып табылады. Соңғы жылдары таяз суға жақындағанда жер үсті суларының динамикасының сандық модельдерін жасау белсенді түрде дамып келеді.

Мақалада дифференциалдық теңдеулерді қолдана отырып, стационарлық емес су ағындарындағы процестердің динамикасын математикалық зерттеулерге шолу, сонымен қатар нақты процестер моделінде шағылысу тұрғысынан осы тәсілдерді бағалау қарастырылған. Жұмыс стационарлық емес су ағындарындағы процестердің динамикасын модельдеудің әртүрлі тәсілдерін талдауға бағытталған. Зерттеу міндеттеріне факторларды, параметрлерді және модельдеу әдістерін ескере отырып, таяз су теңдеуін модельдеудің әртүрлі тәсілдері бар ғылыми жарияланымдарды талдау кіреді.

Түйін сөздер: Сен-Венан теңдеулері, тұрақты емес өзен ағыстарының динамикасы, сандық әдістер, гиперболалық дифференциалдық теңдеулер жүйесі, жоғары өнімді есептеулер

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Краткий обзор современных исследований динамики процессов в неустановившихся течениях воды с помощью уравнения мелкой воды

В гидродинамике (гидравлике) сосуществуют многочисленные подходы к решению проблемы управления динамикой водных потоков в руслах рек и каналов, причем результаты по различным методам различаются, а оценки их достоверности не всегда существуют. Уравнение мелкой воды (или уравнения Сен-Венана в одномерной форме) часто используется инженер-гидротехниками в своей практике. Их кажущаяся простота и их способность достаточно хорошо описывать поведение рек и водотоков делают их полезным инструментом для многих приложений, таких как, например, регулирование судоходных рек и ирригационных сетей в сельском хозяйстве. Основным направлением исследований в области расчета задач, описанных уравнениями Сен-Венана, является разработка численных методов расчета, реализуемых с помощью использования сверхмощных вычислительных компьютеров. Разработки численных моделей динамики поверхностных вод в приближении мелкой воды, активно развиваются в последние годы усилиями исследователей.
Статья посвящена к обзору математических исследований динамики процессов в неустановившихся течениях воды с помощью дифференциальных уравнений, а также оценке этих подходов с точки зрения отражения моделью реальных процессов.
Целью исследования является анализ различных подходов к моделированию динамики процессов в неустановившихся течениях воды. К задачам исследования относится анализ научных публикаций с различными подходами к моделированию уравнения мелкой воды учтенных факторов, параметров и методов моделирования.

Ключевые слова: Уравнения Сен-Венана, динамика неустановившихся течений реки, численные методы, система гиперболических дифференциальных уравнений, высокопроизводительные вычисления.

1 Introduction

Shallow water equations are widely used to model water flow in rivers or lakes. The equations of shallow water, the Saint-Venant equation was proposed in 1871 and the numerical solution of problems for these equations is an urgent problem in computational mathematics. The flows under the surface of the fluid are described by the Saint-Venant equations, which are a system of hyperbolic partial differential equations. This problem includes such important tasks as the problem of floods on rivers, the problem of daily and weekly regulation of productivity, and problems of the outflowing wave upon a dam break. For practical purposes, it is very important to have more accurate methods for solving this problem. Many problems require methods for calculating the flow of water taking into account various physical factors. For example, floods in the coastal zone by storm surge, flooding in rivers, the study of the interfluve problem. Application of the shallow water equation makes it possible to simplify the algorithm, as well as optimize the use of computational resources, which are considered to be a very important factor in solving applied problems and developing software systems.

2 Material and methods

2.1 The formulation of the problem

Two-dimensional Saint-Venant equations are used to control surface flow. These equations are derived from the continuity and momentum equations by depth averaging. The main assumptions used in the derivation of two-dimensional Saint-Venant equations are the distribution of hydrostatic pressure and a small channel slope. The defining equations for the surface flow are obtained as follows:

$$\frac{\partial h}{\partial t}+\frac{\partial (hu_x)}{\partial x}+\frac{\partial (hu_y)}{\partial y}=0, \tag{1}$$

$$\frac{\partial (hu_x)}{\partial t}+\frac{\partial \left(hu_x{}^2+\frac{gh^2}{2}\right)}{\partial x}+\frac{\partial (hu_xu_y)}{\partial y}=hf_x-gh\frac{\partial S}{\partial x}, \tag{2}$$

$$\frac{\partial (hu_x)}{\partial t}+\frac{\partial (hu_xu_y)}{\partial x}+\frac{\partial \left(hu_x{}^2+\frac{gh^2}{2}\right)}{\partial y}=hf_y-gh\frac{\partial S}{\partial y}. \tag{3}$$

