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ON BOUNDED SOLUTIONS OF DIFFERENTIAL SYSTEMS

The question of the existence of bounded solutions on an infinite interval of a linear inhomogeneous system of ordinary differential equations in a finite-dimensional space is considered. The study of bounded solutions of systems of ordinary differential equations is one of the most important problems in the qualitative theory of differential equations. In the study of the asymptotic behavior of solutions to differential systems, the works of A. Poincaré and A.M. Lyapunov. Various conditions for the existence of bounded solutions of a linear system of ordinary differential equations have been obtained by many authors. Note the works of O. Perron, A. Walter, H. Shpet, D. Caligo, N.I. Gavrilova, M. Hukukara, M. Nagumo, M. Caratheodori, U. Barbouti, N.Ya. Lyashchenko, B.P. Demidovich, A. Wintner, R. Bellman, Yu.S. Bogdanov, Z. Vazhevsky, N. Levinson, M. Markus, L. Cesari and others. In this paper, we establish sufficient conditions for the boundedness of all solutions of a linear inhomogeneous system of differential equations on an infinite interval. A coefficient criterion for the boundedness of all solutions on an infinite interval of a linear inhomogeneous system of differential equations in a certain class of differential systems is given. Applied methods of differential equations and function theory. The results obtained are used in applications of differential equations and are of practical value.
Key words: solution, boundedness, system, linear, differential equation.

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Дифференциалдық жүйелердің шектеулі шешімдері туралы

Ақырлы өлшемді кеңістіктегі қарапайым дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің шексіз аралықта шектелген шешімдердің болуы туралы мәселе қарастырылады. Қарапайым дифференциалдық теңдеулер жүйесінің шектеулі шешімдерін зерттеу дифференциалдық теңдеулердің сапалық теориясының маңызды мәселелерінің бірі болып табылады. Дифференциалдық жүйелерге арналған шешімдердің асимптотикалық мінез-құлқын зерттеу барысында А. Пуанкаре мен А.М. Ляпуновтың жұмыстары негізін қалаушы болып табылады. Қарапайым дифференциалдық теңдеулердің сызықтық жүйесінің шектелген шешімдерінің болуының әр түрлі шарттары көптеген авторлармен алынған. О. Перрон, А. Вальтер, Х. Шпет, Д. Калиго, Н.И. Гаврилова, М. Хукукара, М. Нагумо, М. Каратеодори, У. Барбути, Н.Я. Лященко, Б.П. Демидович, А. Винтнер, Р. Беллман, Ю.С. Богданов, З. Важевский, Н. Левинсон, М. Маркус, Л. Сезари және т.б. Бұл жұмыста біз дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің барлық шешімдерінің шексіз аралықта шектелуіне жеткілікті шарттар орнатамыз. Дифференциалдық жүйелердің белгілі бір класындағы дифференциалдық теңдеулердің сызықтық біртекті емес жүйесінің шексіз аралықтағы барлық шешімдердің шектелуінің коэффициент критерийі келтірілген. Дифференциалдық теңдеулер мен функциялар теориясының әдістері қолданылған. Алынған нәтижелер дифференциалдық теңдеулерді қолдану кезінде қолданылады және практикалық маңызы бар.
Түйін сөздер: шешім, шектілік, жүйе, сызықтық, дифференциалдық теңдеу.

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Об ограниченных решениях дифференциальных систем

Рассматривается вопрос существования ограниченных решений на бесконечном интервале линейной неоднородной системы обыкновенных дифференциальных уравнений в конечномерном пространстве. Изучение ограниченных решений систем обыкновенных дифференциальных уравнений является одной из важнейших задач качественной теории дифференциальных уравнений. В исследовании асимптотического поведения решений дифференциальных систем основополагающими являются работы А. Пуанкаре и А.М. Ляпунова. Разнообразные условия существования ограниченных решений линейной системы обыкновенных дифференциальных уравнений получены многими авторами. Отметим работы О. Перрона, А. Вальтера, Х. Шпета, Д. Калиго, Н.И. Гаврилова, М. Хукукара, М. Нагумо, М. Каратеодори, У. Барбути, Н.Я. Лященко, Б.П. Демидовича, А. Винтнера, Р. Беллмана, Ю.С. Богданова, З. Важевского, Н. Левинсона, М. Маркуса, Л. Чезари и другие. В данной работе установлены достаточные условия ограниченности всех решений линейной неоднородной системы дифференциальных уравнений на бесконечном интервале. Приведен коэффициентный признак ограниченности всех решений на бесконечном интервале линейной неоднородной системы дифференциальных уравнений в определенном классе дифференциальных систем. Примененяюся методы дифференциальных уравнений и теории функций. Полученные результаты находят применения в приложениях дифференциальных уравнений и представляет собой, практическую ценность.

Ключевые слова: решение, ограниченность, система, линейная, дифференциальное уравнение.

1 Introduction

The question of the existence of bounded solutions of differential systems on an infinite interval is considered. The study of bounded solutions of systems of ordinary differential equations is one of the most important problems in the qualitative theory of differential equations. In the study of the asymptotic behavior of solutions of differential systems, the works of A. Poincaré [1] and A.M. Lyapunov [2]. Conditions for the existence of bounded solutions of a linear system of ordinary differential equations were obtained by the authors: Dunkel O., Hukuhara M., Nagumo M., Caccioppoli R., Caratheodory M., Dini U., Spath H., Weyl H., Wiman A., Ascoli G ., Gavrilov N.I., Gusarova R.S., Conti R., Barbuti U., Lyashchenko N.Ya., Demidovich B.P., Faedo S., Wilkins I.E, Ghizzetti A., Sobol I.M. , Haupt O., Boas M., Boas R.P., Wintner A., Bellman R., Bogdanov Yu.S., Butlewski Z., Bylov B.F., Wazewski T., Walter A., Caligo D., Kitamura T., Landau E., Levinson N., Marcus M., Perron O., Cesari L., Spath H., Shtokalo I.Z., Sobol I.M., et al. For detailed references, see the book by Cesari L. [3]. General information is available in the books: [4] V.V. Nemytsky and V.V. Stepanov, [5] Erugin N.P., [6] Sansone G., [7] Pliss V.A., [8] Bylov B.F., Vinograd R.E., Grobman D.M., Nemytskiy V.V., [9] Izobov N.A., [10] Coddington E.A. and Levinson N., [11] Demidovich B.P., [12] Lefschetz S., [13] Massera H.L., Scheffer H.H., [14] Bellman R., [15] Coppel W.A., [16] Daletskiy Yu.L., Kerin M.G. We also note the works: [17–20] Wintner A., [21] Yoshizawa T., [22] Bihari I., [23] Hartman Ph., [24] Hale J., Onuchic N.

In this paper, sufficient conditions are established boundedness of all solutions of a linear inhomogeneous system of differential equations on an infinite interval. A coefficient criterion

for the boundedness on an infinite interval of all solutions of a linear inhomogeneous system of differential equations in a certain class of differential systems is given. Applied methods of differential equations and function theory. The results obtained are used in applications of differential equations and are of practical value.

2 Materials and research methods

A linear inhomogeneous system of differential equations is considered

$$\dot{x} = A(t)x + f(t) \tag{1}$$

where

$$t \in I \equiv (1, +\infty), \quad A(t) \in (I), \quad f(t) \in (I);$$

and the corresponding linear homogeneous system of differential equations

$$\dot{x} = A(t)x \tag{2}$$

Theorem 1 *If the conditions*

$$\|A(t)\| \leq K\gamma t^{\gamma-1}, \quad 0 < \gamma < 1, \quad K > 0, \quad \|f(t)\| \leq K\gamma t^{\gamma-1}, \quad t \in [t_0, +\infty); \quad t_0 \in I;$$

and the linear homogeneous system (2) *is generalized correct, has negative upper generalized Lyapunov exponents with respect to* t^γ*, then any solution to the linear inhomogeneous system of differential equations* (1) *on the interval* $[t_0, +\infty)$ *limited.*

Proof. From (1) multiplying scalarly by $x(t)$ get

$$(x', x) = (A(t)x, x) + (f(t), x). \tag{3}$$

From (3) get

$$|(x', x) \leq |(A(t)x, x)| + |(f(t), x)|. \tag{4}$$

From (4) get

$$|(x', x)| \leq \|A(t)\|\|x\|^2 + \|f(t)\|\|x\|. \tag{5}$$

From (5) get

$$-\|A(t)\|v^2 - \|f(t)\|v \leq v'\|A(t)\|v^2 + \|f(t)\|v \tag{6}$$

where $v(t) = \|x(t)\|$. From (6) get

$$de^{-Kt^\gamma} \leq \|x(t)\| \leq De^{Kt^\gamma} \tag{7}$$

where $d > 0$, $D > 0$. From (7) we obtain that any nonzero solution to the linear inhomogeneous system (1) has a finite upper generalized Lyapunov exponent with respect to t^γ. In the linear homogeneous system (2) we take the largest upper generalized Lyapunov

exponent $\lambda_1 < 0$. Let's take $\alpha \in (0, |\lambda_1|)$) and in the linear inhomogeneous system of differential equation (1) we perform the transformation

$$x = ye^{\alpha t^\gamma}, \quad x(t_0) = y(t_0). \tag{8}$$

Then from (1) we obtain

$$\dot{y} = [A(t) + \alpha\gamma t^{\gamma-1}E]y + e^{\alpha t^\gamma} f(t) \tag{9}$$

where the corresponding linear homogeneous system of differential equations

$$\dot{y} = [A(t) + \alpha\gamma t^{\gamma-1}E]y \tag{10}$$

is generalized correct and has negative upper generalized Lyapunov exponents.

From the linear system of differential equations (9) we obtain

$$y(t) = Y(t, t_0)y(t_0) + \int_{t_0}^{t} Y(t, s)e^{\alpha s^\gamma} f(s)ds \tag{11}$$

where $Y(t, t_0) = Y(t)Y^{-1}(t_0)$ – the Cauchy matrix of a linear homogeneous system of differential equations (10). By virtue of the generalized correctness of the linear system (10), for any $\varepsilon \in (0, |\alpha|)$ exists $D_\varepsilon(t_0) > 0$ and the inequality

$$\|Y(t, \tau)\| \le D_\varepsilon(t_0)e^{\varepsilon\tau^\gamma} \tag{12}$$

at $t \ge \tau \ge t_0$. From (11), (12) get

$$\|y(t)\| \le D_\varepsilon(t_0)\|y(t_0)\| + \int_{t_0}^{t} D_\varepsilon(t_0)e^{\varepsilon s^\gamma} K\gamma s^{\gamma-1}ds. \tag{13}$$

From (8), (13) get

$$\|x(t)\| \le e^{-\alpha t^\gamma} D_\varepsilon(t_0)\left(\|x(t_0)\| + K\frac{e^{\varepsilon t^\gamma}}{\varepsilon}\right). \tag{14}$$

From (14), using arbitrary smallness ε, directing $\varepsilon \to 0$ get

$$\|x(t)\| \le e^{-\alpha t^\gamma} D_\varepsilon(t_0)(\|x(t_0)\| + Kt^\gamma) \tag{15}$$

at $t \ge t_0$.

It follows from (15) that any solution of the linear inhomogeneous system of differential equations (1), on the interval $[t_0, +\infty)$ limited. Theorem 1 is proved.

Consider a linear inhomogeneous system of differential equations

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_{ik}(t)y_i + f_i(t), \quad i = \overline{1, n}; \tag{16}$$

where $p_{ik}(t)$, $f_i(t)$, $i = \overline{1, n}$; $k = \overline{1, n}$; continuous real functions on the interval $(1, +\infty)$, $t_0 \in (1, +\infty)$.

Theorem 2 *If for $1 < t_0 \leq t$ conditions are met:*

1) $p_{i-1,i-1}(t) \geq p_{ii}(t) + \beta\gamma t^{\gamma-1}, \quad i = \overline{2,n}; \quad \beta > 0, \gamma > 0;$

2) $\lim t \to +\infty \dfrac{|p_{ik}(t)|}{\gamma t^{\gamma-1}} = 0, \quad i \not\equiv k, \quad i = \overline{1,n}; \quad k = \overline{1,n};$

3) $\lim t \to +\infty \dfrac{1}{t^{\gamma}} \displaystyle\int_{t_0}^{t} p_{ii}(s)ds = \beta_i < 0, \quad i = \overline{1,n};$

4) $|f_i(t)| \leq K\gamma t^{\gamma-1}, \quad i = \overline{1,n}; \quad K > 0;$

then any solution to the linear inhomogeneous system of differential equations (16) *on the interval* $[t_0, +\infty)$ *limited.*

Proof. The corresponding linear homogeneous system of differential equations

$$\frac{dy_i}{dt} = \sum_{k=1}^{n} p_i k(t) y_i, \quad i = \overline{1,n}; \tag{17}$$

under conditions: 1), 2) and 3) is generalized correct and has negative generalized upper Lyapunov exponents with respect to t^{γ}. The largest generalized upper Lyapunov exponent of the linear homogeneous system of differential equations (17) is $\beta_1 < 0$. Using condition 4) and similarly to Theorem 1, we obtain that any solution to the linear inhomogeneous system of differential equations (16) bounded. Theorem 2 is proved.

Let's look at an example. In system $x_1' = -\dfrac{1}{4\sqrt{t}}x_1 + \dfrac{1}{4\sqrt{t}}$, $x_2' = -\dfrac{1}{2\sqrt{t}}x_2 - \dfrac{1}{4\sqrt{t}}$; $1 < t_0 \leq t$; where $\gamma = \dfrac{1}{2}$, $0 < \beta \leq \dfrac{1}{2}$, $\beta_1 = -\dfrac{1}{2}$, $\beta_2 = -1$, $\dfrac{1}{2} \leq K$, the conditions of Theorem 2 are satisfied; therefore, any solution to a linear inhomogeneous system of differential equations is bounded.

3 Result

In this work, sufficient conditions for the boundedness of solutions of a linear inhomogeneous system of differential equations are obtained.

References

[1] Poincaré A. *O krivyh, opredelyaemyh differencial'nymi uravneniyami [About Curves Defined by Differential Equations]* (M.-L.: Gostekhizdat, 1947).

[2] Lyapunov A.M. *Obshchaya zadacha ob ustojchivosti dvizheniya [General problem of motion stability]* (M.-L.: GITTL, 1950): 472.

[3] Cezari L. *Asimptoticheskoe povedenie i ustojchivost' reshenij obyknovennyh differencial'nyh uravnenij [Asymptotic behavior and stability of solutions of ordinary differential equations]* (M.: Mir, 1964): 477.

[4] Nemytsky V.V., Stepanov V.V. *Kachestvennaya teoriya differencial'nyh uravnenij [Qualitative theory of differential equations]* Izd. 2-e, pererab. i dop. (M.-L.: GITTL, 1949): 545.

[5] Erugin N.P. *Linejnye sistemy obyknovennyh differencial'nyh uravnenij [Linear systems of ordinary differential equations]* (Minsk: Izd-vo Akademii nauk BSSR, 1963): 270.

[6] Sansone G. *Obyknovennye differencial'nye uravneniya [Ordinary differential equations]* T.1. (M.: In. lit., 1953): 346; T. 2. (M.: In. lit., 1954): 414.

[7] Pliss V.A. *Nelokal'nye problemy teorii kolebanij [Nonlocal Problems of Oscillation Theory]* (M.: Nauka, 1964): 367.

[8] Bylov B.F., Vinograd R.E., Grobman D.M., Nemyckij V.V. *Teoriya pokazatelej Lyapunova i ee prilozheniya k voprosam ustojchivosti [Theory of Lyapunov exponents and its applications to stability issues]* Monografiya (M.: Nauka, 1966): 576.

[9] Isobov N.A. *Vvedenie v teoriyu pokazatelej Lyapunova [Introduction to the theory of Lyapunov exponents]* (Mn.: BGU, 2006): 319.

[10] Coddington E.A., Levinson N. *Teoriya obyknovennyh differencial'nyh uravnenij [Theory of ordinary differential equations]* (M.: IL, 1958): 475.

[11] Demidovich B.P. *Lekcii po matematicheskoj teorii ustojchivosti [Lectures on the mathematical theory of stability]* (M.: Nauka, 1967): 472.

[12] Lefschetz S. *Geometricheskaya teoriya differencial'nyh uravnenij [Geometric theory of differential equations]* (M.: Izd-vo IL, 1961): 388.

[13] Massera H.L., Schaeffer H.H. *Linejnye differencial'nye uravneniya i funkcional'nye prostranstva [Linear differential equations and function spaces]* Per. s angl. (M.: Mir, 1970): 458.

[14] Bellman R. *Teoriya ustojchivosti reshenij differencial'nyh uravnenij [Stability theory of solutions of differential equations]* (M.: Izd-vo IL, 1954): 216.

[15] Coppel W.A. *Stability and asymptotic behavior of differential equations* (Boston, D.C. Heath, 1965): 166.

[16] Daletskiy Yu.L., Kerin M.G. *Ustojchivost' reshenij differencial'nyh uravnenij v banahovom prostranstve [Stability of solutions of differential equations in a Banach space]* Monografiya (Moskva: Nauka, 1970): 536.

[17] Wintner A., "Asymptotic equilibria", *American Journal of Mathematics* 68(1) (1946): 125-132.

[18] Wintner A., "An Abelian lemma concerning asymptotic equilibria", *American Journal of Mathematics* 68(3) (1946): 451-454.

[19] Wintner A., "Asymptotic integration constants", *American Journal of Mathematics* 68(4) (1946): 553-559.

[20] Wintner A., "On a theorem of Bocher in the theory of ordinary linear differential equations", *American Journal of Mathematics* 76(1) (1954): 183-190.

[21] Yoshizawa T., "Note on the boundedness of solutions of a system of differential equations", *Mem. Coil. Sci. Univ. Kyoto* A 28 (1954): 293-298.

[22] Bihari I., "A generalization of a lemma of bellman and its application to uniqueness problems of differential equations", *Acta Math. Acad. Sci. Hung.* 7 (1956): 81-94.

[23] Hartman Ph. "The existence of large or small solutions of linear differential equations", *Duke Math. J.* 28 (1961): 421-429.

[24] Hale J., Onuchic N. "On the asymptotic behavior of solutions of a class of differential equations", *Contributions to Differential Equations* 1 (1963): 61-75.

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BOUNDARY CONTROL OF ROD TEMPERATURE FIELD WITH A SELECTED POINT

In this paper, we study the issue of boundary control of the temperature field of a rod with a selected point. The main purpose of the work is to clarify the conditions for the existence of a boundary control that ensures the transition of the temperature field from the initial state to the final state. Relations connecting the boundary controls with the initial and final states, as well as with the external temperature field are found. Such boundary controls, generally speaking, constitute an infinite set. For an unambiguous choice of the boundary control, a strictly convex objective functional is chosen. We are looking for a boundary control that minimizes the selected target functional. To do this, we first investigate the existence and uniqueness of solutions to the initial boundary value problem and the conjugate problem. And also, we present the derivation of a system of linear Fredholm integral equations of the second kind, which are satisfied by an optimal boundary control that minimizes a strictly convex target functional on a convex set. Along the way, the linear part of the increment of the target functional is highlighted. Necessary and sufficient conditions for the minimum of a smooth convex functional on a convex set are established. The difference between the results of this work and the available ones is that in the proposed work, the temperature field is given by the heat conduction equation with a loaded term. As a result, the conjugate problem has a slightly different domain of definition than the domain of the conjugate problem in the case of no load.