Here $h(x,y,t)$ is the height of the liquid level above the bottom profile $S(x,y)$, u_x and u_y are velocity components, g is gravity, f_x and f_y are external force components. For flows over a flat surface in the absence of external forces, the system of equations (1) – (3) is called the Saint-Venant equations.

There are a huge number of different methods for solving the shallow Saint-Venant equation. Among them, one can single out such methods as the method of difference schemes, the finite element method, the finite volume method, and others. Until now, the development of new approximation methods for solving such problems continues. This is due to the relevance of the tasks being considered. The main difficulty here consists in obtaining a stable difference solution to the problem. Such problems require special quality requirements for the numerical methods used.

In this paper, we give a brief overview of modern research on the properties of unsteady flows in open flows described by the system of Saint-Venant equations.

3 Literature review

In [1], an implicit difference grid scheme was developed for solving one-dimensional equations of unsteady motion in open rivers, which allows calculations with a large time step. This was especially important for the calculation of floods in large rivers when the duration of the computation process took a long time. The compiled program was widely used at the

State Hydrological Institute of the USSR for numerical experiments and calculations of real objects. However, there is no description of the algorithm and no substantiation of the finite difference method itself.

In [2], a new and simpler mathematical statement of the problem of the wave motion along a dry channel was proposed. In this work, it is assumed that the Saint-Venant equations are valid for the entire flow region, and the boundary conditions at the wavefront are obtained by equating the front propagation speed and the flow velocity near the wave head. A difference scheme is also proposed, which allows to make calculations of unsteady motion in complex channel systems.

In [3], a method was developed for calculating the motion of discontinuous waves in non-prismatic channels with allowance for friction. The numerical method of calculation was based on the representation of Saint-Venant's equations in the so-called form of conservation laws and the use of a difference scheme with recalculation (on the motion of a discontinuous wave without isolating a discontinuity). To calculate the propagation of the discontinuous wave with the separation of the discontinuity, a movable grid was used, which is constructed in the course of the calculation. The application of this method showed satisfactory agreement with the experimental results.

Since 1970, grid methods have begun to appear for numerically solving Saint-Venant's equations. The disadvantage of the method was that the calculation required a large amount of computer time, although the computers of that time worked much slower than modern computers.

The work [4] is devoted to the problem of exponential stability of the solution of nonlinear Saint-Venant equations in differential form. The paper considers the general case when the system includes both arbitrary friction and a slope that changes in space, which leads to non-uniform steady states. An explicit quadratic Lyapunov function is constructed and local exponential stability is proved.

The monograph [5] is devoted to studying mixed problems for one-dimensional hyperbolic systems in canonical form. Lyapunov stability has been established in various functional spaces, in particular, numerous practical models have been considered. The issues of numerical solutions of mixed problems are not considered.

In [6], a discrete Lyapunov function was constructed for the telegraph equation, and its decrease was proved. The use of this approach for the San Venant equation is associated with difficulties that require additional research.

For hyperbolic equations with dissipative boundary conditions the exponential stability of the solution is established by the Lyapunov function method in [7].

In [8], algebraic conditions for the exponential stability of the solution of mixed problems of the linear Saint-Venant equations were obtained. The issue of the numerical solution is not considered.

For one-dimensional quasilinear hyperbolic systems, the problems with dissipative boundary conditions that guarantee the exponential stability of classical solutions are considered in [9]. There is no research on numerical calculations.

In [10], using the Volterra transform of the second kind and the invertible Fredholm transform, optimal control problems for general linear hyperbolic Qsystems are investigated.