Key words: initial-boundary value problem, heat equation, boundary control, Green's function, Fredholm integral equation of the second kind, spectral properties, eigenfunction, eigenvalues.

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Таңдалған нүктесі бар өзекшенің температуралық өрісін шекаралық басқару

Бұл жұмыста таңдалған нүктесі бар өзекшенің температуралық өрісін шекаралық басқару мәселесі зерттеледі. Жұмыстың негізгі мақсаты – температуралық өрістің бастапқы күйден соңғы күйге өтуін қамтамасыз ететін шекаралық басқарудың бар болуы шарттарын анықтау. Шекаралық басқаруларды бастапқы және финалдық күйлермен, сондай-ақ сыртқы температура өрісімен байланыстыратын қатынастар табылды. Мұндай шекаралық басқарулар, жалпы айтқанда, шексіз жиынды құрайды. Шекаралық басқарудың бірегей таңдалуы үшін қатаң дөңес мақсат функционал таңдалады. Таңдалған мақсат функционалды минимумдаушы шекаралық басқару ізделеді. Ол үшін жұмыста алдымен бастапқы-шекаралық есеп пен түйіндес есептің шешімдерінің бар болуы мен жалғыздығын зерттейміз. Сондай-ақ дөңес жиында қатаң дөңес мақсат функционалын минимумдаушы тиімді шекаралық басқарумен қанағаттандырылатын Фредгольмның екінші текті сызықты интегралдық теңдеулер жүйесінің алынуы келтірілген. Осы орайда мақсат функционалдың өсімшесінің сызықтық бөлігі айқындалған. Дөңес жиында тегіс дөңес функционал минимумының қажетті және жеткілікті шарттары анықталған. Жұмыстың нәтижесінің белгілі жұмыстардан айырмашылығы температура өрісі жүктелген мүшесі бар жылуөткізгіштік теңдеуі арқылы берілгендігі.

Нәтижесінде, түйіндес есептің жүктемесі болмаған жағдайда түйіндес есептің облысынан біршама өзгеше анықталу облысы болады.

Түйін сөздер: бастапқы-шекаралық есеп, жылуөткізгіштік теңдеуі, шекаралық басқару, Грин функциясы, екінші текті Фредгольм интегралдық теңдеуі, спектрлік қасиеттер, меншікті функция, меншікті мәндер.

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Граничное управление температурным полем стержня с выделенной точкой

В данной работе изучается вопрос о граничном управлении температурным полем стержня с выделенной точкой. Основная цель работы – выяснение условий существования граничного управления, обеспечивающего переход температурного поля из начального состояния в конечное состояние. Найдены соотношения, связывающие граничные управления с начальным и финальным состояниями, а также внешним температурным полем. Такие граничные управления, вообще говоря, составляют бесконечное множество. Для однозначного выбора граничного управления выбран строго выпуклый целевой функционал. Ищется граничное управление, которое минимизирует выбранный целевой функционал. Для этого в работе сначала исследуются существование и единственность решений начально-граничной задачи и сопряженной задачи. А также, дан вывод системы линейных интегральных уравнений Фредгольма второго рода, которым удовлетворяет оптимальное граничное управление, которое минимизирует строго выпуклый целевой функционал на выпуклом множестве. По пути выделена линейная часть приращения целевого функционала. Установлены необходимые и достаточные условия минимума гладкого выпуклого функционала на выпуклом множестве. Отличие результатов данной работы от имеющихся заключается в том, что в предлагаемой работе температурное поле задается уравнением теплопроводности с нагруженным членом. Вследствие чего сопряженная задача имеет несколько отличительную область определения, чем область определения сопряженной задачи в случае отсутствия нагрузки.

Ключевые слова: начально-граничная задача, уравнение теплопроводности, граничное управление, функция Грина, интегральное уравнение Фредгольма второго рода, спектральные свойства, собственная функция, собственные значения.

1 Introduction

In this paper, we study the issue of boundary control of rod temperature field with a selected point x_0.

$$u_t(x,t) - u_{xx}(x,t) + \alpha u(x_0,t) = f(x,t), \quad (x,t) \in Q, \tag{1}$$

where $Q = \{(x,t):\ 0 < x < b,\ 0 < t < T < +\infty\}$.

It is assumed that at the initial moment $t = 0$ the temperature along the rod of length b is given by law $u(x,0) = u_0(x)$, $0 < x < b$, where $u_0(x)$ is a twice continuously differentiable function. At the moment of time $t = T$ the temperature of the rod is equal to $u(x,T) = \gamma(x)$, $0 < x < b$, where $\gamma(x)$ is also a twice continuously differentiable function. The main purpose of the work is to clarify the conditions for the existence of the boundary control $u(0,t) = \mu(t)$, $u(b,t) = \eta(t)$, which ensures the transition of the temperature field from the state $\{u(x,0) = u_0(x)\}$ to the state $\{u(x,T) = \gamma(x)\}$. Similar problems were considered in [1, 2].

According to the optimization method, we choose the following functional

$$\mathcal{J}[\mu,\eta] = \|u(\cdot,T;\mu,\eta) - \gamma(\cdot)\|^2_{W_2^1(0,b)} + \beta_1 \int_0^T |\mu(t)|^2 dt + \beta_2 \int_0^T |\eta(t)|^2 dt,$$

where β_1, β_2 are positive numbers, γ is a given function from class $W_2^1(0,b)$.

The boundary control problem is as follows: it is required to find boundary controls $(\mu(t), \eta(t))$ and the corresponding solution $u(x,t)$, that satisfies equation (1) with initial boundary controls

$$u(0,t) = \mu(t), \quad u(b,t) = \eta(t), \quad 0 < t < T, \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 < x < b, \tag{3}$$

and minimizes functional $\mathcal{J}[\mu,\eta]$.

Many natural and fundamental physical phenomena can be modeled by partial differential equations (PDEs), such as heat conduction, sound, electrostatics, electrodynamics, fluid flow and quantum mechanics in which states depend on not only time but also space, for example, see [3–5]. In particular, heat diffusion phenomena are extended mainly in describing fluid, thermal, and chemical dynamics, including the wide applications of sea ice melting and freezing [6], lithium-ion batteries [7], etc. The work [8] is concerned with the problem of boundary observer-based finite-time output feedback control for a heat system with Neumann boundary condition and Dirichlet boundary actuator. Finite-time stabilization, which is a key feature in the sliding mode control theory, is investigated in the work [9]. More specifically, finite-time control for the heat equation with Dirichlet boundary condition and the Dirichlet control is investigated in [10]. In work [11] the heat equation with prescribed lateral and final data is studied in half-plane and the uniqueness of the bounded solution is proved. In work [12] the solvability problems of an nonhomogeneous boundary value problem in the first quadrant for a fractionally loaded heat equation are studied. For parabolic equations in a bounded domain, various aspects of inverse source problems has been studied in [13–16], etc.

The paper presents a derivation of a system of linear Fredholm integral equations of the second kind, which optimal boundary control is satisfied. In the proposed work, for the first time, the conjugate problem to a mixed boundary value problem for the heat conduction equation with a loaded term is explicitly written out. As a result, it was possible to obtain more precise information about the solutions of the conjugate problem. We note that in [1], the solution of the mixed boundary value problem for the heat conduction equation is decomposed by the eigenfunctions of a periodic problem with a specially selected period. In [2], the method of work [1] is extended to the heat conduction equation with a loaded term. In this paper, the expansion of the solution to the mixed problem for the heat equation with a loaded term is carried out in terms of the eigenfunctions of the corresponding spectral problem. At the same time, it is necessary to select a period and continue the solution in a wider area. Moreover, the solution of the conjugate problem is carried out similarly to the solution of the mixed problem for the heat conduction equation with a load.

2 Existence and uniqueness of the solutions to the initial-boundary value problem and the conjugate problem

Before studying the boundary control problem, it is necessary to investigate the question of the existence and uniqueness of the solution to problem (1)–(3). To do this, select a function $w(x,t)$ from class $L_2((0,T);W_2^1(0,b))$ such that

$$w(x,t)=\mu(t)+\frac{x}{b}(\eta(t)-\mu(t)).$$

Then, instead of studying problem (1)–(3) it is enough to study the following problem:

$$v_t(x,t)-v_{xx}(x,t)+\alpha v(x_0,t)=F(x,t),\quad (x,t)\in Q, \tag{4}$$

$$v(0,t)=0,\quad v(b,t)=0, \tag{5}$$

$$v(x,0)=v_0(x),\quad 0<x<b, \tag{6}$$

where

$$F(x,t)=f(x,t)-w_t(x,t)-\alpha w(x_0,t),\quad v_0(x)=u_0(x)-\mu(0)-\frac{x}{b}(\eta(0)-\mu(0)).$$

The solution to problem (4)–(6) is sought in the form

$$v(x,t)=\sum_{k\geqslant 1}d_k(t)\varphi_k(t).$$

Here $\{\varphi_k\}$ is the system of root functions of the following eigenvalue problem

$$-\varphi_{xx}(x)+\alpha\varphi(x_0)=\lambda\varphi(x),\quad 0<x<b, \tag{7}$$

$$\varphi(0)=0,\quad \varphi(b)=0. \tag{8}$$

In this case $\varphi_k(x)\equiv\varphi_k(x,\lambda_k)$, where $\{\lambda_k\}$ is a sequence of eigenvalues of (7)–(8). The eigenfunctions $\varphi_k(x)\equiv\varphi(x,\lambda_k)$ and the biorthogonal system of functions $\left\{\psi_k(x)=\frac{\overline{\psi(x,\lambda_k)}}{\langle\varphi(\cdot,\lambda_k),\psi(\cdot,\lambda_k)\rangle}\right\}$ are defined by formulas:

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\alpha\,\frac{\dfrac{\sin\sqrt{\lambda}x_0}{\sqrt{\lambda}}}{\lambda-\alpha(1-\cos\sqrt{\lambda}x_0)}\,(1-\cos\sqrt{\lambda}x),\quad 0<x<b,$$

$$\psi(x,\lambda)=\frac{\sin\sqrt{\lambda}(b-x)}{\sqrt{\lambda}},\quad x_0<x<b,$$

$$\psi(x,\lambda)=\frac{\sin\sqrt{\lambda}(b-x_0)}{\sqrt{\lambda}}\,\cos\sqrt{\lambda}(x_0-x)$$
$$+\frac{\sin\sqrt{\lambda}(x_0-x)}{\sqrt{\lambda}}\left(\cos\sqrt{\alpha}\,(b-x_0)-\overline{\alpha}\frac{1-\cos\sqrt{\lambda}(b-x_0)}{\lambda}\right)-\overline{\alpha}\,\frac{\sin\sqrt{\lambda}(x_0-x)}{\sqrt{\lambda}}$$
$$\times\frac{\cos\sqrt{\lambda}(b-x_0)-\cos\sqrt{\lambda}b-\overline{\alpha}\,\dfrac{(1-\cos\sqrt{\lambda}(b-x_0))(1-\cos\sqrt{\lambda}x_0)}{\lambda}}{\lambda+\overline{\alpha}\,(1-\cos\sqrt{\lambda}x_0)},\ 0<x<x_0,$$

The acceptable values of parameter λ are selected according to the conditions $\lambda - \alpha(1 - \cos\sqrt{\lambda}x_0) \neq 0$, $\lambda + \overline{\alpha}(1 - \cos\sqrt{\lambda}x_0) \neq 0$. Each function $f(x)$ from $L_2(0,b)$ is decomposed into a Fourier series by the system $\{\varphi_k\}$:

$$f(x) = \sum_{k\geqslant 1} C_k(f)\varphi_k(x),$$

where $C_k(f) = \langle f, \psi_k\rangle$, $k \geqslant 1$.

In this case, the Fourier coefficients $\{d_k(t),\ k \geqslant 1\}$ in terms of system $\{\varphi_k(x)\}$ of the solution $v(x,t)$ satisfy equations

$$d_{tk}(t) + \lambda_k d_k(t) = D_k(t), \quad t > 0 \tag{9}$$

and initial conditions

$$d_k(0) = d_k^{(0)}. \tag{10}$$

Here $\{D_k(t)\}$ and $\{d_k^{(0)}\}$ are sequences of Fourier coefficients in terms of system $\{\varphi_k\}$ of functions $F(x,t)$ and $v_0(x)$. Relations (9)–(10) imply the following representation

$$d_k(t) = d_k^{(0)} e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-\tau)} D_k(\tau)d\tau, \quad t > 0. \tag{11}$$

Thus, problem (4)–(6) has a solution $v(x,t)$, representable in the form

$$v(x,t) = \sum_{k\geqslant 1} d_k(t)\varphi_k(x), \tag{12}$$

and the coefficients $d_k(t)$ are calculated by formulas (11). Thus, we can formulate the following statement.

Theorem 1 *Let $v_0(x)$ be a twice continuously differentiable function on a finite segment $[0,b]$, and the matching conditions $v_0(0) = v_0(b) = 0$ are satisfied. Suppose also that $F(x,t) = \mathrm{L}^2((0,T); L_2(0,b))$. Then there is a solution $v(x,t)$ of problem (4)–(6), which can be represented as a Fourier series (12), the coefficients $\{d_k(t)\}$ of which are found by formulas (11).*

Remark 1 *Note that $v_0(x)$ is decomposed into a Fourier series by the system $\{\varphi_k\}$ and the corresponding Fourier series on $[0,b]$ converges uniformly. This follows from the fact that $v_0(x)$ belongs to the domain of operator B. The monograph [17] contains theorems on the uniform convergence of spectral decompositions in such cases.*

We denote by $G(x,\xi,t) = \sum\limits_{k\geqslant 1} e^{-\lambda_k t}\varphi_k(x)\overline{\psi_k(\xi)}$, the function that represents the Green function of the Dirichlet problem for the heat equation with the selected point [18]. Then the statement follows.

Corollary 1 *Let the conditions of Theorem 1 be satisfied. Then there exists a solution $u(x,t)$ of problem* (1)–(3)*, which can be represented in the form*

$$u(x,t) = \int_0^b u_0(\xi)G(x,\xi,t)d\xi + \int_0^t d\tau \int_0^b f(\xi,\tau)G(x,\xi,t-\tau)d\xi - u_0(0)\int_0^b G(x,\xi,t)d\xi$$
$$- \frac{u_0(b)-u_0(0)}{b}\int_0^b \xi G(x,\xi,t)d\xi - \alpha\int_0^t \mu(\tau)d\tau \int_0^b G(x,\xi,t-\tau)d\xi$$
$$- \frac{\alpha}{b}\int_0^t (\eta(\tau)-\mu(\tau))\,d\tau \int_0^b \xi G(x,\xi,t-\tau)d\xi.$$

We now formulate and prove a uniqueness theorem for a solution.

Theorem 2 *Let the conditions of Theorem 1 be satisfied. Then problem* (4)–(6) *has a unique solution.*

Proof 1 *The idea of this proof is borrowed from the work of V.A. Il'in [19]. Let $r(x)$ be one of the eigenfunctions of operator B^*. We denote by $\Phi(x,t)$ any of the functions of the form*

$$\Phi(x,t) = r(x)f(t),$$

where $f(t)$ is a function that is continuously differentiable on the entire numerical axis, which is equal to zero for all $t > t_0$, where t_0 is some number less than T.

Let $v(x,t)$ be a solution to problem (4)–(6) *for $F \equiv 0$, $v_0 \equiv 0$. Let consider integral*

$$0 = \int_0^b \int_0^T (v_t(x,t) - v_{xx}(x,t) + \alpha v(x_0,t))\,\Phi(x,t)dtdx = \int_0^b r(x)dx \int_0^T v_t(x,t)f(t)dt$$
$$+ \int_0^T f(t)dt \int_0^b Bv(x,t)r(x)dx = \int_0^b r(x)dx \left(\varphi(x,t)f(t)\Big|_0^T - \int_0^T v(x,t)f_t(t)dt\right)$$
$$+ \int_0^T f(t)dt \int_0^b v(x,t)B^*r(x)dx = -\int_0^b \int_0^T v(x,t)r(x)f_t(t)dtdx$$
$$+ \overline{\lambda}\int_0^b \int_0^T v(x,t)r(x)f(t)dtdx, \quad (13)$$

where $\overline{\lambda}$ is the eigenvalue of operator B^* corresponding to eigenfunction $r(x)$.

Let us continue $v(x,t)$ on domain $t < 0$, by setting it equal to zero there. Then, taking into account that $f(t) = 0$ for $t > t_0$, relation (13) *can be rewritten in the form*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)\left(-f_t(t) + \overline{\lambda}f(t)\right)dtdx = 0. \quad (14)$$

We fix any $\xi \geqslant 0$. Then function $f(\xi+t)$ is a priori equal to zero for $t \geqslant t_0$. In equality (14) *we substitute $f(\xi+t)$ instead of $f(t)$. Then for all $\xi \geqslant 0$ we have equality*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)\left(-f_\xi(\xi+t) + \overline{\lambda}f(\xi+t)\right)dtdx = 0. \quad (15)$$

From (15) *it follows*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)f(\xi+t)dtdx = c\cdot e^{\overline{\lambda}\xi}, \quad \xi \geqslant 0. \tag{16}$$

However, for $\xi \geqslant t_0$ *and* $t>0$ *the function* $f(\xi+t)\equiv 0$. *Therefore, it follows from relation* (16) *that* $c=0$. *Therefore, we have the equality*

$$\int_0^b \int_{-\infty}^{\infty} v(x,t)r(x)f(\xi+t)dtdx = 0, \quad \xi \geqslant 0. \tag{17}$$

Since the system of eigenfunctions $\{\overline{\psi(x,\lambda_k)},\ k\geqslant 1\}$ *of operator* B^* *is complete in space* $L_2(0,b)$, *equalities* (17) *imply*

$$\int_0^{\infty} v(x,t)f(\xi+t)dt \equiv 0 \text{ in space } L_2(0,b).$$

In particular, for $\xi=0$ *we find that*

$$\int_0^T v(x,t)f(t)dt = 0.$$

The latter equality holds for any function $f(t)$, *that has the properties described above. Therefore* $v(x,t)\equiv 0$ *for* $0<x<b$, $0<t<T$. *Theorem 2 is completely proved.*

Therefore, the conjugate problem to problem (4)–(6) takes the form

$$-\Psi_t(x,t) - \Psi_{xx}(x,t) = E(x,t), \quad (x,t)\in Q, \tag{18}$$

$$\Psi(0,t)=0, \quad \Psi(b,t)=0, \quad t>0, \tag{19}$$

$$\begin{cases} \Psi(x_0+0,t) = \Psi(x_0-0,t), \quad t>0, \\ \Psi_x(x_0+0,t) = \Psi_x(x_0-0,t) + \overline{\alpha}\int_0^b \psi(x,t)dx, \quad t>0, \end{cases} \tag{20}$$

$$\Psi(x,T) = \Psi_T(x), \quad 0<x<b. \tag{21}$$

Thus, we can formulate the following statement.

Theorem 3 *Let* $\Psi_T(x)$ *be a twice continuously differentiable function on a finite segment* $[0,b]$, *moreover, for* $\Psi_T(x)$ *the matching conditions* (19)–(20) *are satisfied. Suppose also that* $E(x,t)\in \mathrm{L}^2((0,T);L_2(0,b))$. *Then there is a solution* $\Psi(x,t)$ *to problem* (18)–(21), *which can be represented as a Fourier series dual to series* (12).

Theorem 3 implies the following statement.

Corollary 2 *Let the conditions of Theorem 3 be satisfied. Then there exists a solution* $\Psi(x,t)$ *to problem* (18)–(21), *which can be represented in the form*

$$\Psi(x,t) = \int_0^b \Psi_T(\xi)\overline{G(\xi,x,T-t)}d\xi + \int_t^T d\tau \int_0^b E(\xi,\tau)\overline{G(\xi,x,T-\tau)}d\xi.$$

Now we formulate and prove the uniqueness theorem.

Theorem 4 *Let the conditions of Theorem 3 be satisfied. Then problem* (18)–(21) *has a unique solution.*

The proof of Theorem 4 repeats the proof of Theorem 2. Let $r(x)$ be one of the eigenfunctions of operator B. We denote by $\Phi(x,t)$ any function of the form

$$\Phi(x,t) = r(f)f(t),$$

where $f(t)$ is a continuously differentiable function on the entire numerical axis, which is equal to zero for all $t < t_0$, where t_0 is some positive number. Further, the reasoning from the proof of Theorem 2 is repeated.

3 Necessary conditions for maintaining the final temperature regime

In this section, we study the boundary control problem I:

$$W_t(x,t) - W_{xx}(x,t) + \alpha W(x_0,t) = f(x,t), \quad (x,t) \in Q. \tag{22}$$

$$W(x,T) = u_T(x), \quad 0 < x < b, \tag{23}$$

Statement of the boundary control Problem I:

Let $W(x,t;f,u_T)$ be an arbitrary solution of problem (22)–(23). We denote the boundary controls corresponding to $W(x,t;f,u_T)$, by $\mu(t) = W(0,t;f,u_T)$ and $\eta(t) = W(b,t;f,u_T)$, as well as by $u_0(x) = W(x,0;f,u_T)$ the initial temperature regime.

What necessary conditions do $\mu(t)$, $\eta(t)$, $u_0(x)$, satisfy if $W(x,t;f,u_T)$ satisfies (22)–(23)?

This boundary control Problem I corresponds to a given final temperature regime $u_T(x)$. To answer the question posed, it is convenient to introduce solutions $\Psi(x,t) = \Psi(x,t;\Psi_T)$ to conjugate equation

$$-\Psi_t(x,t) - \Psi_{xx}(x,t) = 0, \quad (x,t) \in Q, \quad x \neq x_0, \tag{24}$$

with conditions

$$\Psi(0,t) = 0, \quad \Psi(b,t) = 0, \quad t > 0, \tag{25}$$

$$\begin{cases} \Psi(x_0+0,t) = \Psi(x_0-0,t), \\ \Psi'(x_0+0,t) = \Psi'(x_0-0,t) + \overline{\alpha}\int_0^b \Psi(\xi,t)d\xi, \quad t > 0, \end{cases} \tag{26}$$

and the final temperature distribution

$$\Psi(x,T) = \Psi_T(x), \quad 0 < x < b \tag{27}$$

for an arbitrary function $\Psi_T(x)$ from class $W_2^2[0,b]$.

Lemma 1 *For an arbitrary solution $u(x,t) \equiv u(x,t;f,\mu,\eta,u_0)$ of equation*

$$u_t(x,t) - u_{xx}(x,t) + \alpha u(x_0,t) = f(x,t), \quad (x,t) \in Q, \tag{28}$$

with boundary conditions

$$u(0,t)=\mu(t),\quad u(b,t)=\eta(t),\quad T>t>0, \tag{29}$$

and with the initial temperature distribution

$$u(x,0)=u_0(x), 0<x<b, \tag{30}$$

the following integral relation is valid

$$\int_0^T\int_0^b f(x,t)\overline{\Psi(x,t)}dxdt=\int_0^b\left(u(x,T)\overline{\Psi_T(x)}-u_0(x)\overline{\Psi(x,0)}\right)dx$$
$$-\int_0^T\mu(t)\overline{\Psi_x(0,t)}dt+\int_0^T\eta(t)\overline{\Psi_x(b,t)}dt,$$

where $\Psi(x,t)\equiv\Psi(x,t;\Psi_T)$ *is the solution to conjugate problem* (24)–(27) *for an arbitrary* $\Psi_T(x)\in W_2^2[0,b]$.

Let us formulate another useful lemma.

Lemma 2 *For an arbitrary solution* $v(x,t)\equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *of equation*

$$v_t(x,t)-v_{xx}(x,t)+\alpha v(x_0,t)=f(x,t),\quad (x,t)\in Q, \tag{31}$$

with boundary conditions

$$v(0,t)=\mu_1(t),\quad v(b,t)=\eta_1(t),\quad T>t>0, \tag{32}$$

and with the final temperature distribution

$$v(x,T)=v_T(x),\quad 0<x<b, \tag{33}$$

the following integral relation is valid

$$\int_0^T\int_0^b f(x,t)\overline{\Psi(x,t)}dxdt=\int_0^b\left(v_T(x)\overline{\Psi_T(x)}-v(x,0)\overline{\Psi(x,0)}\right)dx$$
$$-\int_0^T\mu_1(t)\overline{\Psi_x(0,t)}dt+\int_0^T\eta_1(t)\overline{\Psi_x(b,t)}dt,$$

where $\Psi(x,t)\equiv\Psi(x,t;\Psi_T)$ *is the solution to conjugate problem* (24)–(27) *for an arbitrary* $\Psi_T(x)\in W_2^2[0,b]$.

We now formulate an important statement.

Theorem 5 *For the solution* $u(x,t)\equiv u(x,t;f,\mu,\eta,u_0)$ *to problem* (28)–(30) *and for the solution* $v(x,t)\equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *to problem* (31)–(33) *the following integral identity is valid*

$$\int_0^T(\eta_1(t)-\eta(t))\,G_x(\xi,b,T-t)dt-\int_0^T(\mu_1(t)-\mu(t))\,G_x(\xi,0,T-t)dt$$
$$\equiv\int_0^b(v(x,0)-u_0(x))\,G(\xi,x,T)dx,\quad\forall\,\xi\in(0,b), \tag{34}$$

where $G(x,\xi,t)=\sum\limits_{k\geqslant1}e^{-\lambda_k t}\varphi_k(x)\overline{\psi_k(\xi)}$ *is a Green's function.*

Proof 2 *Lemmas 1 and 2 imply the integral identity*

$$0 = -\int_0^b (v(x,0) - u_0(x))\,\overline{\Psi(x,0)}dx$$
$$- \int_0^T (\mu_1(t) - \mu(t))\,\overline{\Psi_x(0,t)}dt + \int_0^T (\eta_1(t) - \eta(t))\,\overline{\Psi_x(b,t)}dt, \tag{35}$$

for all $\Psi(x,t)$ *at any* $\Psi_T(x)$. *Corollary 2 implies that*

$$\Psi(x,t) = \int_0^b \Psi_T(\xi)\overline{G(\xi,x,T-t)}d\xi.$$

Therefore, relation (35) *takes the form*

$$-\int_0^b (v(x,0) - u_0(x))\,dx \int_0^b \overline{\Psi_T(\xi)}G(\xi,x,T)d\xi$$
$$-\int_0^T (\mu_1(t) - \mu(t))\,dt \int_0^b \overline{\Psi_T(\xi)}G_x(\xi,0,T-t)d\xi$$
$$+\int_0^T (\eta_1(t) - \eta(t))\,dt \int_0^b \overline{\Psi_T(\xi)}G_x(\xi,b,T-t)d\xi.$$

Rearranging the order of the integrals, we obtain the equality

$$\int_0^b \overline{\Psi_T(\xi)}\Bigg\{ \int_0^T (\eta_1(t) - \eta(t))\,G_x(\xi,b,T-t)dt$$
$$- \int_0^T (\mu_1(t) - \mu(t))\,G_x(\xi,0,T-t)dt - \int_0^b (v(x,0) - u_0(x))\,G(\xi,x,T)dx \Bigg\} d\xi = 0.$$

Since $\Psi_T(\xi)$ *is an arbitrary function from* $W_2^2[0,b]$, *then relation* (34) *follows from the last equality. Theorem 5 is completely proved.*

This implies the following statement.

Corollary 3 *Let* $u(x,t) \equiv u(x,t;f,\mu,\eta,u_0)$ *and* $v(x,t) \equiv v(x,t;f,\mu_1,\eta_1,v_T)$ *are solutions to problems* (28)–(30) *and* (31)–(33), *respectively. If* $u_0 = v(x,0)$, $x \in (0,b)$, *then the following identity is valid*

$$\int_0^T (\eta_1(t) - \eta(t))\,G_x(\xi,b,T-t)dt - \int_0^T (\mu_1(t) - \mu(t))\,G_x(\xi,0,T-t)dt \equiv 0, \quad \forall\,\xi \in (0,b).$$

4 Optimality criteria

In this section, the target functional is investigated.

$$\mathcal{J}[\mu,\eta] = \int_0^b |u(x,T;\mu,\eta) - \gamma(x)|^2 dx$$
$$+ \int_0^b |u'(x,T;\mu,\eta) - \gamma'(x)|^2 dx + \beta_1 \int_0^T \mu^2(t)dt + \beta_2 \int_0^T \eta^2(t)dt.$$

Let us take arbitrary controls $(\mu(t), \eta(t))$ and $(\mu(t) + h(y), \eta(t) + q(t))$, where $h(0) = 0$, $q(0) = 0$. We denote the corresponding solutions of problem (1)–(3) by $u(x, t; \mu, \eta)$ and $u(x, t; \mu + h, \eta + q)$. Let us introduce the notation

$$\Delta u(x,t) = u(x,t;\mu+h,\eta+q) - u(x,t;\mu,\eta).$$

Then from (1)–(3) it follows

$$\frac{\partial}{\partial t}\Delta u - \frac{\partial^2}{\partial x^2}\Delta u + \alpha \Delta u(x_0,t) = 0, \quad (x,t) \in Q, \tag{36}$$

$$\Delta u\big|_{x=0} = h(t), \quad \Delta u\big|_{x=b} = q(t), \quad t > 0, \tag{37}$$

$$\Delta u\big|_{t=0} = 0, \quad 0 < x < b. \tag{38}$$

Arguing as in the proof of Theorem 1, we obtain the representation

$$\Delta u(x,t) = \frac{1}{b}\int_0^t (q(\tau) - h(\tau))\frac{\partial}{\partial \tau}K_1(x,t-\tau)d\tau - \frac{\alpha x_0}{b}\int_0^t q(\tau)K_0(x,t-\tau)d\tau$$
$$- \alpha\left(1 - \frac{x_0}{b}\right)\int_0^t h(\tau)K_0(x,t-\tau)d\tau + \int_0^t h(\tau)\frac{\partial}{\partial \tau}K_0(x,t-\tau)d\tau,$$

where $K_0(x,t) = \sum\limits_{k\geqslant 1}\beta_k^{(0)}e^{-\lambda_k t}\cdot\varphi_k(x)$, $K_1(x,t) = \sum\limits_{k\geqslant 1}\beta_k^{(1)}e^{-\lambda_k t}\cdot\varphi_k(x)$.

Consider the increment of the target functional $\mathcal{J}[\mu,\eta]$.

$$\Delta\mathcal{J}[\mu,\eta] = 2\int_0^b \mathrm{Re}\left(\left(\overline{u(x,T;\mu,\eta)} - \overline{\gamma(x)}\right)\Delta u(x,T)\right)dx$$
$$+ 2\int_0^b \mathrm{Re}\left(\left(\overline{u_x(x,T;\mu,\eta)} - \overline{\gamma_x(x)}\right)\frac{\partial}{\partial x}\Delta u(x,T)\right)dx + 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt$$
$$+ 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt + \overline{\overline{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2)dt}\right)$$
$$= 2\int_0^b \mathrm{Re}\left(\left(\overline{u(x,T;\mu,\eta)} - \overline{\gamma(x)}\right)\Delta u(x,T)\right)dx$$
$$+ 2\mathrm{Re}\left(\left(\overline{u_x(x,T;\mu,\eta)} - \overline{\gamma_x(x)}\right)\Delta u(x,T)\right)\Big|_{x=0}^{x=b}$$
$$- 2\int_0^b \mathrm{Re}\left(\left(\overline{u_{xx}(x,T;\mu,\eta)} - \overline{\gamma_{xx}(x)}\right)\frac{\partial}{\partial x}\Delta u(x,T)\right)dx$$
$$+ 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt + 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt + \overline{\overline{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2)dt}\right)$$
$$= 2\mathrm{Re}\left(\left(\overline{u_x(b,T;\mu,\eta)} - \overline{\gamma_x(b)}\right)q(T)\right) - 2\mathrm{Re}\left(\left(\overline{u_x(0,T;\mu,\eta)} - \overline{\gamma_x(0)}\right)h(T)\right)$$
$$+ 2\int_0^b \mathrm{Re}\left(\Delta u(x,T)\left[-\left(\overline{u_{xx}(x,T;\mu,\eta)} - \overline{\gamma_{xx}(x)}\right) + \left(\overline{u(x,T;\mu,\eta)} - \overline{\gamma(x)}\right)\right]\right)dx$$

$$+2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right)dt+2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right)dt+\overline{\overline{o}}\left(\sqrt{\int_0^T(|h(t)|^2+|q(t)|^2)dt}\right), \tag{39}$$

where

$$\lim_{q,h\to 0}\frac{\overline{\overline{o}}\left(\sqrt{\int_0^T(|h(t)|^2+|q(t)|^2)dt}\right)}{\int_0^T(|h(t)|^2+|q(t)|^2)dt}=0.$$

Let us introduce the solution $\Psi(x,t;\mu,\eta)$ of the following conjugate problem (18)–(21) at $E(x,t)\equiv 0$, $\Psi_T(x)=\left(-\frac{d^2}{dx^2}+I\right)(u(x,T;\mu,\eta)-\gamma(x))$. For further purposes, we transform the integral

$$\begin{aligned}
&\int_0^b \Delta u(x,T)\left[-\left(u_{xx}(x,T;\mu,\eta)-\gamma_{xx}(x)\right)+\left(u(x,T;\mu,\eta)-\gamma(x)\right)\right]dx\\
&=\int_0^T\frac{\partial}{\partial t}\left(\int_0^b \Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}dx\right)dt\\
&=\int_0^T dt\int_0^{x_0}\left[\frac{\partial^2}{\partial x^2}\Delta u(x,t)-\alpha\Delta u(x_0,t)\right]\overline{\Psi(x,t;\mu,\eta)}dx\\
&+\int_0^T dt\int_{x_0}^b\left[\frac{\partial^2}{\partial x^2}\Delta u(x,t)-\alpha\Delta u(x_0,t)\right]\overline{\Psi(x,t;\mu,\eta)}dx\\
&+\int_0^T dt\int_0^b \Delta u(x,t)\frac{\partial}{\partial t}\overline{\Psi(x,t;\mu,\eta)}dx\\
&=\int_0^T dt\left\{\left[\frac{\partial}{\partial x}\Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}-\Delta u(x,t)\frac{\partial}{\partial x}\overline{\Psi(x,t;\mu,\eta)}\right]_{x=0}^{x=x_0-0}\right.\\
&\left.+\left[\frac{\partial}{\partial x}\Delta u(x,t)\overline{\Psi(x,t;\mu,\eta)}-\Delta u(x,t)\frac{\partial}{\partial x}\overline{\Psi(x,t;\mu,\eta)}\right]_{x=x_0+0}^{x=b}\right\}\\
&+\alpha\int_0^T \Delta u(x_0,t)\overline{\Psi(x,t;\mu,\eta)}dt\\
&+\int_0^T dt\int_{x_0}^b \Delta u(x,t)\left(\frac{\partial^2\overline{\Psi(x,t;\mu,\eta)}}{\partial x^2}+\frac{\partial\overline{\Psi(x,t;\mu,\eta)}}{\partial t}\right)dx\\
&=\int_0^T h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}dt-\int_0^T q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}dt.
\end{aligned}$$

Thus, the following relation is true

$$\begin{aligned}
&2\int_0^b \mathrm{Re}\left(\Delta u(x,T)\left[-\frac{d^2}{dx^2}+I\right](u(x,T;\mu,\eta)-\gamma(x))\right)dx\\
&=2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}\right)dt-2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}\right)dt.
\end{aligned}$$

From the last relation and equality (39) it follows that the increment of the target functional will take the form

$$\Delta\mathcal{J}[\mu,\eta] = 2\mathrm{Re}\left(\left[\overline{u_x(b,T;\mu,\eta)} - \overline{\gamma_x(b)}\right] q(T)\right) - 2\mathrm{Re}\left(\left[\overline{u_x(0,T;\mu,\eta)} - \overline{\gamma_x(0)}\right] h(T)\right)$$
$$+ 2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu,\eta)}\right) dt - 2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu,\eta)}\right) dt$$
$$+ 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu(t)}h(t)\right) dt + 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta(t)}q(t)\right) dt + \overline{\overline{o}}\left(\sqrt{\int_0^T (|h(t)|^2 + |q(t)|^2)dt}\right).$$

Thus, we were able to isolate the linear part of the increment of the target functional $\Delta\mathcal{J}[\mu,\eta]$. Necessary and sufficient conditions for the minimum of a smooth convex functional $\mathcal{J}[\mu,\eta]$ on a convex set $U = \{\mu(t),\eta(t) : \mu,\eta \in W_2^1[0,T]\}$ [20] are given in the following statement.

Theorem 6 *Let $(\mu^*(t),\eta^*(t)) \in U$ and gives a minimum to functional $\mathcal{J}[\mu,\eta]$. Then the following inequality must be*

$$2\mathrm{Re}\left(\left[\overline{u_x(b,T;\mu^*,\eta^*)} - \overline{\gamma_x(b)}\right] q(T)\right) - 2\mathrm{Re}\left(\left[\overline{u_x(0,T;\mu^*,\eta^*)} - \overline{\gamma_x(0)}\right] h(T)\right)$$
$$+ 2\int_0^T \mathrm{Re}\left(h(t)\frac{\partial}{\partial x}\overline{\Psi(0,t;\mu^*,\eta^*)}\right) dt - 2\int_0^T \mathrm{Re}\left(q(t)\frac{\partial}{\partial x}\overline{\Psi(b,t;\mu^*,\eta^*)}\right) dt$$
$$+ 2\beta_1\int_0^T \mathrm{Re}\left(\overline{\mu^*(t)}h(t)\right) dt + 2\beta_2\int_0^T \mathrm{Re}\left(\overline{\eta^*(t)}q(t)\right) dt = 0.$$

for all $h,q \in W_2^1[0,T]$ with conditions $h(0) = q(0) = 0$. Moreover, since $\mathcal{J}[\mu,\eta]$ is convex on U, the above necessary condition is also sufficient for $(\mu^*(t),\eta^*(t)) \in U$.