We note that the papers [4], [5], [7]- [10] study questions related to the theoretical aspects of the solvability and stability of mixed problems for hyperbolic systems and do not consider the issues of constructing numerical solutions and the stability of difference schemes. In the numerical calculation of mixed problems for hyperbolic systems, the reason for the above is that the dimension of the system of linear algebraic equations increases with the size of the considered region. This leads to an unreasonably large volume of computations and requires the involvement of high-performance computing technology.

In [11], the authors propose a class of difference schemes for hyperbolic systems of equations that have several notation forms. The stability of the proposed difference schemes is investigated using the energy integrals technique. However, their application for the study of exponential stability is a rather difficult task.

The work [12] is devoted to the study of initial-boundary value problems for a class of quasilinear hyperbolic systems. An a priori estimate is obtained for the solution of the problem by the method of the integral of energy. The issues of the numerical solution and its stability are not studied.

In [13], a linear initial-boundary value problem of the dynamics of fluid-saturated porous media, described by three elastic parameters in a reversible hydrodynamic approximation, was numerically solved. The adequacy of the computational model remains open.

The book [14] is devoted to methods for solving high-order algebraic systems arising from the application of the grid method to problems of mathematical physics. Along with the iterative methods, which are most widely used in computational practice for solving these problems, direct methods are also presented.

Much progress has been made in the study of general properties and patterns of modeling shallow water in systems with open channels and reservoirs. However, the study of specific hydrological objects is important. Modelling river systems and reservoirs seem to be the most difficult. As positive examples of constructing multidimensional models for this kind of object, let us point out the lower course of the river Bureya beyond the Bureyskaya hydroelectric power station [15], as well as others.

Formally, Saint-Venant's equations are valid for the case when the height of the liquid level is much less than the characteristic dimensions of the problem, and the bottom shape is a sufficiently smooth function. The problem with a discontinuous bottom profile is being intensively studied in the framework of the Saint-Venant approximation. The construction of analytical solutions for such problems is rather laborious even for plane one-dimensional flows, and there is extensive literature on these issues [16]- [17]. In [18], the author constructed analytical solutions of the Saint-Venant equations for five characteristic problems of the decay of a discontinuity over a step and bottom step. Analytical solutions are used both to assess situations that arise in several practical cases and to test numerical algorithms aimed at calculating such flows. Such currents include, in particular, currents on the thresholds of sluices, currents when sluice gates are destroyed or when overflowing through the crest of a dam, currents in narrow sea straits with a complex bottom shape, and some other problems.

Difficulties in the numerical simulation of such flows are caused by the appearance of a complex configuration of discontinuities in the solution, caused both by the nonlinearity of the equations themselves and by the discontinuous profile of the underlying surface. In [19], methods are proposed to overcome these problems by isolating the discontinuity line associated with the position of the boundary of a step or step and modifying the system of Saint Venant equations. Using this approach makes the numerical algorithm more accurate, but deprives it of homogeneity. The latter is not always convenient when calculating practical problems. Another way to solve this problem is to use two-layer shallow water equations [20]. The construction of a homogeneous numerical algorithm for solving problems with bottom discontinuities also seems to be relevant.

In [21], based on the laws of conservation of momentum and mass of a liquid, a derivation of the hypothetical equations of Saint-Venant from the Navier-Stokes equations is given. When deriving the Saint-Venant equations, the authors used assumptions (the length of the watercourse is much greater than its depth and width, the channel is elongated in a straight line, the bottom slope is small, etc.).

In [22], a central-upwind difference scheme was proposed using the two-dimensional central approach developed in [23]. The simplest semi-discrete schemes in the center upstream were obtained in [24]- [26].

In the monograph [27], effective algorithms for the numerical solution of shallow water equations are presented, of which, first of all, it is necessary to single out the method of decay of an arbitrary discontinuity taking into account the discontinuous bottom, which ensures the existence and uniqueness of the solution for any initial data, as well as an algorithm for the equations of viscous shallow water and non-negative "algorithm for equations in the diffusion approximation. Thoroughly performed testing demonstrates the high accuracy and reliability of the proposed methods.

In [28], explicit schemes of the Lax-Wendroff and McCormack type of the second order of approximation were used for the numerical solution of the shallow water equations.

A numerical algorithm for solving problems of regulated shallow water equations on unstructured grids is described in [29]. In the above equations of shallow water, a small value of τ is introduced, which is called the regularization parameter or temporal smoothing and has the dimension of time. The terms with the coefficient τ are the regulating additives to the shallow water equations. The algorithm was tested and it was shown that the constructed solution is in good agreement with the calculation results by known numerical methods.

In [30], [31], a new numerical method for solving the shallow water equation was proposed and tested, based on the smoothing of the classical equations over some small time intervals. This procedure leads to the appearance of regulatory additives that ensure the stability of the numerical solution of the problem in a wide range of Froude numbers. This makes it possible to use the non-difference grid approximation and apply the flow form of equations without linearizing the original equations, which ensures strict observance of the laws of conservation of mass and momentum in the absence of external forces. This algorithm is universal for solving a wide class of problems; it allows us to calculate flows with moving areas of a dry bottom. Moreover, it is easy to parallelize and generalize to unstructured mesh design.

For the numerical solution of the shallow water equation, the regularization method, that is, the averaging of the equations over a certain short time interval was used in [32].

In [33], the long-range transport of impurities along the length of the Novosibirsk reservoir is investigated. The computational algorithm is constructed using a different scheme of increased accuracy for modeling the long-range transport of an impurity. It should be taken into account that a formal increase in the order of approximation of the advective terms of the equations leads to oscillating numerical solutions in the aeas of sharp changes in gradients. This led to the development of numerical algorithms adaptive to the solution, which ensure the preservation of the monotonicity of the numerical solution and a high order of numerical schemes.

In [34], using the Lax-Wendroff scheme as an example, it is shown that the main reason for the reduction inaccuracy to the first order and below in TVD schemes in the calculation is that monotonicity in them is achieved by using various minimax procedures, which reduce the smoothness of the difference operators of flows. It is shown theoretically and numerically that the Lax-Wendroff scheme, in contrast to its TVD modifications, approximates the ε-Hugoniot conditions with the second-order at the fronts of non-stationary shock waves. At the same time, the Lax-Wendroff scheme reduces the order of convergence to the first one in the vicinity of the gradient catastrophe point, in which the discontinuous wave is formed. This decrease in convergence happens because this scheme, in contrast to compact schemes with artificial viscosities of a higher order of divergence, has only the first order of weak approximation on discontinuous solutions. In [35], 2D models of the Saint-Venant equation are considered and devoted to surface flows to study the behavior of flood waves. Open channel water runoff in drains and rivers is considered taking into account the fact that such streams are a source of flash floods. To predict and simulate flood behavior, a mathematical model is established with initial and boundary conditions using Saint-Venant 2D PDEs. The corresponding model is then discretized using an explicit finite difference method and implemented in MATLAB. For testing and implementation, a simple rectangular flow channel is considered.

4 Results and discussion

The first computer algorithms are associated with the processing of techniques and methods used in manual calculations. The method of explicit difference schemes and the method of characteristics require too much computation, especially when modeling river flows in channels with a complex form for calculations, which makes them ineffective. We should note that the shortcomings in the computations of riverbeds with complex shapes were associated with significant difficulties and led to an unjustifiably large amount of calculations.

The transition to the study of unsteady currents in rivers in the presence of discontinuities (discontinuous waves) naturally brings about new difficulties. Therefore, the first works in this direction were associated with several very serious simplifying assumptions.

Despite the successes achieved in this area, several problems remain unresolved here, and little attention is paid to some areas of research. Thus, in essence, there are no reliable methods for calculating the movement of the flood flow when the water leaves the floodplain. Methods for calculating the propagation of discontinuous waves in channels with a complex outline, especially natural ones, need further improvement.

However, the simulation results are useful for understanding and predicting flood behavior at various locations in the flow channel at specific time intervals and can be useful in early warning systems for floods. It is also suggested that combining an underground flow with a surface flow may provide an even better approximation for flood circulation.

5 Conclusion

It can be said that the problem of unsteady movement in open channels has been and is being given great attention, as evidenced by several reports and scientific works. But, the theory of unsteady flows in eroded channels is being developed relatively poorly. The issues of exponential stability of numerical solutions to the shallow water equation, etc., have not been investigated.

In conclusion, we note that this review does not claim to be complete and deals mainly with the computational aspects of various problems for the Saint-Venant equation.