5 System of linear integral equations for optimal boundary control

In this section, we derive a system of linear Fredholm integral equations of the second kind, which are satisfied by the optimal boundary control

$$(\mu^*(t),\eta^*(t)) \in U.$$

Since in Theorem 6 the functions $h(t)$ and $q(x)$ are arbitrary from $W_2^1[0,T]$ and independent of each other, we can write the system of relations

$$\begin{cases} \dfrac{\partial}{\partial x}\Psi(0,t;\mu^*,\eta^*) + \beta_1\mu^*(t) = 0, \quad T > t > 0, \\ -\dfrac{\partial}{\partial x}\Psi(b,t;\mu^*,\eta^*) + \beta_2\eta^*(t) = 0, \quad T > t > 0, \end{cases} \tag{40}$$

$$\begin{cases} u_x\Psi(b,t;\mu^*,\eta^*) = \gamma_x(b), \\ u_x\Psi(0,t;\mu^*,\eta^*) = \gamma_x(0). \end{cases} \tag{41}$$

Now, according to Corollary 2, we have the representation

$$u(x,T;\mu^*,\eta^*)=\int_0^T \mathrm{H}_1(x,T-\tau)\mu^*(\tau)d\tau+\int_0^T \mathrm{H}_2(x,T-\tau)\eta^*(\tau)d\tau+\mathrm{H}_3(x),$$

where

$$\mathrm{H}_1(x,t-\tau)=-\alpha\int_0^b\left(1-\frac{\xi}{b}\right)G(x,\xi,t-\tau)d\xi,\quad \mathrm{H}_2(x,t-\tau)=-\frac{\alpha}{b}\int_0^b G(x,\xi,t-\tau)d\xi,$$

$$\begin{aligned}\mathrm{H}_3(x,t)=&\int_0^b u_0(\xi)G(x,\xi,t)d\xi+\int_0^T d\tau\int_0^b f(\xi,\tau)G(x,\xi,t-\tau)d\xi\\&-u_0(0)\int_0^b G(x,\xi,T)d\xi-\frac{u_0(b)-u_0(0)}{b}\int_0^b \xi G(x,\xi,T)\xi.\end{aligned}$$

Further, Corollary 3 implies the representation

$$\begin{aligned}\Psi(x,t)=&\int_0^b\left(-\frac{d^2}{d\xi^2}+I\right)(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\overline{G(\xi,x,T-t)}d\xi\\=&\Big(-\frac{d}{d\xi}(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\overline{G(\xi,x,T-t)}\\&+(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\,\frac{d}{d\xi}\overline{G(\xi,x,T-t)}\Big)\Big|_{\xi=0}^{\xi=b}\\&+\int_0^b(u(\xi,T;\mu^*,\eta^*)-\gamma(\xi))\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}d\xi\\=&\int_0^T Q_1(T-\tau,x,T-t)\mu^*(\tau)d\tau+\int_0^T Q_2(T-\tau,x,T-t)\eta^*(\tau)d\tau+Q_3(x,t),\end{aligned}$$

where

$$\begin{aligned}Q_1(T-\tau,x,T-t)&=\textstyle\int_0^b\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_1(\xi,T-\tau)d\xi,\\Q_2(T-\tau,x,T-t)&=\textstyle\int_0^b\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_2(\xi,T-\tau)d\xi,\\Q_3(x,t)&=\textstyle\int_0^b\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\mathrm{H}_3(\xi)d\xi-\int_0^b\gamma(\xi)\left(-\frac{\partial^2}{\partial\xi^2}+I\right)\overline{G(\xi,x,T-t)}\xi.\end{aligned}$$

Thus, relations (40) and (41) imply the required system of linear integral equations with respect $(\mu^*(t),\eta^*(t))$.

$$\begin{cases}\beta_1\mu^*(t)+\int_0^T P_1^0(T-\tau,T-t)\mu^*(\tau)d\tau+\int_0^T P_2^0(T-\tau,T-t)\eta^*(\tau)d\tau=P_3^0(t),\\ \beta_2\eta^*(t)+\int_0^T P_1^b(T-\tau,T-t)\mu^*(\tau)d\tau+\int_0^T P_2^b(T-\tau,T-t)\eta^*(\tau)d\tau=P_3^b(t),\end{cases}$$

where

$$P_1^a(T-\tau,T-t)=\frac{\partial}{\partial x}Q_1(T-\tau,x,T-t)\Big|_{x=a},$$

$$P_2^a(T-\tau,T-t)=\frac{\partial}{\partial x}Q_2(T-\tau,x,T-t)\Big|_{x=a},\quad P_3^a(t)=\frac{\partial}{\partial x}Q_3(x,t)\Big|_{x=a},$$

6 Conclusion

The results of the work, the boundary control of the temperature field of the bar with a selected point can be useful in solving the problem of stabilization a loaded parabolic equation using boundary control, which can be used in problems of mathematical modeling using controlled loaded differential equations.

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References

[1] A. V. Fursikov, "Stabilizability of quasi linear parabolic equation by feedback boundary control," Sbornik Mathematics, London Mathematical Society, United Kingdom, **192** (4), 593–639 (2001).

[2] M. Jenaliyev, K. Imanberdiyev, A. Kassymbekova and K. Sharipov, "Stabilization of solutions of two-dimensional parabolic equations and related spectral problems," Eurasian Math. J., **11** (1), 72–85 (2020).

[3] J. L. Wang and H. N. Wu, "Passivity of delayed reaction-diffusion networks with application to a food web model," Appl. Math. Comput. **219** (24), 11311–11326 (2013).

[4] B. Tang, G. Sapiro and V. Caselles, "Color image enhancement via chromaticity diffusion," IEEE Trans. Image Process. (10), 701–707 (2001).

[5] M. Mitra and S. Gopalakrishnan, "Wave propagation in imperfectly bonded single walled carbon nanotube-polymer composites," J. Appl. Phys. (102), 084301 (2007).

[6] J.S. Wettlaufer, "Heat flux at the ice-ocean interface," J. Geophys. Res. Oceans **96** (C4), 7215–7236 (1991).

[7] S. Koga, L. Camacho-Solorio and M. Krstic, "State estimation for lithium ion batteries with phase transition materials," ASME 2017 Dynamic Systems and Control Conference, Tysons, Virginia, USA, 1–8 (2017).

[8] N. Ghaderi, M. Keyanpour and H. Mojallali, "Observer-based finite-time output feedback control of heat equation with Neumann boundary condition," Journal of the Franklin Institute **357** (14), 9154–9173 (2020).

[9] A. Polyakov and L. Fridman, "Stability notions and lyapunov functions for sliding mode control system," J. Francl. Inst. **351** (4), 1831–1865 (2014).

[10] A. Polyakov, J. M. Coron and L. Rosier, "On boundary finite-time feedback control for heat equation," 20th IFAC World Congress, Toulouse, France, (2017).

[11] A. Hasanov and M. Slodicka, "An analysis of inverse source problems with final time measured output data for the heat conduction equation: A semigroup approach, Applied Mathematics Letters 26 (2013) 207–214.

[12] M. I. Ramazanov, M. T. Kosmakova and L. Zh. Kasymova, "On a Problem of Heat Equation with Fractional Load," Lobachevskii Journal of Mathematics **41**, 1873–1885 (2020).

[13] L. Beilina and M. V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems* (Springer, New York, 2012).

[14] A. L. Bukhgeim and M. V. Klibanov, "Uniqueness in the large of a class of multidimensional inverse problems," Soviet Mathematics-Doklady (17), 244–247 (1981).

[15] V. Isakov, "Inverse Source Problems," in: Mathematical Surveys and Monographs, Vol. 34, American Mathematical Society, Providence-RI, (1990).

[16] V. Isakov, "Inverse parabolic problems with the final overdetermination," Communications on Pure and Applied Mathematics (54), 185–209 (1991).

[17] M. A. Naimark, *Lineinye differentsial'nye operatory (2-ye izd.) [Linear Differential Operators (2nd ed.)]* (Nauka, Main Edition of Physical and Mathematical Literature, Moscow, 1969) (in Russian).

[18] L. C. Evans, *Partial Differential Equations* (American Mathematical Society, 1998).

[19] V. A. Il'in, "The solvability of mixed problems for hyperbolic and parabolic equations," Uspekhi Mat. Nauk, **15** 2(92), 97–154 (1960); Russian Math. Surveys, **15** (1), 85–142 (1960).

[20] F. P. Vasiliyev, *Metody resheniya ekstremal'nykh zadach [Methods for solving extremal problems]* (Nauka, Main Edition of Physical and Mathematical Literature, Moscow, 1981) (in Russian).

[21] T. Kato, *Theory of perturbations of linear operators* (Mir, Moscow, 1972) (in Russian).

Список литературы

[1] A. V. Fursikov, "Stabilizability of quasi linear parabolic equation by feedback boundary control," Sbornik Mathematics, London Mathematical Society, United Kingdom, **192** (4), 593–639 (2001).

[2] M. Jenaliyev, K. Imanberdiyev, A. Kassymbekova and K. Sharipov, "Stabilization of solutions of two-dimensional parabolic equations and related spectral problems," Eurasian Math. J., **11** (1), 72–85 (2020).

[3] J. L. Wang and H. N. Wu, "Passivity of delayed reaction-diffusion networks with application to a food web model," Appl. Math. Comput. **219** (24), 11311–11326 (2013).

[4] B. Tang, G. Sapiro and V. Caselles, "Color image enhancement via chromaticity diffusion," IEEE Trans. Image Process. (10), 701–707 (2001).

[5] M. Mitra and S. Gopalakrishnan, "Wave propagation in imperfectly bonded single walled carbon nanotube-polymer composites," J. Appl. Phys. (102), 084301 (2007).

[6] J.S. Wettlaufer, "Heat flux at the ice-ocean interface," J. Geophys. Res. Oceans **96** (C4), 7215–7236 (1991).

[7] S. Koga, L. Camacho-Solorio and M. Krstic, "State estimation for lithium ion batteries with phase transition materials," ASME 2017 Dynamic Systems and Control Conference, Tysons, Virginia, USA, 1–8 (2017).

[8] N. Ghaderi, M. Keyanpour and H. Mojallali, "Observer-based finite-time output feedback control of heat equation with Neumann boundary condition," Journal of the Franklin Institute **357** (14), 9154–9173 (2020).

[9] A. Polyakov and L. Fridman, "Stability notions and lyapunov functions for sliding mode control system," J. Francl. Inst. **351** (4), 1831–1865 (2014).

[10] A. Polyakov, J. M. Coron and L. Rosier, "On boundary finite-time feedback control for heat equation," 20th IFAC World Congress, Toulouse, France, (2017).

[11] A. Hasanov and M. Slodicka, "An analysis of inverse source problems with final time measured output data for the heat conduction equation: A semigroup approach, Applied Mathematics Letters 26 (2013) 207–214.

[12] M. I. Ramazanov, M. T. Kosmakova and L. Zh. Kasymova, "On a Problem of Heat Equation with Fractional Load," Lobachevskii Journal of Mathematics **41**, 1873–1885 (2020).

[13] L. Beilina and M. V. Klibanov, *Approximate Global Convergence and Adaptivity for Coefficient Inverse Problems* (Springer, New York, 2012).

[14] A. L. Bukhgeim and M. V. Klibanov, "Uniqueness in the large of a class of multidimensional inverse problems," Soviet Mathematics-Doklady (17), 244–247 (1981).

[15] V. Isakov, "Inverse Source Problems," in: Mathematical Surveys and Monographs, Vol. 34, American Mathematical Society, Providence-RI, (1990).

[16] V. Isakov, "Inverse parabolic problems with the final overdetermination," Communications on Pure and Applied Mathematics (54), 185–209 (1991).

[17] M. A. Naimark, *Lineinye differentsial'nye operatory (2-ye izd.) [Linear Differential Operators (2nd ed.)]* (Nauka, Main Edition of Physical and Mathematical Literature, Moscow, 1969) (in Russian).

[18] L. C. Evans, *Partial Differential Equations* (American Mathematical Society, 1998).

[19] V. A. Il'in, "The solvability of mixed problems for hyperbolic and parabolic equations," Uspekhi Mat. Nauk, **15** 2(92), 97–154 (1960); Russian Math. Surveys, **15** (1), 85–142 (1960).

[20] F. P. Vasiliyev, *Metody resheniya ekstremal'nykh zadach [Methods for solving extremal problems]* (Nauka, Main Edition of Physical and Mathematical Literature, Moscow, 1981) (in Russian).

[21] T. Kato, *Theory of perturbations of linear operators* (Mir, Moscow, 1972) (in Russian).

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VAN DER CORPUT LEMMA WITH BESSEL FUNCTIONS

In this article, we study analogues of the van der Corput lemmas [19] involving Bessel functions. In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals. This estimate was first obtained by the Dutch mathematician Johannes Gaultherus van der Corput. Van der Corput interested in the behavior for large positive λ of the oscillatory integral $\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx$, where ϕ is a real-valued smooth function (the phase) and ψ is complex valued smooth function (amplitude). In case $a=-\infty, b=+\infty$, it is assumed that ψ has a compact support in $\mathbb{R}$. In our case we replace the exponential function with the Bessel functions, to study oscillatory integrals appearing in the analysis of wave equation with singular damping. More specifically, we study integral of the form $I(\lambda)=\int_a^b J_n(\lambda\phi(x))\psi(x)dx$ for the range $n=0$, where $\psi\in\mathcal{C}$ and smooth, and λ is a positive real number that can vary.The generalisations of the van der Corput lemma is proved. As an application of the above results, the generalised Riemann-Lebesgue lemma is considered.
Key words: van der Corput lemma, Bessel function, asymptotic estimate,wave equation,oscillatory integrals.

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Бессель функциялары қатысқан Ван дер Корпут леммасы

Бұл мақалада біз Ван дер Корпуттың Бессель функцияларын қамтитын леммасының аналогтарын зерттейміз. Гармоникалық талдауда ең маңызды бағалаулардың бірі - Ван дер Корпут леммасы, ол тербелмелі интегралдарды бағалау болып табылады. Бұл бағалауды алғаш рет голланд математигі Иоганнес Голтерус Ван дер Корпут алған. Ван дер Корпут $\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx$ тербеліс интегралының λ үлкен оң болғандағы әрекетіне қызығушылық танытты. ϕ- нақты тегіс функция (фаза), ал ψ- күрделі тегіс функция (амплитуда). $a=-\infty, b=+\infty$ жағдайында, ψ- $\mathbb{R}$ ішінде компактілі үйірткілі болады деп болжанады. Біздің жағдайда экспоненциалды функцияны Бессель функцияларымен ауыстырамыз, сингулярлы сөнген толқындық теңдеудің талдауында пайда болатын тербелмелі интеграл зерттеледі. Нақтырақ айтқанда, $n=0$ диапазоны үшін $I(\lambda)=\int_a^b J_n(\lambda\phi(x))\psi(x)dx$ түріндегі тербелмелі интегралды зерттейміз, мұндағы $\psi\in\mathcal{C}$ және тегіс, ал λ- өзгере алатын оң нақты сан. Ван дер Корпут леммасының жалпылауы дәлелденеді. Жоғарыда алынған нәтижелердің қолданысы ретінде жалпыланған Риман-Лебег леммасы қарастырылады.
Түйін сөздер: Ван дер Корпут леммасы, Бессель функциясы, асимптотикалық бағалау,толқындық теңдеу, тербелмелі интегралдар.

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Лемма Ван дер Корпута с функциями Бесселя

В данной статье мы изучаем аналоги лемму Ван дер Корпута [19] с функциями Бесселя. В гармоническом анализе одной из важнейших оценок является лемма Ван дер Корпута, которая является оценкой осциллирующих интегралов. Эта оценка впервые была получена голландским математиком Йоханнесом Голтерусом ван дер Корпутом. Ван дер Корпут интересовался поведением при больших положительных λ осциллирующего интеграла $\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx$,, где ϕ - вещественная гладкая функция (фаза), а ψ - комплексная гладкая функция (амплитуда). В случае $a=-\infty, b=+\infty$ предполагается, что ψ имеет компактный носитель в $\mathbb{R}$. В нашем случае показательная функция заменяется функциями Бесселя, чтобы изучить осциллирующие интегралы, возникающие при анализе волнового уравнения с сингулярным затуханием. В частности, мы изучаем интеграл вида $I(\lambda)=\int_a^b J_n(\lambda\phi(x))\psi(x)dx$ для диапазона $n=0$, где $\psi\in\mathcal{C}$ и гладкие, а λ - положительное действительное число, которое может меняться. Доказаны обобщения леммы Ван дер Корпута. В качестве приложения полученных результатов рассматривается обобщенная лемма Римана-Лебега.

Ключевые слова: лемма Ван дер Корпута, функция Бесселя, асимптотическая оценка, волновое уравнение, осциллирующие интегралы.

1 Introduction

In harmonic analysis, one of the most important estimates is the van der Corput lemma, which is an estimate of the oscillatory integrals.

This estimate was first obtained by the Dutch mathematician **Johannes Gaultherus van der Corput** (4 September 1890 – 16 September 1975) and named in his honour. While the paper [1] was published in Mathematische Annalen in 1921. Johannes Gaultherus van der Corput introduced the van der Corput lemma, a technique for creating an upper bound on the measure of a set drawn from harmonic analysis, and the van der Corput theorem on equidistribution modulo 1. He became member of the Royal Netherlands Academy of Arts and Sciences in 1929, and foreign member in 1953. He was a Plenary Speaker of the ICM in 1936 in Oslo.

Van der Corput interested in the behavior for large positive λ of the oscillatory integral

$$\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx,$$

where ϕ is a real-valued smooth function (the phase) and ψ is complex valued smooth function (amplitude). In case $a=-\infty, b=+\infty$, it is assumed that ψ has a compact support in $\mathbb{R}$.

Such integrals arise in the study of decay estimates of solutions of the Schrödinger and the wave equations.

Indeed, the estimate obtained by van der Corput, following Proposition 1 and Proposition 2 in Chapter VIII of [2], can be stated as follows:

Lemma 1 *Suppose ϕ is a real-valued and smooth function in $[a,b]$. If ψ is a smooth function, ϕ' is monotonic, $|\phi'|\geq 1$ for all $x\in(a,b)$, then*

$$\left|\int\limits_a^b e^{i\lambda\phi(x)}\psi(x)dx\right|\leq C\lambda^{-1},\ \ \lambda>0.$$

Various generalizations of the van der Corput lemmas have been investigated over the years. One dinemsional and multidimensional analogues of the van der Corput lemmas were studied in [3-8] while in [9] the multi-dimensional van der Corput lemma was obtained with constants independent of the phase and amplitude. We also note that in [10-12] optimal constants were found for various versions of van der Corput's lemmas. The main goal of the present paper is to study van der Corput lemmas for the oscillatory integral defined by

$$I(\lambda) = \int_a^b J_0(\lambda\phi(x))\psi(x)dx, \tag{1}$$

where $\psi \in C$ and smooth, and λ is a positive real number that can vary.

Recently, the attention of many mathematicians has been attracted by Van der Corput's estimates for integrals with special functions [13-18]. For example, in [13] mainly focus on numerical evaluation of highly oscillatory Bessel transforms and presented based on the multiple integral, the schemes for computing this class of the transform. In work [14] presented van der Corput-type lemmas for Bessel and Airy transforms with shifted Jacobi weight functions. These results on the asymptotic orders of the highly oscillatory integrals on the frequency are optimal. Furthermore, from these estimates, the convergence rates on Filon-type methods are easily derived.