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References

[1] Vasil'yev O.F., Godunov S.K., Pritvits N.A., Temnoyeva T.A., Fryazinova I.L., Shugrin S.M. Chislennyy metod rascheta rasprostraneniya dlinnykh voln v otkrytykh ruslakh i prilozheniye yego k zadache o pavodke[A numerical method for calculating the propagation of long waves in open channels and its application to flood problem] // Reports of the USSR Academy of Sciences. - 1963. - №151(3). - C. 525-527

[2] Shugrin S.M. O neodnorodnykh raznostnykh skhemakh[On inhomogeneous difference schemes] // Journal of Computational Mathematics and Mathematical Physics. - 1966. - T. 6, № 1. - C. 184-185. Q

[3] Vasil'yev O.F., Gladyshev M.T., Sudobicher V.G. Chislennoye resheniye zadach o teche niyakh s preryvnymi volnami v otkrytykh ruslakh[Numerical solution of problems on flows with discontinuous waves in open channels] // Numerical methods of continuum mechanics. – 1970. – T. 1. – № 5. – C. 3

[4] Hayat A., Shang P. A quadratic lyapunov function for saint-venant equations with arbitrary friction and space-varying slope // Automatica. - 2019. - V. 100. - P. 52-60.

[5] Bastin G., Coron J.M. Stability and Boundary Stabilization of 1-D Hyperbolic Systems. - Itemirkhauser Basel. - 2016. - 307 p.

[6] GoEttlich S., Schillen P. Numerical discretization of boundary control problems for systems of balance laws: Feedback stabilization // European Journal of Control. - 2017. - V. 35. - P. 11-18.

[7] Bastin G., Coron J.M. A quadratic Lyapunov function for hyperbolic density–velocity systems with nonuniform steady states // Systems & Control Letters. - 2017. - V. 104. - P. 66-71.

[8] Bastin G., Coron J.M., Novel, B.D. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel // Networks and Heterogeneous Media. - 2009. - V. 4. - P. 177-187.

[9] Coron J.M., Bastin G. Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the C^1-norm // SIAM Journal on Control and Optimization. - 2015. - V. 53, № 3. - P. 1464-1483.

[10] Coron J.M., Hu Long, Olive G. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation // Automatica Journal IFAC. - 2017. - V. 84. - P. 95-100.

[11] Blokhin A.M., Aloev R.D., Hudayberganov M.U. One Class of Stable Difference Schemes for Hyperbolic System // American Journal of Numerical Analysis. - 2014. - V. 6, № 3. - P. 85-89.

[12] Aloev R.D., Khudoyberganov M. U., Blokhin A.M. Construction and research of adequate computational models for quasilinear hyperbolic systems // Numerical Algebra, Control, and Optimization. - 2018. - V. 8, № 3. - P. 287-299.

[13] Berdyshev A.S., Imomnazarov Kh.Kh., Jian-Gang T., Mikhailov A. The Laguerre spectral method is applied to the numerical solution of a two-dimensional linear dynamic seismic problem for porous media // Open Computer Science. - 2016. - V. 1. - P. 202-212.

[14] Samarskiy A.A., Nikolayev Ye.S. Metody resheniya setochnykh uravneniy [Methods for solving grid equations]. - Наука. - 1978. - 532 c.

[15] Klimovich V.I., Petrov O.A. Chislennoye modelirovaniye techeniy pri rabote vodoslivnoy plotiny Bureyskoy GES[Numerical modeling of currents during the operation of the spillway dam of the Bureyskaya HPP] // Izvestia of the All-Russian Scientific Research Institute of Hydraulic Engineering named after V.I. B.E. Vedeneeva. - 2012.— T. 266. — C. 22–37.

[16] Alcrudoa F., Benkhaldoun F. Exact solutions to the Riemann problem of the shallow water equations with a bottom step // Computers & Fluids. - 2001. - V. 30, № 6. - P. 643–671

[17] Han E., Warnecke G. Exact Riemann solutions to shallow water equations // Quarterly of Applied Mathematics. - 2014. - V. 72, № 3. - P. 1-44

[18] Bulatov O.V. Analiticheskiye i chislennyye resheniya uravneniy Sen-Venana dlya nekotorykh zadach o raspade razryva nad ustupom i stupen'koy dna[Analytical and numerical solutions of the Saint-Venant equations for some problems on the decay of a discontinuity over a step and bottom step] // Journal of Computational Mathematics and Mathematical Physics. - 2014. - T. 54, № 1. - C. 149-163

[19] Belikov V.V., Borisova N.M., Ostapenko V.V. Sovershenstvovaniye metodov chislennogo modelirovaniya gidrotekhnicheskikh sooruzheniy s rezkimi perepadami otmetok dna[Improvement of methods for numerical modeling of hydraulic structures with sharp drops in bottom elevations] // Safety of energy structures. - 2007. - T.16. -C. 79–89.

[20] Petrosyan A.S. Dopolnitel'nyye glavy gidrodinamiki tyazheloy zhidkosti so svobodnoy granitsey[Additional chapters of the hydrodynamics of a heavy fluid with a free boundary]. - Rotaprint IKI RAS. - 2010. - 65 c.

[21] Abduraimov M.G., Muzafarov KH.A., Puttiyev A.A. Dvizheniye vod v otkrytykh ruslakh (uravneniya Sen-Venana)[Abduraimov M.G., Muzafarov Kh.A., Puttiev A.A. Water movement in open channels (Saint-Venant equations)] // Math modeling. - 1998. - T. 10, № 6. -C. 97-106.

[22] Kurganov A. Finite-volume schemes for shallow-water equations // Acta Numerica. - 2008. - V. 27, № 6. -P. 289-351.

[23] Jiang G.S., Tadmor E. Nonoscillatory central schemes for multidimensional hyperbolic conservation laws // SIAM J. Sci. Comput. - 1998. - V. 19. -P. 1892-1917.

[24] Kurganov A., Levy D. Central-upwind schemes for the Saint-Venant system // M2AN Math. Model. Numer. Anal. - 2002. - V. 36. -P. 397-425.

[25] Kurganov A., Tadmor E. New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations // J. Comput. Phys. - 2000. - V. 160. -P. 241-282.

[26] Shirkhani H., Mohammadian A., Seidou O., Kurganov A. A well-balanced positivity-preserving central-upwind scheme for shallow water equations on unstructured quadrilateral grids // Comput. & Fluids. - 2016. - V. 126. -P. 25-40.

[27] Belikov V.V., Aleksyuk A.I. Modeli melkoy vody v zadachakh rechnoy gidrodinamiki[Shallow water models in problems of river hydrodynamics]. - RAS. - 2020. - 345 c.

[28] Yan M., Cundy T.W. Modeling of Two-Dimensional Overland Flow // Water Resour.Res. - 1989. - V. 25, № 9. -P. 2019-2035.

[29] Yelizarova T.G., Bulatov O.V. Chislennyy algoritm resheniya regulyarizovannykh uravneniy melkoy vody na nestrukturirovannykh setkakh[Numerical algorithm for solving regularized shallow water equations on unstructured grids]. - Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences. - 2014. - 25 c.

[30] Bulatov O.V., Yelizarova T.G. Regulyarizovannyye uravneniya melkoy vody i effektivnyy metod chislennogo modelirovaniya techeniy v neglubokikh vodoyemakh[Regularized shallow water equations and an effective method for numerical simulation of currents in shallow water bodies] // Journal of Computational Mathematics and Mathematical Physics. - 2011. - T. 51, № 1.- C. 7-184.

[31] Yelizarova T.G., Ivanov A.V. Chislennyy algoritm resheniya regulyarizovannykh uravneniy melkoy vody na nestrukturirovannykh setkakh[Numerical algorithm for solving regularized shallow water equations on unstructured grids]. - Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences. - 2016. - 27 c.

[32] Kvon V.I., Kvon D.V., Zonov S.D., Karamyshev V.B. Chislennyy raschet techeniy i dal'nego perenosa primesi v ravninnykh rechnykh vodokhranilishchakh [Numerical calculation of currents and long-range transport of impurities in lowland river reservoirs] // Applied Mechanics and Technical Physics. - 2003. - T. 44, № 6.- C. 158-163.