2 Material and methods

In this section we consider $I(\lambda)$,defined by (1), that is

$$I(\lambda) = \int\limits_a^b J_0(\lambda\phi(x))\psi(x)dx.$$

From the Chapter II, §11 (1) of [20] the Bessel function is

$$J_\alpha(\lambda\phi(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\alpha)!}\left(\frac{\lambda}{2}\right)^{2m+\alpha} \phi^{2m+\alpha}(x), \alpha \in Z. \tag{2}$$

The behavior of the function $J_0(\lambda\phi(x))$ which we derived from Chapter III, §36 (1) of [20] for any values of $\phi(x)$ ($\phi(x) = 0$ excepted) and for all $\lambda \in C$ is

$$|J_0(\lambda\phi(x))| \leq 1. \tag{3}$$

As for small λ the integral (1) is just bounded, we consider the case $\lambda \geq 1$.

Lemma 2 *Let the function $\phi(x)$ is in $\in C^2[a,b]$, and $\phi(x) \neq 0, \phi'(x) \neq 0$, then*

$$J_0(\lambda\phi(x)) = -\frac{1}{\lambda^2} \cdot \frac{1}{(\phi(x)} \cdot \frac{1}{(\phi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))]. \tag{4}$$

Proof. We consider the function (2) in the case $\alpha = 0$

$$J_0(\lambda\phi(x)) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\lambda\phi(x)}{2}\right)^{2m},$$

and we will differentiate the function $J_0(\lambda\phi(x))$ once

$$\begin{aligned} \frac{d}{dx} J_0(\lambda\phi(x)) &= 2\sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m-1)!} \phi^{2m-1}(x)\phi'(x) \left(\frac{\lambda}{2}\right)^{2m} \\ &= 2\sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!m!} \phi^{2m+1}(x)\phi'(x) \left(\frac{\lambda}{2}\right)^{2m+2} \\ &= -\frac{\lambda^2}{2}\phi'(x) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!} \phi^{2m+1}(x) \left(\frac{\lambda}{2}\right)^{2m}. \end{aligned} \tag{5}$$

Then, transforming the (5), we will differentiate the function bellow

$$\frac{d}{dx}\left(\frac{\phi(x)}{\phi'(x)}\frac{d}{dx} J_0(\lambda\phi(x))\right) = -\frac{\lambda^2}{2} \cdot 2\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!m!}(m+1)\phi^{2m+1}(x)\phi'(x)\left(\frac{\lambda}{2}\right)^{2m}$$

$$= -\lambda^2\phi'(x)\phi(x) \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \phi^{2m+1}(x) \left(\frac{\lambda}{2}\right)^{2m}$$

$$= -\lambda^2\phi'(x)\phi(x) J_0(\lambda\phi(x)).$$

Thus, by differentiating twice we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x} J_0(\lambda\phi(x))\right) + \lambda^2\phi'(x)\phi(x) J_0(\lambda\phi(x)) = 0.$$

The proof is complete.

3 Results and discussion

We formulate our result in the form of theorem and show its application bellow

Theorem 1 *Let $-\infty \le a < b \le \infty$ and $\alpha = 0$. Let $\phi(x) \in C^2[a,b]$ and $\psi \in C^1[a,b]$. If $|\phi(x)| > 1, |\phi'(x)| > 1, |\phi''(x)| > 1$ for all $x \in [a,b]$, then we shall now prove that*

$$|I(\lambda)| \le \frac{1}{\lambda^2}\left|\frac{\phi(b)}{\phi'(b)} D(b) - \frac{\phi(a)}{\phi'(a)} D(a)\right|.$$

Proof. We put the function $J_0(\lambda\phi(x))$ to the integral (1) and integrating it by parts we find that

$$I(\lambda) = \int_a^b J_0(\lambda\phi(x))\psi(x)dx = -\frac{1}{\lambda^2}\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x} J_0(\lambda\phi(x)\right) \cdot \frac{\psi(x)}{\phi(x)\phi'(x)} dx$$

$$= -\frac{1}{\lambda^2}\left[\frac{\psi(x)}{\phi'^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))|_{x=a}^{x=b} - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x))\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)dx\right]$$

$$= -\frac{C}{\lambda^2}\int_a^b J_0(\lambda\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)dx,$$

where

$$C = \left[\frac{\psi(x)}{\phi'^2(x)}\frac{\mathrm{d}}{\mathrm{d}x}J_0(\lambda\phi(x)) - \frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)J_0(\lambda\phi(x))\right]|_{x=a}^{x=b}.$$

Applying the expression (3) and suppose that $\phi(x) \in C^2[a,b]$, $\psi \in C^1[a,b]$,$|\phi(x)| > 1, |\phi'(x)| > 1, |\phi''(x)| > 1$ for all $x \in R$ we have

$$|I(\lambda)| = \left|-\frac{C}{\lambda^2}\int_a^b J_0(\lambda\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)dx\right|$$

$$= \frac{1}{\lambda^2}\int_a^b |J_0(\lambda\phi(x)|\left|\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)\right)\right|dx$$

$$\leq \frac{C}{\lambda^2}\left|\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\phi(x)}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)dx\right)\right|$$

$$\leq \frac{C}{\lambda^2}\left|\frac{\phi(b)}{\phi'(b)}D(b) - \frac{\phi(a)}{\phi'(a)}D(a)\right|,$$

where

$$D(b) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)'|_{x=b}.$$

and

$$D(a) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\psi(x)}{\phi(x)\phi'(x)}\right)'|_{x=a}.$$

The proof is complete.

Remark 1 *If $[a,b] = R := (-\infty, +\infty)$, then assume that $\psi \in C_0^1(R)$ is the function with compact support.*

3.1 Aplication. Generalised Rieman-Lebesgue lemma

The Riemann-Lebesgue lemma is the classical result of harmonic and asymptotic analysis. The simplest form of the Riemann-Lebesgue lemma states that for a function $f \in C^1([a,b])$ we obtain

$$\int_a^b e^{ikx}f(x)dx = O\left(\frac{1}{k}\right), \quad \text{at} \quad k \to \infty.$$

We consider the following integral of Fourier-Bessel transform

$$\int_a^b J_0(kx)f(x)dx.$$

If $f \in C^2([a,b])$, then from the van der Corput lemma and by Theorem 1 we have

$$\int_a^b J_0(kx)f(x)dx = O\left(k^{-2}\right).$$

4 Conclusion

Thus, in this paper, we consider analogues of the van der Corput lemma involving Bessel functions. The main result of the work is to study oscillatory integrals appearing in the analysis of wave equation with singular damping. We have proved the behavior of the oscillatory integral for large positive λ. Therefore, the estimates, which we got, can be used to for proofs of generalised Riemann-Lebesgue lemma.

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References

[1] J.G. van der Corput, Zahlentheoretische Abschätzungen , *Math. Ann.* 84: 1-2 (1921): 53–79.

[2] E.M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals" , *Princeton Mathematical Series.*, Vol. 43., Princeton Univ. Press, Princeton, 1993.

[3] E. M. Stein, S. Wainger,"The estimation of an integral arising in multiplier transformations" , *Studia Math.*, 35 (1970): 101-104.

[4] D. H. Phong, E. M. Stein,"Operator versions of the van der Corput lemma and Fourier integral operators" , *Math. Res. Lett.*, 1(1994): 27–33.

[5] D. H. Phong, E. M. Stein,"Models of degenerate Fourier integral operators and Radon transforms" , *Ann. Math.*, 140:3 (1994): 703-722.

[6] A. Carbery, M. Christ, J. Wright, "Multidimensional van der Corput and sublevel set estimates" , *J. Amer. Math. Soc.*, 1999: 981-1015.

[7] M. Christ, X. Li, T. Tao, C. Thiele, "On multilinear oscillatory integrals, nonsingular and singular" , *Duke Math. J.*, 130, 2 (2005):321–351.

[8] J. Bourgain, L. Guth., "Bounds on oscillatory integral operators based on multilinear estimate" , *Geom. Funct. Anal.*, 21, 6 (2011): 1239-1295.

[9] M. Ruzhansky, "Multidimensional decay in the van der Corput lemma" , *Studia Math.*, 208 (2012): 1-10.

[10] M. Greenblatt,"Sharp estimates for one-dimensional oscillatory integral operators with phase" , *Amer. J. Math.*,127,3 (2005): 659-695.

[11] K. M. Rogers, "Sharp van der Corput estimates and minimal divided differences" , *Proc. Amer. Math. Soc.*, 133, 12 (2005): 3543-3550.

[12] I. R. Parissis, A sharp bound for the Stein-Wainger oscillatory integral // Proc. Amer. Math. Soc.–136:3 (2008), 963–972.

[13] R. Chen, G. Yang, "Numerical evaluation of highly oscillatory Bessel transforms" , *Comput. Appl. Math.*,342(2018): 16-24.

[14] S. Xiang, J. Comput, "On van der Corput-type lemmas for Bessel and Airy transforms and applications" , *Appl. Math.*, 351 (2019): 179-185.

[15] M. Ruzhansky and B.T.Torebek, "Van der Corput lemmas for Mittag-Leffler functions" ,2020, 1-32, arXiv:2002.07492.

[16] M. Ruzhansky and B.T.Torebek, "Van der Corput lemmas for Mittag-Leffler functions, II, α-directions" , *Bulletin des Sciences Mathеʼmatiques*, 171, 103016 (2021): 1–23.

[17] S. Zaman, S.Siraj-ul-Islam, I. Hussain, J. Comput, "Approximation of highly oscillatory integrals containing special functions" , *Appl. Math.*, 2020, 365, 112372.

[18] M. Ruzhansky and B.T. Torebek,"Multidimensional Van der Corput-type estimates involving Mittag-Leffler functions" , *Fractional Calculus and Applied Analysis*, 23, 6 (2020): 1663–1677.

[19] N.N. Lebedev,"Special functions and their applications, 2nd rev.ed." , *Fiz-matgiz, Moscow*,1963; , *English transl., Prentice-Hall , Englewood Cliffs,* 1965.

[20] G.N. Watson, "A treatise on the theory of Bessel functions " , *Foreign literature publishing*, Moscow, 1945: 49.

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ON THE SCHWARZ PROBLEM FOR THE MOISIL–TEODORESCU SYSTEM IN A SPHERICAL LAYER AND IN THE INTERIOR OF A TORUS

The theory of analytic functions is a classical direction in the study of elliptic equations and equations of mixed type in the plane. In a three-dimensional bounded domain $\Omega \subseteq \mathbb{R}^3$, we consider the elliptic system

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0,$$

where $u(x) = (u_0, u_1, u_2, u_3)$ is the desired vector function $u \in C^1(\Omega)$. Such a system is called the Moisil-Teodorescu system. For solutions of this system, the basic facts of the theory of analytic functions on the plane are valid, including the integral theorem and Cauchy's formula, Morera's theorem, and others. Doubly connected regions play a significant role in fluid mechanics. For example, the flow created by a long solid cylinder moving in the direction of the normal to its axis occurs precisely in a doubly connected region. In this paper, we write out the fundamental solution of the differential operator $M(\partial/\partial x)$ in the space $\mathbb{R}^3$ and present well-posed problems for the Moisil–Teodorescu system in the case of a spherical layer and the interior of a torus. The results of this work show a significant difference between the well-posed problem in a spherical layer and a similar problem in a torus.

Key words: Cauchy–Riemann system, Moisil–Teodorescu system, Schwartz problem, spherical layer, torus interior, solvability of the problem.

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Сфералық қабаттағы және тордың ішкі бөлігіндегі Моисил–Теодореско жүйесі үшін Шварц есебі туралы

Аналитикалық функциялар теориясы жазықтықтағы эллиптикалық теңдеулер мен аралас типтес теңдеулерді зерттеуде классикалық бағыт болып табылады. Үшөлшемді шенелген $\Omega \subseteq \mathbb{R}^3$ обылысында келесі эллиптикалық жүйесін қарастырамыз

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0,$$

мұндағы $u(x) = (u_0, u_1, u_2, u_3)$ вектор- функциясы $C^1(\Omega)$ класынан. Мұндай жүйе Моисил-Теодореско жүйесі деп аталады. Бұл жүйенің шешімдері үшін жазықтықтағы аналитикалық функциялар теориясының негізгі фактілері, соның ішінде Кошидің интегралдық теоремасы мен формуласы, Морер теоремасы және басқалар. Екі байланысты облыстар сұйықтықтар механикасында маңызды рөл атқарады. Мысалы, өз осімен нормаль бағытта қозғалатын ұзын тұтас цилиндрден жасалған ағын дәл екі байланысты облыста жүзеге асады. Бұл жұмыста $M(\partial/\partial x)$ дифференциалдық операторы үшін $\mathbb{R}^3$ кеңістігінде іргелі шешімі жазылған және сфералық қабаттағы және тордың ішкі бөлігіндегі Моисил–Теодореско жүйесі үшін тиянақты есептер келтірілген. Моисил–Теодореско жүйесі эллиптикалық Коши–Риман жүйесінің жалпыланған мысалы болып табылады. Бұл жұмыстың нәтижелерінен сфералық қабаттағы тиянақты қойылған есеп пен тордың ішкі бөлігінде қойылған есептің арасында айтарлықтай айырмашылықты көреміз.

Түйін сөздер: Коши–Риман жүйесі, Моисил–Теодореско жүйесі, Шварц есебі, сфералық қабат, тордың ішкі бөлігі, есептің шешімділігі.

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О задаче Шварца для системы Моисила–Теодореску в шаровом слое и во внутренности тора

Теория аналитических функции является классическим направлением в изучении эллиптических уравнений и уравнений смешанного типа на плоскости. В трехмерной ограниченной области $\Omega \subseteq \mathbb{R}^3$ рассматривается эллиптическая система

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0,$$

где $u(x) = (u_0, u_1, u_2, u_3)$ – искомая вектор- функция $u \in C^1(\Omega)$. Такая система называется системой Моисила – Теодореску. Для решений этой системы справедливы основные факты теории аналитических функций на плоскости, включая интегральную теорему и формулу Коши, теорему Морера и другие. Двусвязные области играют значительную роль в механике жидкости. К примеру течение, создаваемое длинным твердым цилиндром, движущегося в направлении нормали к своей оси, происходит именно в двусвязной области. В данной работе выписан фундаментальное решение дифференциального оператора $M(\partial/\partial x)$ в пространстве $\mathbb{R}^3$ и приведены корректные задачи для системы Моисила–Теодореску в случае шарового слоя и внутренности тора. Из результатов данной работы видно существенное отличие корректной задачи в шаровом слое от аналогичной задачи в торе.

Ключевые слова: система Коши–Римана, система Моисила–Теодореску, задача Шварца, шаровой слой, внутренность тора, разрешимость задачи.

1 Introduction

Complex analysis methods constitute a classical direction in the study of elliptic equations and equations of mixed type on the plane. At present, active research is being carried out in this direction in many mathematical centers of the world.

Multiply connected (in particular, doubly-connected) domains play an important role in fluid mechanics. For example [1] the flow created by a long solid cylinder moving in the direction of the normal to its axis, occurs precisely in a two-connected domain. From the fact that certain closed curves in such a domain are non-contractible to a point, it follows that the presence of lifting power. Another example [1] is the motion of a smoke ring in the outside of the torus. Thus, it makes sense to study the well-posed formulation of the problems for elliptic systems in multiply connected domains. Plane multiply connected domains are usually described by the number of connected components of the boundary of the domain. Spatially multiply connected domains already require a large number of topological characteristics. For spatial multiply connected domains along with the number of connected components of the boundary of the domain, it is convenient to consider also so-called the order of connectedness of the domain [2, 3].

In this paper, we denote the number of connected components of the boundary of the domain by n, and the order of connectedness of the domain is denoted by m. For example, for a spherical layer in a three-dimensional space $n = 2, m = 1$, and for the interior of a torus in the same space $n = 1, m = 2$.

It is noted that the formulation of well-posed problems for first order elliptic systems depend on the numbers n, m in [4–6].

This paper presents the well-posed problems for the Moisil–Theodorescu system in the case of the spherical layer and the interior of the torus. The results of this work show a significant difference between well-posed problem in the spherical layer and the similar problem in the torus. A more general investigation of the Fredholm property of boundary value problems of first order elliptic systems in multiply connected domains can be found in the papers of A.P. Soldatov [7–9]. Moreover the index of the studied problems is calculated in [7].

Materials and methods

2 Cauchy–Riemann and Moisil–Teodorescu systems

In a flat bounded domain $\Omega \subseteq \mathbb{R}^2$, we consider the elliptic system

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} \partial/\partial x_1 & -\partial/\partial x_2 \\ \partial/\partial x_2 & -\partial/\partial x_1 \end{pmatrix} u(x) = 0,$$

where $u(x) = (u_1, u_2)$ is the desired vector function $u \in C^1(\Omega)$. Such a system is called a Cauchy-Riemann system.

In a three-dimensional bounded domain $\Omega \subseteq \mathbb{R}^3$, we consider the elliptic system

$$M(\partial/\partial x)u(x) \equiv \begin{pmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial/\partial x_1 & 0 & -\partial/\partial x_3 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_3 & 0 & -\partial/\partial x_1 \\ \partial/\partial x_3 & -\partial/\partial x_2 & \partial/\partial x_1 & 0 \end{pmatrix} u(x) = 0, \tag{1}$$

where $u(x) = (u_0, u_1, u_2, u_3)$ is the desired vector function $u \in C^1(\Omega)$. Such a system is called the Moisil-Teodorescu system.

For the solutions of this system, the basic facts of the theory of analytic functions on the plane are valid, including the integral theorem and the Cauchy formula, the Morera theorem, etc. The foundations of this theory were laid in the works of G.K. Moisil and N. Teodorescu [10]. It is easy to show that all components u_j of the solution $u = (u_0, u_1, u_2, u_3)$ of system (1) are harmonic functions. In this sense, it is an example of a multidimensional generalized Cauchy-Riemann system [11].

This theory was further developed in the works of A.V. Bitsadze [12,13]. In particular, he introduced the concept of a Cauchy-type integral for system (1) and pointed out its various applications.

The fundamental solution of the differential operator $M(\partial/\partial x)$ in the space $\mathbb{R}^3$ is the matrix function $M^\top(x)/|x|^3$, where $\top$ – matrix transposition symbol. In these notations, the integral

$$(I\varphi)(x) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-x)}{|y-x|^3} M[n(y)]\varphi(y)d_2y, \quad x \notin \Gamma, \tag{2}$$

where d_2y is the area element on the surface $\Gamma = \partial D$ and $n(y)$ is the unit normal, determines the solution of system (1). The choice of density in the form $M[n(y)]\varphi(y)$ is dictated by the fact that it ensures the validity of the analogue of the Sokhotsky-Plemelja formulas.