[33] Zonov S.D., Kvon D.V., Kvon V.I. Chislennoye modelirovaniye protsessov perenosa primesi v pribrezhnoy zone vodokhranilishcha[Numerical modeling of pollutant transport processes in the coastal zone of the reservoir] // Environmental analysis of the region (theory, methods, practice). -2000. -C. 212–220.

[34] Kamboh S.A., Izzatul N.S, Labadin J., Monday O.E. Simulation of 2D Saint-Venant equations in an open channel by using MATLAB // Journal of IT in Asia. - 2015. - V. 5, № 1. - P. 15–22.

[35] Blokhin A.M., Alayev R.D. Integraly energii i ikh prilozheniya k issledovaniyu ustoychivosti raznostnykh skhem[Energy integrals and their applications to the study of the stability of difference schemes]. - Novosibirsk University Publishing House. - 1993. - 224 c.

Список литературы

[1] Васильев О.Ф., Годунов С.К., Притвиц Н.А., Темноева Т.А., Фрязинова И.Л., Шугрин С.М. Численный метод расчета распространения длинных волн в открытых руслах и приложение его к задаче о паводке // Доклады АН СССР. - 1963. - №151(3). - С. 525-527

[2] Шугрин С.М. О неоднородных разностных схемах // Журнал вычислительной математики и математической физики. - 1966. - Т. 6, № 1. - С. 184-185. Q

[3] Васильев О.Ф., Гладышев М.Т., Судобичер В.Г. Численное решение задач о тече ниях с прерывными волнами в открытых руслах // Численные методы механики сплошной среды. – 1970. – Т. 1. – № 5. – С. 3

[4] Hayat A., Shang P. A quadratic lyapunov function for saint-venant equations with arbitrary friction and space-varying slope // Automatica. - 2019. - V. 100. - P. 52-60.

[5] Bastin G., Coron J.M. Stability and Boundary Stabilization of 1-D Hyperbolic Systems. - Itemirkhauser Basel. - 2016. - 307 p.

[6] GoEttlich S., Schillen P. Numerical discretization of boundary control problems for systems of balance laws: Feedback stabilization // European Journal of Control. - 2017. - V. 35. - P. 11-18.

[7] Bastin G., Coron J.M. A quadratic Lyapunov function for hyperbolic density–velocity systems with nonuniform steady states // Systems & Control Letters. - 2017. - V. 104. - P. 66-71.

[8] Bastin G., Coron J.M., Novel, B.D. On Lyapunov stability of linearised Saint-Venant equations for a sloping channel // Networks and Heterogeneous Media. - 2009. - V. 4. - P. 177-187.

[9] Coron J.M., Bastin G. Dissipative boundary conditions for one-dimensional quasi-linear hyperbolic systems: Lyapunov stability for the C^1-norm // SIAM Journal on Control and Optimization. - 2015. - V. 53, № 3. - P. 1464-1483.

[10] Coron J.M., Hu Long, Olive G. Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation // Automatica Journal IFAC. - 2017. - V. 84. - P. 95-100.

[11] Blokhin A.M., Aloev R.D., Hudayberganov M.U. One Class of Stable Difference Schemes for Hyperbolic System // American Journal of Numerical Analysis. - 2014. - V. 6, № 3. - P. 85-89.

[12] Aloev R.D., Khudoyberganov M. U., Blokhin A.M. Construction and research of adequate computational models for quasilinear hyperbolic systems // Numerical Algebra, Control, and Optimization. - 2018. - V. 8, № 3. - P. 287-299.

[13] Berdyshev A.S., Imomnazarov Kh.Kh., Jian-Gang T., Mikhailov A. The Laguerre spectral method is applied to the numerical solution of a two-dimensional linear dynamic seismic problem for porous media // Open Computer Science. - 2016. - V. 1. - P. 202-212.

[14] Самарский А.А., Николаев Е.С. Методы решения сеточных уравнений. - Наука. - 1978. - 532 с.

[15] Климович В.И., Петров О.А. Численное моделирование течений при работе водосливной плотины Бурейской ГЭС // Известия Всероссийского научно-исследовательского института гидротехники им. Б.Е. Веденеева. - 2012.— Т. 266. — С. 22–37.