Namely, if the function φ satisfies the Holder condition and the surface Γ is a Lyapunov surface, then there exist limit values

$$u^\pm(y_0) = \lim_{x\to y_0, x\in D^\pm} u(x), \quad y_0 \in \Gamma,$$

for which the analogue of the Sokhotsky-Plemelya formulas is valid

$$u^\pm = \pm\varphi + u^*. \tag{3}$$

Here $D^+ = D,\ D^- = \mathbb{R}^3 \setminus \overline{D}$, the normal n is assumed to be external to D and the function $u^* = I^*\varphi$ is defined by the singular integral

$$(I^*\varphi)(y_0) = \frac{1}{2\pi}\int_\Gamma \frac{M^\top(y-y_0)}{|y-y_0|^3} M[n(y)]\varphi(y)d_2y, \tag{4}$$

which is understood as the limit at $\varepsilon \to 0$ of integrals over $\Gamma \cap \{|y-y_0| \geq \varepsilon\}$. These formulas were first obtained by A.V. Bitsadze [12]. From the point of view of the minimum requirements for surface smoothness, this result was refined in [14]: if Γ belongs to the class $C^{1,\nu},\ 0 < \nu < 1$, then the operator I is bounded $C^\mu(\Gamma) \to C^\mu(\overline{D}),\ \ 0 < \mu < \nu$. Here and below, by $C^\mu(G)$ we mean the Banach Holder space defined by the usual norm

$$|\varphi|_{\mu,G} = |\varphi|_{0,G} + [\varphi]_{\mu,G} \quad [\varphi]_{\mu,G} = \sup_{x\neq y, x,y\in G} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\mu},$$

where $|\varphi|_{0,G}$ means sup $-$norm. Similar meaning has the space $C^{1,\mu}(\overline{D})$ continuously - differentiable functions and the class of $C^{1,\mu}$ surfaces.

In terms of the integral (2), the Cauchy integral formula for solutions $u \in C^{\mu}(\overline{D})$ of system (1) in a finite domain D can be written as

$$u(x) = (Iu^{+})(x) \quad x \in D. \tag{5}$$

In this case, the Cauchy theorem gives the equality

$$(Iu^{+})(x) = 0, \quad x \in D^{-}. \tag{6}$$

If the domain D is infinite, then under the additional assumption

$$u(x) = o(|x|^{-1}) \tag{7}$$

for $|x| \to \infty$ these formulas remain valid.

Let the function $u(x)$ be given and be a solution to (1) in each component of the complement to Γ, satisfies the Holder condition in their closure and condition (7) at infinity. Then from the formulas (5) and (6) applied in these components are completely similarly to the case of analytic functions, we derive the representation

$$u = I(u^{+} - u^{-}) \tag{8}$$

general solution in the form of a Cauchy-type integral. Taking into account the Sokhotsky-Plemelya formulas (3), this representation allows the problem of linear conjugation

$$u^{+} - Gu^{-} = f$$

with a given invertible $(4 \times 4)-$ matrix $G \in C^{\mu}(\Gamma)$ reduce to an equivalent two-dimensional singular integral equation

$$(\varphi + I^{*}\varphi) + G(\varphi - I^{*}\varphi) = f.$$

Results and discussion

3 The Schwartz problem for the Moisil–Theodorescu system in the spherical layer

Let $\Omega = \{x \in \mathbb{R}^3 : 0 < r_1 < |x| < r_2\}$, where r_1, r_2 are some positive numbers. We denote by Γ the boundary of the domain Ω, i.e. $\Gamma = \{x \in \mathbb{R}^3 : |x| = r_1\} \cup \{x \in \mathbb{R}^3 : |x| = r_2\}$. It is required to find the vector-function $u = (u_0, u_1, u_2, u_3) = (u_0, \widetilde{u})$ that satisfies the Moisil–Theodorescu system

$$\begin{cases} div\, \tilde{u} = 0, \quad x \in \Omega, \\ grad\, u_0 + rot\, \tilde{u} = 0, \quad x \in \Omega, \end{cases} \tag{9}$$

and Schwartz conditions

$$\begin{cases} u_0^{+}(y) = f_1(y), \quad y \in \Gamma, \\ \tilde{u}^{+}(y)\, n(y) = f_2(y), \quad y \in \Gamma, \end{cases} \tag{10}$$

where $\vec{n}(y)$ is the outer normal to the boundary Γ at the point y.

Here, in what follows, we will use the operations of a vector field, which for the vector function $u = (u_1, u_2, u_3) \in C^1(\Omega)$ and the scalar function $w \in C^1(\Omega)$ are defined by the equalities

$$div\,\tilde{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, gradw = \left(\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_3}\right),$$

and

$$rot\,\tilde{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

It is directly verified that the system (9) is an elliptic system. The boundary conditions (10) satisfy the complementarity condition [5-7,11]. Therefore the problem (9),(10) has a Fredholm property. Necessary and sufficient conditions for the solvability of problem (9),(10) are noted in [7]. To describe the condition of solvability of (9),(10), we need the following constructions.

We choose an open covering $S_k, k = 1, 2, 3, 4$ of the surface Γ and the unit tangent vectors $\vec{p}_k(y), \vec{q}_k(y)$ to Γ from the class $C^\mu(S_k)$ so that at each point $y \in S_k$ the vectors $\vec{p}_k(y), \vec{q}_k(y)$ and $\vec{n}(y)$ were pairwise orthogonal.

Since Γ is the union of two spheres, it follows that such a choice is possible. By the results of [7], we introduce the conjugate problem to (9),(10)

$$\begin{cases} div\,\tilde{v} = 0, \quad x \in \Omega, \\ grad\,v_0 + rot\,\tilde{v} = 0, \quad x \in \Omega, \end{cases} \tag{11}$$

$$\begin{cases} \tilde{v}^+(y)\,p_k(y) = 0, \quad y \in \Gamma, \\ \tilde{v}^+(y)\,q_k(y) = 0, \quad y \in S_k, \quad k = 1, 2, 3, 4. \end{cases} \tag{12}$$

Proposition 1 *[7] The nonhomogeneous problem (9),(10) is solvable in the class $C^\mu(\overline{\Omega})$ if and only if the orthogonality condition*

$$\begin{cases} \int_{|y|=r_1} [-yf_1(y)\,\tilde{v}^+(y) + r_1 f_2(y)\,v_0^+(y)]d_2y = 0, \\ \\ \int_{|y|=r_2} [yf_1(y)\,\tilde{v}^+(y) + r_2 f_2(y)\,v_0^+(y)]d_2y = 0, \end{cases} \tag{13}$$

holds for all $(v_0, \tilde{v})$ representing the solutions of the homogeneous problem (11),(12).

Further we assume that the orthogonality requirements (13) for the data f_1, f_2 are satisfied. So, problem (9),(10) is solvable (can be ambiguously solvable). One of the possible solutions of the problem (9),(10) is denoted by $(w_0, \tilde{w})$, $x \in \Omega$.

We formulate the following statement that is useful for further investigation.

Lemma 1 *The first component $u_0(x)$ of the vector-function $u = (u_0, u_1, u_2, u_3)$ represents the solution of the Dirichlet problem for the Laplace equation*

$$\begin{cases} \Delta u_0 = 0, \quad x \in \Omega, \\ u_0^+(y) = f_1(y), \quad y \in \Gamma. \end{cases} \tag{14}$$

Since the Dirichlet problem for the Laplace equation (14) has a unique solution, then previously introduced $w_0(x) \equiv u_0(x)$ for all $x \in \Omega$.

The second equation of system (9) implies that

$$grad\,u_0 + rot\,\tilde{u} \equiv 0, \quad x \in \Omega,$$

$$grad\, w_0 + rot\, \tilde{w} \equiv 0, \quad x \in \Omega.$$

Subtracting one equality from the other, we obtain the following equation

$$rot\, (\tilde{u} - \tilde{w}) = 0, \quad x \in \Omega.$$

By the same way we can write down the boundary condition

$$(\tilde{u}^+ - \tilde{w}^+) \cdot \vec{n}(y) = 0, \quad y \in \Gamma.$$

The difference $\tilde{u}(x) - \tilde{w}(x)$ we denote by $\tilde{\theta}(x)$. Hence, it follows that $\tilde{\theta}(x)$ is the solution of the homogeneous problem

$$\begin{cases} div\, \tilde{\theta} = 0, \quad x \in \Omega, \\ rot\, \tilde{\theta} = 0, \quad x \in \Omega, \\ \tilde{\theta}^+ n = 0, \quad y \in \Gamma. \end{cases} \tag{15}$$

Lemma 2 *To solve the inhomogeneous problem*

$$\begin{cases} div\, \tilde{u} = 0, \quad x \in \Omega, \\ rot\, \tilde{u} = -\theta(x), \quad x \in \Omega, \\ \tilde{u}^+ n = 0, \quad y \in \Gamma \end{cases} \tag{16}$$

the relation

$$\int_{l'_{y_0,y}} \tilde{u}^+(y)\, e(y) d_1 y - \int_{l^{-1}_{y_0,y}} \tilde{u}^+(y)\, e(y) d_1 y = -\int_S \theta(x)\, n(x) d_2 x,$$

for any $y \in \Omega$ and for any $l_{y_0,y}$.

Proof 1 *Let's fix a point $y_0 \in D$ and choose an arbitrary $y \in \Omega$. Let $l_{y_0,y}, l'_{y_0,y} \subset D$ be arbitrary paths that connecting the point y_0, y and lying entirely in this region. These paths $l_{y_0,y} \sim l'_{y_0,y}$ are homotopic in Ω,, since the domain Ω is a spherical layer. Let $L = l_{y_0,y} \cup l'_{y_0,y}$ and denote by S the surface that formed by the closed contour L.*
Therefore, the closed-loop integral L by the Stokes formula is equal to

$$\int_L \tilde{u}^+(y) e(y) d_1 y = \int_S (\mathrm{rot}\, \tilde{u})^+(x) n(x) d_2 x,$$

where $e(y)$ is a unit tangent vector to the contour $\partial\Gamma_0$, oriented positively with respect to n (i.e. the traversal of this contour, as viewed from the end of the vector n, is carried out counterclockwise). According to the second equation of system (16), we have relation

$$\int_L \tilde{u}^+(y) e(y) d_1 y = -\int_S \theta(x) n(x) d_2 x.$$

Since $L = l'_{y_0,y} \cup l^{-1}_{y_0,y}$, then we rewrite the last relation in the form

$$\int_{l'_{y_0,y}} \tilde{u}^+(y) e(y) d_1 y - \int_{l^{-1}_{y_0,y}} \tilde{u}^+(y) e(y) d_1 y = -\int_S \theta(x) n(x) d_2 x.$$

This is true for any $y \in D$ and for any $l_{y_0,y}$.

The following statement is proved in [7].

Theorem 1 *[7] The homogeneous problem (15) defined in the spherical layer has a unique solution belonging to the class $C^{\mu}(\overline{\Omega})$.*

Proposition 1 implies that the nonhomogeneous problem (9),(10) has a solution if the requirements (13) hold. Thus, the results of [7] imply the existence of a single well-posed problem for system (9) in the spherical layer.

4 The Schwartz problems for the Moisil–Theodorescu system in the interior of a torus in three-dimensional space

Let $\Omega = \{(x_1, x_2, x_3) : x_1 = rcos\varphi, \quad x_2 = rcos\theta(3 + sin\varphi), \quad x_3 = rsin\theta(3 + sin\varphi), \quad r < 1, 0 \le \varphi \le 2\pi, 0 \le \theta \le 2\pi\}$ presents the interior of the torus in three-dimensional space.

By Γ we denote the boundary of the domain Ω, namely $\Gamma = \{(y_1, y_2, y_3) : y_1 = cos\varphi, y_2 = cos\theta(3+sin\varphi), y_3 = sin\theta(3+sin\varphi), 0 \le \varphi \le 2\pi, 0 \le \theta \le 2\pi\}$. It is required to find the scalar function $u_0(x)$ and the vector-function $\tilde{u} = (u_1, u_2, u_3)$ that satisfy the Moisil–Theodorescu system with Schwartz conditions

$$\begin{cases} div\,\tilde{u} = 0, \quad x \in \Omega, \\ gradu_0 + rot\,\tilde{u} = 0, \quad x \in \Omega, \end{cases} \tag{17}$$

$$\begin{cases} u_0^+(y) = f_1(y), \quad y \in \Gamma, \\ \tilde{u}^+(y)\,n(y) = f_2(y), \quad y \in \Gamma, \end{cases} \tag{18}$$

$$\int_{-\pi}^{\pi}[-u_2(0, 3cos\theta, 3sin\theta)sin\theta + u_3(0, 3cos\theta, 3sin\theta)cos\theta]d\theta = \alpha(u_0^+, \tilde{u}^+n), \tag{19}$$

where the quantity α represents an arbitrary linear continuous functional in the space $C^1(\Gamma)\times C^1(\Gamma)$.

The Fredholm index of the problem (17),(18) (without condition (19)) is calculated in [7]. By the results of the work [7] the nonhomogeneous problem (17),(18) (without condition (19)) is solvable in $C^{\mu}(\overline{\Omega})$ if and only if the orthogonality condition

$$\int_{\Gamma} f_2(y)d_2y = 0. \tag{20}$$

holds.

Further we assume that the orthogonality condition (20) holds.

Proposition 2 *If the condition (19) holds, then the problem (17),(18) is uniquely solvable.*

Proof 2 *We will prove this proposition by contradiction. Suppose that there exist two solutions of the problem (17),(18). We denote them by $u_0(x)$, $\tilde{u}(x)$ and $w_0(x)$, $\tilde{w}(x)$.*

It is clear that $u_0(x) = w_0(x)$, $x \in \Omega$. The similar statement is proved in section 3. The difference $\tilde{u}(x) - \tilde{w}(x)$ we denote by $\check{\theta}(x)$. Thus, $\check{\theta}(x)$ is a solution of the problem

$$\begin{cases} div\,v(x) = 0, \quad x \in \Omega, \\ rot\,v(x) = 0, \quad x \in \Omega, \end{cases}$$

$$v(y)^{+}n(y) = 0, \quad y \in \Gamma,$$
$$\int_{-\pi}^{\pi} [-v_2(0, 3cos\theta, 3sin\theta)sin\theta + v_3(0, 3cos\theta, 3sin\theta)cos\theta]d\theta = 0, \tag{21}$$

By the results of the work [7] there exists a harmonic function $\varphi(x)$ such that $\check{\theta}(x) = grad\,\varphi(x)$. On the boundary Γ the harmonic function $\varphi(x)$ satisfies the following condition

$$\frac{\partial\varphi}{\partial n} = 0, \quad y \in \Gamma.$$

The third condition in (21) means that

$$\lim_{\theta\to\pi} \varphi(0, 3cos\theta, 3sin\theta) = \lim_{\theta\to-\pi} \varphi(0, 3cos\theta, 3sin\theta). \tag{22}$$

It was proved in [7] that the difference of the limits

$$\lim_{x_3\to 0^+, x_2<0} \varphi(x_1, x_2, x_3) - \lim_{x_3\to 0^-, x_2<0} \varphi(x_1, x_2, x_3)$$

does not depend on the points $(x_1, x_2, x_3) \in \Omega,\ x_2 < 0, x_3 = 0$ for $(x_1, x_2, x_3) \in \Omega$.

Then (22) implies that

$$\lim_{x_3\to 0^+, x_2<0} \varphi(x_1, x_2, x_3) - \lim_{x_3\to 0^-, x_2<0} \varphi(x_1, x_2, x_3) = 0$$

for $(x_1, x_2, x_3) \in \Omega$, $x_2 < 0$. In this case we conclude that $\varphi(x) = const$ for all $x \in \Omega$. Consequently, $\check{\theta}(x) \equiv 0$ for all $x \in \Omega$.

We now state the main result of this section.

Theorem 2 *Let f_1 and f_2 be arbitrary functions in $C^1(\Gamma)$, and (20) holds for f_2. Then the problem (17),(18) has a unique solution for arbitrary linear continuous functional $\alpha(f_1, f_2)$ in $C^1(\Gamma) \times C^1(\Gamma)$.*

Remark 1 *The functional $\alpha(f_1, f_2)$ can be defined by the formula*

$$\alpha(f_1, f_2) = \int_\Gamma f_1(y)\mu_1(y)d_2y - \int_\Gamma f_2(y)\mu_2(y)d_2y,$$

where $\mu_1(\cdot), \mu_2(\cdot)$ are continuous functions on the surface Γ.

In this case, in the problem (17),(18) condition (19) is a nonlocal boundary condition. Nonlocal boundary value problems for differential equations have been studied by many authors. In particular, in the work [16] systematically studied solutions of nonlocal problems for pseudo-hyperbolic equations.

In [17–19] works questions of the Fredholm solvability of the Neumann problem for a higher order elliptic equation on the plane were studied, and the equivalence of the solvability condition for the generalized Neumann problem with the complementary condition (the Shapiro-Lopatinsky condition) was proved.

Conclusion. Thus, in this paper, we considered the Moisil – Teodorescu elliptic system $M(\partial/\partial x)u(x) = 0$ in a three-dimensional bounded domain $\Omega \subseteq \mathbb{R}^3$. For solutions of this system, the basic facts of the theory of analytic functions on the plane are valid, including the integral theorem and Cauchy's formula, Morera's theorem, and others. In this paper, we write out the fundamental solution of the differential operator $M(\partial/\partial x)$ in the space $\mathbb{R}^3$ and present well-posed problems for the Moisil–Teodorescu system in the case of a spherical layer and the interior of a torus. The results of this work show a significant difference between the well-posed problem in a spherical layer and a similar problem in a torus.

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References

[1] Batchelor G.K., An Introduction to Fluid Dynamics, Cambridge University Press. 2000.

[2] Hatcher A., Algebraic Topology, Cambridge University Press. New Yourk. 2002.

[3] Simmons G.F., Introduction to Topology and Modern Analysis, McGraw Hill Book Company. 1968.

[4] Agmon S., Douglis A., Nirenberg L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II, Comm. Pure and Appl. Math., 17 (1964), 35—92.

[5] Shevchenko V.I., Some boundary value problem for holomorphic vector, Sborn. Math. Phys. Kiev, 8 (1970), 172—187. (in Russian)

[6] Yao Y., The Riemann boundary value problem for the Moisil-Theodorescu system, In Tournal-Sichuan Normal University. Natural Science Edition, 29 (2006), 188—191.

[7] Soldatov A.P., The Riemann-Hilbert problem for the Moisil-Theodorescu system, Izv. RAN, Ser. Math. (2018) (in Russian).

[8] Polunin V.A., Soldatov A.P., Riemann-Hilbert problem for the Moisil-Teodorescu system in multiply connected domains, Electronic Journal of Differential Equations, 310 (2016), 1–5.

[9] Polunin V.A., Soldatov A.P., On the integral represenation of solutions of the Moisil-Theodorescu system in multiply connected domains, Dokl. RAN, 475:4 (2017), 192–201.

[10] Moisil Gr. C., Theodoresco N., Fonctions holomorphes dans l'espace, Mathematica, 5 (1931), 142-153.

[11] Stein I., Weiss G., Introduction to harmonic analysis on Euclidean spaces, Moscow. Mir. 1974.

[12] Bitsadze A.V., Spatial analogue of the Cauchy-type integral and some of its applications, Izv. USSR Academy of Sciences, ser. math., 17:6 (1953), 525-538.

[13] Bitsadze A.V., Boundary value problems for second order elliptic equations, Moscow. Mir. 1966.

[14] Polunin V.A., Soldatov A.P., Three-dimensional analogue of the Cauchy-type integral, Differ. equations, 47:3 (2011), 366-375.

[15] Nazarov S., Plamenevsky B.A., Elliptic problems in domains with piecewise smooth boundaries. Walter de Gruyter, 2011.

[16] Kozhanov A.I., Koshanov B.D., Sultangazieva Zh.B., New boundary value problems for fourth-order quasi-hyperbolic equations, Siberian Electronic Mathematical Reports, 16 (2019), 1383-1409. http://dx.doi.org/10.33048/semi.2019.16.098

[17] Koshanov B.D., Soldatov A.P., Boundary value problem with normal derivatives for a higher order elliptic equation on the plane, Differential Equations, 52:12 (2016), 1594-1609. https://doi.org/10.1134/S0012266116120077

[18] Koshanov B.D., Soldatov A.P., On the Solvability of the Boundary Value Problems for the Elliptic Equation of High Order on a Plane, Bulletin of the Karaganda University. Mathematics series, 91:3 (2018), 24-31. https://doi.org/10.31489/2018M3/24-30

[19] Koshanov B.D., Equivalence of the Fredholm solvability condition for the Neumann problem to the complementarity condition, Journal of Mathematics, Mechanics and Computer Science, 109:1 (2021), 34-54. https://doi.org/10.26577/JMMCS.2021.v111.i3.04

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THE STATE OF THE PROBLEM OF THE JOINT MOVEMENT OF FLUID IN THE PORE SPACE

This article discusses the problems of studying the issue of joint motion of liquids in the porous space. The article provides the construction of a mathematical model of the theory of filtration, which describes phase transitions. The main difficulty in constructing this model is associated with the fact that free interphase boundaries create regions that change over time, and it is required to find the temperature or concentration fields of substances in them. In this case, the coordinates of the considered phase boundaries are not initially specified and must be calculated already in the process of solving. For this, a derivation of the averaged equation for the problem of finding the rupture surface during the movement of two incompressible viscous liquids in the pores of the soil skeleton was proposed. The article deals with the case when the skeleton is an absolutely rigid body. The rationale was given for the choice of an averaged filtration model instead of a microscopic one. The main research methods are classical methods of mathematical physics, functional analysis and computation methods of the theory of partial differential equations, as well as difference methods. The formulation of the problem is given, and the definition of a generalized solution for solving the problem is provided. Next, an averaged model is derived and the existence of at least one generalized solution to the problem is proved.
Key words: Stefan problem, difference scheme, numerical methods, phase boundary, sorption, adsorption, surfactant, relaxation time, averaged model, microscopic model, macroscopic model, joint motion of liquids.

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Кеуек кеңістігіндегі сұйықтықтардың бірлескен қозғалысы туралы

Мақалада сипатталған зерттеудің мақсаты кеуекті кеңістіктегі сұйықтықтардың бірлескен қозғалысы туралы мәселені зерттеу болып табылады. Мақалада фазалық ауысуларды сипаттайтын сүзу теориясының математикалық моделінің құрылысы қарастырылады.
Бұл модельді құрудағы басты қиындық бос интерфазалық шекаралар уақыт өте келе өзгеретін айма қтарды құрайтындығына байланысты және олар температура өрістерін немесе заттардың концентрациясын табуды қажет етеді. Бұл жағдайда фазалық бөлімнің қарастырылған шекараларының координаттары бастапқыда орнатылмаған және шешім барысында есептелуі керек.
Ол үшін топырақ қаңқасының тесіктерінде екі сығылмайтын тұтқыр сұйықтықтың қозғалысы кезінде сыну бетін табу есебінің орташа теңдеуін алу ұсынылды. Бұл модельді құрудағы басты қиындық бос интерфазалық шекаралар уақыт өте келе өзгеретін аймақтарды құрайтындығына байланысты және олардағы заттардың температурасы мен концентрациясының өрістерін табу керек. Бұл жағдайда фазалардың қарастырылған шекараларының координаттары бастапқыда көрсетілмеген және оларды шешу барысында есептелуі керек. Ол үшін топырақ қаңқасының тесіктерінде екі сығылмайтын тұтқыр сұйықтықтың қозғалысы кезінде жыртылу бетін табу үшін орташа теңдеуді алу ұсынылды. Мақалада қаңқа мүлдем қатты болған жағдайы қарастырылған. Негізгі зерттеу әдістеріне математикалық физиканы классикалық әдістері, функционалдық талдау және дербес дифференциалдық теңдеулер теориясының есептеу әдістері, сонымен қатар айырымдық әдістері жатады.

Түйін сөздер: Стефан есебі, айырмашылық схемасы, сандық әдістер, фазалық шекара, сорбция, адсорбция, беттік-белсенді зат, релаксация уақыты, орташа модель, микроскопиялық модель, макроскопиялық модель.

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Состояние вопроса совместного движения жидкостей в поровом пространстве

Целью исследования, описанного в данной статье, является изучение вопроса о совместном движении жидкостей в пористом пространстве. В статье рассматривается построение математической модели теории фильтрации, описывающей фазовые переходы. Основная трудность при построении этой модели связана с тем, что свободные межфазные границы образуют области, изменяющиеся во времени, и они требуют нахождения полей температуры или концентрации веществ.
При этом координаты рассматриваемых границ раздела фаз изначально не заданы и должны быть рассчитаны в ходе решения. Для этого был предложен вывод усредненного уравнения задачи о нахождении поверхности разрыва при движении двух несжимаемых вязких жидкостей в порах скелета грунта. В данной статье рассматриваются вопросы изучения совместного движения жидкостей в пористом пространстве. В статье приведено построение математической модели теории фильтрации, описывающей фазовые переходы.
Основная трудность при построении данной модели связана с тем, что свободные межфазные границы образуют области, изменяющиеся во времени, и необходимо найти в них поля температуры и концентрации веществ. При этом координаты рассматриваемых границ фаз изначально не указаны и должны быть вычислены в процессе их решения. Для этого было предложено получить усредненное уравнение для задачи нахождения поверхности разрыва при движении двух несжимаемых вязких жидкостей в отверстиях почвенного скелета. В статье рассмотрен случай, когда скелет является абсолютно твердым телом. Основными методами исследования являются классические методы математической физики, функциональный анализ и вычислительные методы теории уравнений частных производных, а также разностные методы.

Ключевые слова: Задача Стефана, разностная схема, численные методы, граница раздела фаз, сорбция, адсорбция, поверхностно-активное вещество, время релаксации, усредненная модель, микроскопическая модель, макроскопическая модель.

1 Introduction

For a better and more complete understanding of the processes that occur during oil production, it is necessary to simulate liquid flow in porous media. Modeling is commonly used to develop optimal reservoir development methods, as well as to select suitable well locations, and of course to test various oil recovery technologies. Mathematical models of filtration are based on the laws of mechanics of multiphase media and contain systems of partial differential equations.
As a rule, the mathematical model is also supplemented with auxiliary equations depending on the properties of the porous medium.
A numerical study of liquid filtration has been carried out in many works. It can be pointed out that the main problems of such problems are associated, first, with the nonlinearity of the obtained systems of equations. If we turn to the definition, then the theory of poroelasticity studies the joint mechanism of fluid flow and the change in porous media. In this case, the

main mathematical models of the theory of filtration, as a rule, are supplemented by the Lame elasticity equation for the displacements of the medium. [1] Such mathematical models of poroelasticity contain systems of nonlinear, nonstationary systems of partial differential equations. For the approximate solution of boundary value problems, as a rule, numerical methods are used.
The equations of poroelasticity, which were obtained by M. Biot and C. von Terzaghi, for a certain time served as the basis for solving problems in the field of poroelasticity. These equations take into account the movement of not only the fluid in the pores, but also the solid skeleton. Later, some authors such as R. Burridge and J. Keller, E. Sanchez-Palencia and T. Levy, proposed the derivation of the poroelasticity equations, which are based on the laws of continuum mechanics and averaging methods. First, using the classical laws of continuum mechanics, the joint motion of the elastic skeleton and fluid in the pores is described at the microscopic level, and then approximating models are found using the averaging theory.

2 Materials and methods

The main methods of this research are the classical methods of mathematical physics, computational methods of the theory of partial differential equations, functional analysis, as well as difference methods. In practice, methods are also widely used that explicitly track the movement of interphase boundaries. All these methods are based on the use of the finite difference method, in this case, the calculations are carried out on uniform or non-uniform grids. [2] It is always determined between which nodes of the computational grid the moving border is at the moment, or through which node the border passes. The joint motion of elastic skeleton and fluid in pores in the area Ω is described by R. Burridge and J. Keller, T. Levy using the following mathematical model:

$$\frac{\partial}{\partial t}(\rho v) + \nabla(\rho v \oplus v - \chi P_f + (1-\chi)P_s) = \rho F, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho v) = 0 \tag{2}$$

where $\nabla \cdot u$ is the divergence u, the matrix $a \oplus b$ is defined as $(a \oplus b)c = a(bc)$ for vectors a, b and c, χ is the characteristic function of the pore space, Ω, P_f, P_s is the stress tensors of the liquid and solid components, v is the velocity of the medium, ρ is the density of the medium and F is the given vector distributed mass forces. Equations (1) and (2) are understood as integral identities and contain dynamic equations for the liquid component:

$$\rho\frac{dv}{dt} = \nabla P_f \ + \ \rho F, \ \ \frac{d\rho}{dt} \ + \ \rho\nabla v = 0 \tag{3}$$

in Ω_f for $t > 0$, the dynamic equations for the solid component are given below:

$$\rho\frac{dv}{dt} = \nabla P_s \ + \ \rho F, \ \ \frac{d\rho}{dt} + \rho\nabla v \ = \ 0 \tag{4}$$

in Ω_s) for $t > 0$, and the condition for the continuity of normal stresses then looks like this:

$$(P_s - P_f) \cdot n \ = \ 0$$

on the common boundary "porous space rigid skeleton"$\Gamma(t)$, where $n$is the unit normal to $\Gamma(t)$. To describe the joint motion of two inhomogeneous fluids in an elastic skeleton, we will supplement our dynamic system with the transport equation for the density $\rho^\epsilon(x,t)$ of a mixture of liquid and solid components:

$$\frac{d\rho}{dt} = 0. \tag{5}$$

We also supplement this system with the initial condition:

$$\rho(x,0) = \rho_s, x \in \Omega_s, \quad \rho(x,0) = \rho_f^{\pm}, \quad x \in \Omega_{if}^{\pm}. \tag{6}$$

The resulting problem is highly nonlinear and contains an unknown quantity, that is, the interface between the pore space and the rigid skeleton. [3] In our case, the solid and liquid components do not mix. Therefore, the free boundary $\Gamma(t)$ is a contact discontinuity surface, and it can be determined from the Cauchy problem:

$$\frac{\partial \ \chi}{dt} \ \equiv \frac{\partial \chi}{\partial t} + \nabla \chi v \ \ = \ 0, \ \ \chi(x,0) \ = \ \chi_0(x) \tag{7}$$

is true for the characteristic function χ in the region Ω for $t > 0$.

Theorem Let $B_0, B.B_1$ be three Banach spaces, where

$$B_0 \subset B \subset B_1.$$

B_0, B_1 are reflective. Nesting $B_0 \subset B$is compact Then let

$$W = \left\{ v \middle| v \in L_{p_0}(0,T,B_0), \frac{\partial v}{\partial t} \in L_{p_1}(0,T,B_1) \right\}.$$

Proof. We use the norm of the space W

$$\|v\| \, L_{p_0}(0,T,B_0) + \left\| \frac{\partial v}{\partial t} \right\| L_{p_1}(0,T,B_1)$$

Then we get a Banach space. It's obvious that $W \subset L_{p_0}(0,T,B)$ Then the nesting $W \subset L_{p_0}(0,T,B)$ is compact.

3 Problem statement

If problem (1), (6), (7) can be solved, then such a given mathematical model will be useless for practical use, since the function χ changes its values from 0 to 1 on a scale of several microns. Although, the problem, in general, should be considered in an area of about several

tens or hundreds of meters [4,5] In this case, you can consider and apply the averaging of this model. But then our problem (1), (2), (7) will become unsolvable. In this case, we propose to apply the linearization of the main dynamical system according to the scheme proposed by R. Burridge and E. Sanchez-Palencia, that is, we approximate the characteristic function χ of the liquid part Ω_f by its value at the initial moment of time, as well as the free boundary $\Gamma(t)$ by its initial position Γ_0. In what follows, we suppose that $v \sim \frac{\partial w}{\partial t}$, where w is the vector of displacement of the medium, we get:

$$\frac{\partial}{\partial t}(pv) \simeq \rho_f \chi_0 \ + \ \rho_s(1-\chi_0)\ \frac{\partial^2 w}{\partial t^2}$$

where ρ_f, ρ_s are the densities of the liquid in the pores and the solid skeleton:

$$P_f \ = \ 2\mu D(x,v) - \rho II.$$

$$P_s \ = \ 2\lambda D(x,w) - \rho II.$$

Here $D(x,v)$ is the symmetric part of ∇v, II is the unit tensor, w is the vector of displacement of the medium, as μ we denote the dynamic viscosity, through v we denote the bulk viscosity, and λ is the Lame elastic constant. [6] Let $\xi(x)$ be the characteristic function of the region Ω. Then the resulting $\chi^\epsilon(x) \ = \ \xi(x)\chi(\frac{x}{\epsilon})$ will be the characteristic function of the liquid region Ω_f^ϵ in dimensionless variables

$$x \to \frac{x}{L}, \quad w \to \frac{w}{L}, \quad t \to \frac{t}{\tau}, \quad F \to \frac{F}{g}$$

where L is the characteristic size of the physical area, τ is the characteristic time of the physical process, and g is the value of the acceleration of gravity. In this case, our dynamic system will take the following form:

$$a_\tau \varrho^\epsilon \frac{\partial^2 w}{dt^2} = \nabla P + \varrho^\epsilon F. \tag{8}$$

$$P \ = \ \chi^\epsilon a_\mu D(x, \frac{\partial w}{\partial t}) \ + \ (1-\chi^\epsilon) a_\lambda D(x,w) - \rho II. \tag{9}$$

$$\nabla w \ = \ 0. \tag{10}$$

Special cases of linearization of problem (1) (2), (7) have been studied by many scientists, such as, for example, Buckingham, Buchanan-Gilbert-Lin, Keller, Levy, Sanchez-Hubert, Sanchez-Palencia. The problem of averaging for compressible mean linearized systems was most fully investigated in the works of the scientist A.M. Meirmanov. [3, 7, 8] He proposed a classification based on the dependence on the values of dimensionless criteria, which are presented below:

$$\lim_{\epsilon \to 0} a_\tau(\epsilon) \ = \ \tau_0.$$

$$\lim_{\epsilon\to 0} a_\mu(\epsilon) \;=\; \mu_0.$$

$$\lim_{\epsilon\to 0} a_\lambda(\epsilon) \;=\; \lambda_0.$$

Filtration of a liquid is a very slow process, the medium speed is usually between 3 and 5 meters per year. Therefore, the process time is just very long and $a_\tau \sim 0$. And, for example, for fast processes such as water hammer, $a_\tau \sim 1$, *or* $a_\tau \sim \infty$.
In this case, we can neglect the inertial terms in (9) and restrict ourselves to the following equation:

$$\nabla P \;+\; \varrho^\epsilon F \;=\; 0. \tag{11}$$

In order to describe the joint motion of two inhomogeneous fluids, we supplement the system of equations (9) - (11) with the following transport equation:

$$\frac{\partial \rho^\epsilon}{\partial t} \;+\; v\nabla\rho^\epsilon \;=\; 0, \quad v = \frac{\partial w}{\partial t}. \tag{12}$$

Supplement with the initial condition the equation for the density ρ^ϵ of a mixture of liquid and solid components:

$$\rho^\epsilon(x,0) \;=\; \rho_s, \quad x \in \Omega_s, \quad \rho^\epsilon(x,0) = \rho_f^\pm, \quad x \in \Omega_f^\pm \tag{13}$$

The simplest case of our system (9) - (11) will consider the case when a rigid skeleton is an absolutely rigid body. Then it is characterized by the following equality:

$$\lambda_0 \;=\; \infty.$$

Then the system of equations consists of the Stokes equations:

$$\nabla v \;=\; 0, \tag{14}$$

$$\nabla(a_\mu D(x,v) - \rho II) + \varrho_f F \;=\; 0 \tag{15}$$

for the pressure ρ and velocity v of the fluid in the region Ω_f at $t > 0$ and the equality

$$v \;=\; 0 \tag{16}$$

in a solid skeleton Ω_s.

4 Formulation of main result

Combining all the results, we formulate them together in the form of one theorem.

Theorem *Let the triple* $(\omega^\epsilon(x,t), \rho^\epsilon(x,t), \rho^\epsilon(x,t))$ *be a generalized solution of the MM model.*Then:

1) the sequences $\{\omega^\epsilon\}$, $\{\nabla\omega^\epsilon\}$, $\{v^\epsilon\}$, $\{\nabla v^\epsilon\}$, $\{\rho^\epsilon\}$, and $\{\nabla\rho^\epsilon\}$ converge weakly in $L_2(\Omega_T)$ to the functions ω,$\nabla\omega$, $v = E_{\Omega_f^\epsilon}(\partial\omega/\partial t)$, $\nabla v = \nabla(E_{\Omega_f^\epsilon}(\partial\omega/\partial t))$, ρ, p, $\nabla\rho$ respectively;
2) the limit functions are a solution of the averaged system of equations in the Ω_T region, consisting of the continuity equation

Let $\Omega \in R^2$ be a bounded region with boundary S, which was obtained by periodic repetition of the unit cell ϵY, where $\epsilon > 0$ is a small parameter,

$$Y = Y_f \cup Y_s \cup \gamma \cup \partial Y, \quad Y = (0,1)\times(0,1), \quad \epsilon Y = (0,\epsilon)\times(0,\epsilon)$$

where $\gamma = \partial Y_f \cup \partial Y_s$ is the Lipschitz boundary between two sets Y_f and Y_s. Let $\overline{\Omega}_f^\epsilon$ be the periodic repetition of unit cell $\epsilon\overline{Y_f}$, and $\overline{\Omega_s^\epsilon}$ is the periodic repetition of $\epsilon\overline{Y_s}$. Then

$$\Omega = \Omega_f^\epsilon \cup \Omega_s^\epsilon \cup \Gamma^\epsilon$$

where $\Gamma^\epsilon = \partial\ \Omega_f^\epsilon \cap \partial\ \Omega_s^\epsilon$ is a periodic repetition of the boundary $\epsilon\gamma$. Let the region Y_s be completely surrounded by the region Y_f, that is

$$Y_s \cap \partial\, Y = 0.$$

In the region Ω, the mathematical model of the joint motion of an incompressible fluid and an elastic incompressible skeleton at the microscopic level has the form

$$\nabla\cdot(\chi^\epsilon\mu_0 D(x,\frac{\partial\omega^\epsilon}{\partial t}) \ + \ (1-\chi^\epsilon)\lambda_0 D(x,\omega^\epsilon) - \qquad \rho^\epsilon I) + \rho^\epsilon F \ = \ 0. \tag{17}$$

$$\nabla\cdot\omega^\epsilon \ = \ 0, \quad x\in\Omega, \quad t>0. \tag{18}$$

$$x\in\Omega, \quad t>0.$$

$$\frac{d\rho^\epsilon}{dt} \equiv \frac{\partial\rho^\epsilon}{\partial t} \ + \ \frac{\partial\omega^\epsilon}{\partial t}\cdot\nabla\rho^\epsilon \ = \ 0, \quad x\in\Omega, \ t>0. \tag{19}$$

$$\chi^\epsilon\omega^\epsilon(x,0) \ = \ 0 \ \ at \ \ x\in\Omega. \tag{20}$$

$$\omega^\epsilon(x,t) \ = \ 0 \ \ at \ \ x\in S = \partial\Omega, \quad t>0. \tag{21}$$

$$\chi^\epsilon \rho^\epsilon(x,0) = \rho_0(x), \quad x \in \Omega. \tag{22}$$

where $\omega^\epsilon(x,t) = (\omega_l^\epsilon(x,t), \omega_2^\epsilon(x,t))$ is the vector of displacement of the continuous medium, $\rho^\epsilon(x,t)$ is the pressure in the continuous medium, $D(x,\omega)$ is the symmetric part of the gradient of the vector ω (stress tensor), I is the unit matrix, $\chi^\epsilon(x)$ is the characteristic function of the pore space, $\chi^\epsilon(x)$ μ_0 is the dimensionless viscosity of the fluid, λ_0 is the dimensionless Lam constant.

$$\nabla \cdot \omega \ = \ 0 \tag{23}$$

averaged equations of angular momentum

$$\nabla \cdot \tilde{P} + \rho F = 0. \tag{24}$$

Where

$$P = n_1 : D(x, \frac{\partial \omega}{\partial t}) + n_2 : D(x,\omega) + \int_0^t n_3(t-\tau : D(x,\omega(x,\tau))d\tau - \rho I$$

and the averaged transport equation

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \frac{\partial \omega}{\partial t} \cdot \nabla \rho = 0, \quad \rho = m\rho_f + (1-m)\rho_s \tag{25}$$

supplemented by boundary

$$\omega(x,t) = 0, \quad x \in S, \quad t \in (0,T).$$

$$\frac{\partial \rho}{\partial n} = 0, \quad x \in S, \quad t \in (0,T).$$

and initial conditions

$$\omega(x,0) = 0, \quad x \in \Omega.$$

$$\rho(x,0) = m\rho_0(x), \quad x \in \Omega.$$

where n_1 is a fourth rank tensor, is symmetric and positive definite, n is the unit outward normal vector to the S boundary. The system of equations (23) - (25), supplemented with boundary and initial conditions, is nothing more than Musket's averaged model of the joint motion of fluid and pore space.