[16] Alcrudoa F., Benkhaldoun F. Exact solutions to the Riemann problem of the shallow water equations with a bottom step // Computers & Fluids. - 2001. - V. 30, № 6. - P. 643–671

[17] Han E., Warnecke G. Exact Riemann solutions to shallow water equations // Quarterly of Applied Mathematics. - 2014. - V. 72, № 3. - P. 1-44

[18] Булатов О.В. Аналитические и численные решения уравнений Сен-Венана для некоторых задач о распаде разрыва над уступом и ступенькой дна // Журнал вычислительной математики и математической физики. - 2014. - Т. 54, № 1. - С. 149-163

[19] Беликов В.В., Борисова Н.М., Остапенко В.В. Совершенствование методов численного моделирования гидротехнических сооружений с резкими перепадами отметок дна // Безопасность энергетических сооружений. - 2007. - Т.16. -С. 79–89.

[20] Петросян А.С. Дополнительные главы гидродинамики тяжелой жидкости со свободной границей. - Ротапринт ИКИ РАН. - 2010. - 65 с.

[21] Абдураимов М.Г., Музафаров Х.А., Путтиев А.А. Движение вод в открытых руслах (уравнения Сен-Венана) // Математическое моделирование. - 1998. - Т. 10, № 6. -С. 97-106.

[22] Kurganov A. Finite-volume schemes for shallow-water equations // Acta Numerica. - 2008. - V. 27, № 6. -P. 289-351.

[23] Jiang G.S., Tadmor E. Nonoscillatory central schemes for multidimensional hyperbolic conservation laws // SIAM J. Sci. Comput. - 1998. - V. 19. -P. 1892-1917.

[24] Kurganov A., Levy D. Central-upwind schemes for the Saint-Venant system // M2AN Math. Model. Numer. Anal. - 2002. - V. 36. -P. 397-425.

[25] Kurganov A., Tadmor E. New high-resolution central schemes for nonlinear conservation laws and convection-diffusion equations // J. Comput. Phys. - 2000. - V. 160. -P. 241-282.

[26] Shirkhani H., Mohammadian A., Seidou O., Kurganov A. A well-balanced positivity-preserving central-upwind scheme for shallow water equations on unstructured quadrilateral grids // Comput. & Fluids. - 2016. - V. 126. -P. 25-40.

[27] Беликов В.В., Алексюк А.И. Модели мелкой воды в задачах речной гидродинамики. - РАН. - 2020. - 345 с.

[28] Yan M., Cundy T.W. Modeling of Two-Dimensional Overland Flow // Water Resour.Res. - 1989. - V. 25, № 9. -P. 2019-2035.

[29] Елизарова Т.Г., Булатов О.В. Численный алгоритм решения регуляризованных уравнений мелкой воды на неструктурированных сетках. - ИПМ им. М. В. Келдыша РАН. - 2014. - 25 с.

[30] Булатов О.В., Елизарова Т.Г. Регуляризованные уравнения мелкой воды и эффективный метод численного моделирования течений в неглубоких водоемах// Журнал вычислительной математики и математической физики. - 2011. - Т. 51, № 1.- С. 7-184.

[31] Елизарова Т.Г., Иванов А.В. Численный алгоритм решения регуляризованных уравнений мелкой воды на неструктурированных сетках. - ИПМ им. М. В. Келдыша РАН. - 2016. - 27 с.

[32] Квон В.И., Квон Д.В., Зонов С.Д., Карамышев В.Б. Численный расчет течений и дальнего переноса примеси в равнинных речных водохранилищах // Прикладная механика и техническая физика. - 2003. - Т. 44, № 6.- С. 158-163.

[33] Зонов С.Д., Квон Д.В., Квон В.И. Численное моделирование процессов переноса примеси в прибрежной зоне водохранилища // Экологический анализ региона (теория, методы, практика). -2000. -С. 212–220.

[34] Kamboh S.A., Izzatul N.S, Labadin J., Monday O.E. Simulation of 2D Saint-Venant equations in an open channel by using MATLAB // Journal of IT in Asia. - 2015. - V. 5, № 1. - P. 15–22.

[35] Блохин А.М., Алаев Р.Д. Интегралы энергии и их приложения к исследованию устойчивости разностных схем. - Издательство Новосибирского университета. - 1993. - 224 с.

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