5 Conclusions

When studying the Rayleigh-Taylor instability in hydrodynamics, the following stages are traced: linear, asymptotic, intermediate, regular, and turbulent. The most investigated is the Rayleigh-Taylor instability for the case where there is a flat interface. The linear stage is well studied in the works of Rayleigh, Taylor and Lewis, the regular asymptotic stage is studied in the works of Birkhoff. But the analytical apparatus of mathematics for the analysis of the Rayleigh-Taylor instability is not enough. Experimental studies are very laborious and can be obtained only numerically. [9, 10] Numerical approaches are based on the use of velocity-pressure variables and current-vortex velocity or velocity-vortex velocity variables. Also, the formulation of the problem makes it easy to extend the numerical methods for calculating plane flows to the three-dimensional case. But the continuity equation for an incompressible fluid contains velocity components, so there is no direct relationship with pressure. In the course of computer simulation, it was revealed that the motion of fluids, which is described by the system of equations (17) - (19), depends on the following parameters: the ratios $\delta = \rho_f^+/\rho_f^-$, where ρ_f^+ *and* ρ_f^-, are the densities of the upper and lower liquids, respectively, μ^+, μ^- the viscosity of the liquids, λ_0 the Lamй elastic coefficient, and the pore size ϵ, that is, the Rayleigh-Taylor instability is observed, as in the case when the walls of the region are a solid. Numerical calculations were carried out for various values of λ, δ, and a constant value of the viscosity of liquids and for the same ϵ, $\delta = 1.25$ unit cell size:
-for $\epsilon = 2 \cdot 10^{-5}$, $\lambda \to 0$ there is a change in the interface between the liquids;
-for $\epsilon = 2 \cdot 10^{-5}$, $\lambda = 0.5$, $\delta \to \infty$ there is a change in the interface between the liquids;
-for $\epsilon = 2 \cdot 10^{-5}$, $\lambda = 0.5$, $\delta \to 1$ no change in the interface between the liquids is observed.

The state of the issue of joint motion of liquids in the porous space was investigated. The rationale was given for the choice of an averaged filtration model instead of a microscopic one. New microscopic mathematical models of the motion of viscous incompressible liquids of various viscosities in the pore space are derived.

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References

[1] Kaliev I.A., Mukhambetzhanov S.T., Sabitova G.S. Numerical modeling of the non-equilibrium sorption process // Ufa Mathematical Journal. - 2016. -V. 8 (2). -P. 39-43. https://doi.org/10.13108/2016-8-2-39

[2] Ahmed-Zaki D.Zh., Mukhambetzhanov S.T., Imankulov T.S. Design of i-fields system component: Computer model of oil-recovery by polymer flooding // Informatics in Control, Automation and Robotics (ICINCO), 12th International Conference, Colmar, France, -2015. -P. 510-516.

[3] Meirmanov A.M., Mukhambetzhanov S.T., Nurtas, M. Seismic in composite media: Elastic and poroelastic components// Siberian Electronic Mathematical Reports. - 2016. https://doi.org/10.17377/semi.2016.13.006

[4] Kenzhebayev T.S., Mukhambetzhanov S.T. Numerical Solution of the inverse problem of filtration theory by modulating functions// Far East Journal of Mathematical Sciences. - 2016. -V. 99 (12). -P. 1779

[5] Smagulov S. H., Mukhambetzhanov S. T., Baymirov K. M. Difference schemes for modeling two-dimensional Musket-Leverett equations on an irregular grid// Reports of the 3rd Kazakhstan-Russian scientific-practical conference. -2016. -P. 43-48.

[6] Zhumagulov B. T., Mukhambetzhanov S. T., Shyganakov N. A. Modeling oil displacement taking into account mass transfer processes. -2016. -P.416

[7] Kaliev I.A.„ Sabitova G.S. ?The third boundary value problem for the system of equations of non-equilibrium sorption?// Sib. elektron. matem. izv.. -2018. -V. 15. -P. 1857-1864

[8] Kaliev I.A.„ Sabitova G.S. Neumann boundary value problem for system of equations of non-equilibrium sorption// Ufa Math. J.. -2019. -V. 11(4). -P. 33-39

[9] Choi J. H., Edwards P., Ko K., Kim Y. S. Definition and classification of fault damage zones: a review and a new methodological approach.// Earth-Science Rev. -2019. -V.152. -P.70-87 https://doi.org /10.1016/j.earscirev.2015.11.006

[10] Feng J., Ren Q., Xu K. Quantitative prediction of fracture distribution using geomechanical method.//- 2018. -V.162. -P. 22-34. https://doi: 10.1016/j.petrol.2017.12.006

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ON THE BOUNDEDNESS OF THE RIESZ POTENTIAL AND ITS COMMUTATOR'S IN THE GLOBAL MORREY TYPE SPACES WITH VARIABLE EXPONENTS

The paper considers the global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents $p(.)$, $\theta(.)$, where $\Omega \subset R^n$ is an unbounded domain. The questions of boundedness of the Riesz potential and its commutator in these spaces are investigated. We give the conditions for variable exponents $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ and on the functions $(w_1(.), w_2(.))$ under which the Riesz potential I^α , will be bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. The same conditions are obtained for the boundedness of the commutator of the Riesz potential in these spaces. In the case when the exponents p, θ constant numbers, the questions of boundedness of the Riesz potential and its commutator in global Morrey spaces were previously studied by other authors. There are also well-known results on the boundedness of the Riesz potential in global Morrey-type spaces with variable exponents, when the domain $\Omega \subset R^n$ is bounded.

Key words: global Morrey type spaces, variable exponent, Riesz potential, commutator of Riesz potential, boundedness of operator .

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Көрсеткіштері айнымалы глобальді Морри типтес кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы

Бұл жұмыста $p(.)$, $\theta(.)$ көрсеткіштері айнымалы глобальді Морри типтес кеңістіктер $GM_{p(.),\theta(.),w(.)}(\Omega)$ қарастырылады, мұндағы $\Omega \subset R^n$-шенелмеген облыс. Көрсетілген кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы сұрақтар зерттеледі. $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ көрсеткіштері және $(w_1(.), w_2(.))$ функцияларына I^α Рисс потенциалы $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ кеңістігінен $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$ кеңістігіне шенелген болуының шарттары алынды. Рисс потенциалының коммутаторына да көрсетілген кеңістіктерде дәл осы сияқты шенелгендігік шарттары алынды. p, θ көрсеткіштері тұрақты болатын жағдайда Морри типтес кеңістіктердегі Рисс потенциалы және оның коммутаторының шенелгендігі туралы сұрақтарын басқа авторлар бұрын зерттеген. $\Omega \subset R^n$ шенелген облыс жағдайындағы көрсеткіштері айнымалы глобальді Морри типтес кеңістіктердегі Рисс потенциалының шенелгендік шарттары туралы да белгілі.
Түйін сөздер: глобальді Морри типтес кеңістіктер, айнымалы көрсеткіш, Рисс потенциалы, Рисс потенциалының коммутаторы, оператордың шенелгендігі.

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Об ограниченности потенциала Рисса и его коммутатора в глобальных пространствах типа Морри с переменным показателем

В работе рассматриваются глобальные пространства типа Морри $GM_{p(.),\theta(.),w(.)}(\Omega)$ с переменными показателями $p(.)$, $\theta(.)$, где $\Omega \subset R^n$ - неограниченная область. Исследуются вопросы ограниченности потенциала Рисса и его коммутатора в указанных пространствах. Получены условия на переменные показатели $(p_1(.), p_2(.))$ и $(\theta_1(.), \theta_2(.))$ и на функции $(w_1(.), w_2(.))$ при которых потенциал Рисса I^α, будет ограничен из $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ в $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$. Такие же условия получены для ограниченности коммутатора потенциала Рисса в рассматриваемых пространствах. В случае, когда показатели p, θ постоянные числа, вопросы ограниченности потенциала Рисса и его коммутатора в глобальных пространствах Морри ранее были исследованы другими авторами. Так же известны результаты об ограниченности потенциала Рисса в глобальных пространствах типа Морри с переменными показателями, когда область $\Omega \subset \mathbb{R}^n$ ограниченная.
Ключевые слова: Глобальные пространства типа Морри, переменный показатель, потенциал Рисса, коммутатор потенциала Рисса, ограниченность операторов.

1 Introduction

1.1 Review of studies by other authors

The Morrey space $M_{p,\lambda}$ was introduced in [1] in connection with the study solutions of differential equations with partial derivatives. The boundedness of integral classical operators of harmonic analysis in global Morrey-type spaces $GM_{p,\theta,w}$ with constant exponents p, θ was well studied ([2]-[5]). The boundedness of classical integral operators in the Lebesgue spaces wih variable exponent was studied in [6]-[7]).

The Morrey-type space $\mathcal{M}_{p(.),\lambda(.)}$ with variable exponents is also well studied in [8]. The generalized Morrey-type space $M_{p(.),w(.)}(\Omega)$ with variable exponent in the case of a bounded domain $\Omega \subset \mathbb{R}^n$ were introduced and studied in [9] and [10], in the case of an unbounded domain $\Omega \subset \mathbb{R}^n$ were studied in [11].

The Riesz potential I^α with exponent α is defined by :

$$I^\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, 0 < \alpha < n.$$

The boundedness of the Riesz potential in generalized Morrey-type spaces with variable exponent was studied in [9] and [10] in the case of a bounded domain $\Omega \subset \mathbb{R}^n$ and in [11] in the case of an unbounded domain $\Omega \subset \mathbb{R}^n$.

Here and below, we denote by $B(x,r)$ the ball with center $x \in \mathbb{R}^n$ and radius $r > 0$, $\tilde{B}(x,r) = B(x,r) \cap \Omega$, $\Omega \subset \mathbb{R}^n$.

The space $BMO(\Omega)$ is defined as the space of all integrable functions f with finite norm

$$\|f\|_{BMO} = \|f\|_* = \sup_{x\in\Omega, r>0} |B(x,r)|^{-1} \int_{\tilde{B}(x,r)} |f(y) - f_{\tilde{B}(x,r)}| dy,$$

where $f_{\tilde{B}(x,r)} = |\tilde{B}(x,r)|^{-1} \int_{\tilde{B}(x,r)} f(y)dy$.

Let $b \in BMO(\Omega)$. The commutator of the Riesz potential is defined by

$$[b, I^\alpha]f = I^\alpha(bf) - b(I^\alpha f) = \int_{R^n} \frac{(b(y) - b(x))}{|x-y|^{n-\alpha}} f(y)dy, 0 < \alpha < n.$$

The boundedness of the commutator of the Riesz potential in weighted Lebesgue spaces with variable exponent was studied in [12].

1.2 Basic definitions. Preliminary results.

Let $p(x)$ be a measurable function on $\Omega \subset \mathbb{R}^n$ with values on $(1, \infty)$. Assume that

$$1 < p_- \leq p(x) \leq p_+ < \infty \tag{1}$$

where

$$p_- = p_-(\Omega) = \underset{x\in\Omega}{esinf}\, p(x),$$
$$p_+ = p_+(\Omega) = \underset{x\in\Omega}{essup}\, p\,(x)\,.$$

We denote by $L_{p(.)}(\Omega)$ the space of all functions $f(x)$ measurable on Ω such that

$$J_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty,$$

where the norm is defined as follows

$$||f||_{p(.)} = \inf\left\{\eta > 0, J_{p(.)}\left(\frac{f}{\eta}\right) \leq 1\,\right\}.$$

For details on the Lebesgue space with variable exponent, see [6]. $\mathcal{P}(\Omega)$ is the set of measurable functions $p(x)$ for which $p : \Omega \to [1, \infty)$, $\mathcal{P}^{\log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying the local logarithmic condition:

$$|p(x) - p(y)| \leq \frac{A_p}{-ln\,|x-y|}, |x-y| \leq \frac{1}{2}, x, y \in \Omega,$$

where the constant number A_p does not depend on x and y. $\mathbb{P}^{\log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying (1) and local logarithmic condition. In the case where Ω is an unbounded set, we denote by $\mathbb{P}^{log}_\infty(\Omega)$ a subset of the set $\mathbb{P}^{\log}(\Omega)$ satisfying the logarithmic conditions at infinity:

$$|p(x) - p(\infty)| \leq A_\infty ln(2 + |x|), x \in \mathbb{R}^n.$$

Let Ω be a bounded open set, $p \in \mathbb{P}^{\log}(\Omega)$, and $\lambda(x)$ a function measurable on Ω with values on $[0, n]$. Morrey spaces $\mathcal{L}_{p(.),\lambda(.)}(\Omega)$ with variable exponents $p(.)$, $\lambda(.)$ were introduced [8] with the norm

$$\|f\|_{\mathcal{L}_{p(.),\lambda(.)}(\Omega)} = \underset{x\in\Omega, t>0}{sup}\, t^{-\frac{\lambda(x)}{p(x)}} \|f\|_{L_{p(.)}(\tilde{B}(x,t))}.$$

Let $w(x, r)$ be a positive measurable function on $\Omega \times (0, l)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $l = diam\Omega$.The generalized Morrey space $M_{p(.),w(.)}(\Omega)$ with variable exponents on a bounded domain $\Omega \subset \mathbb{R}^n$ were defined in [9] with norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \underset{x\in\Omega, r>0}{sup} \frac{r^{-\frac{n}{p(x)}}}{w(x,r)} ||f||_{L_{p(.)}(\tilde{B}(x,r))}.$$

Let $w(x, r)$ be a measurable function : $\Omega\times(0, l) \to [0, \infty)$, where $\Omega \subset \mathbb{R}^n$ bounded domain, $l = diam\Omega$, measurable function $\theta(r) : (0, l) \to [1, \infty]$. Morrey type spaces $M_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponent on a bounded domain $\Omega \subset \mathbb{R}^n$ were defined in [10] with the norm

$$||f||_{M_{p(.),\theta(.),w(.)}(\Omega)} = \underset{x\in\Omega}{sup} ||w(x,r) r^{-\frac{n}{p(x)}} ||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\delta)}.$$

Let $w(x,r)$ be a positive measurable function on an unbounded domain $\Omega \subset \mathbb{R}^n$.The generalized Morrey space $M_{p(.),w(.)}(\Omega)$ with variable exponent was defined in [11] with the norm

$$||f||_{M_{p(.),w(.)}(\Omega)} = \sup_{x\in\Omega, r>0} \frac{||f||_{L_{p(.)}(\tilde{B}(x,r))}}{w(x,r)}.$$

We introduce global Morrey-type spaces with variable exponents on unbounded domains.

Let's put

$$\eta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & \text{if } r \le 1; \\ \frac{n}{p(\infty)}, & \text{if } r > 1. \end{cases}$$

Let $p \in P_\infty^{log}(\Omega)$, $w(x,r)$ be a positive measurable function on $\Omega \times [0,\infty]$, where $\Omega \in R^n$, the measurable function $\theta(r)$:$(0,\infty) \to [1,\infty)$. Global Morrey space with variable exponents $GM_{p(.),\theta(.),w(.)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ unbounded domain, defined as the set of functions $f \in L^{loc}_{p(.)}(\Omega)$ with finite norm

$$||f||_{GM_{p(.),\theta(.),w(.)}(\Omega)} = \sup_{x\in\Omega} ||w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)},$$

for $1 \le \theta(r) < \infty$, with finite norm

$$||f||_{GM_{p(.),\infty,w(.)}(\Omega)} = ||f||_{M_{p(.),w_1(.)}(\Omega)} = \sup_{x\in\Omega, r>0} w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))},$$

for $\theta(r) = \infty$.

Note that the space $GM_{p(.),\infty,w(.)}(\Omega)$ coincides with the generalized Morrey-type space $M_{p(.),w_1(.)}(\Omega)$ with variable exponent, where $w_1(x,r) = \frac{r^{\eta_p(x,r)}}{w(x,r)}$.

In the case of $w(x,r) = r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}$ we denote the indicated space by via $GM^{\lambda(.)}_{p(.),\theta(.)}$:

$$GM^{\lambda(.)}_{p(.),\theta(.)}(\Omega) = GM_{p(.),w(.),\theta}\big|_{w(x,r)=r^{-\frac{\lambda(x)}{p(x)}+\eta_p(x,r)}},$$

$$||f||_{GM^{\lambda(.)}_{p(.),\theta(.)}(\Omega)} = \sup_{x\in\Omega} ||r^{-\frac{\lambda(x)}{p(x)}}||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)}.$$

If $p(.) = p = const$, $\theta(x) = \theta = const$, then the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ coincides with the well-known global Morrey space $GM_{p,\theta,w}(\Omega)$ (see, for example, [4]). The following lemma gives a sufficient condition under which the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ is not trivial.

Lemma 2.1. Let

$$\sup_{x\in\Omega} ||w(x,r)||_{L_{\theta(.)}(0,\infty)} < \infty.$$

Then the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ is not empty.

Proof. It suffices to show that the space contains bounded functions. Let $|f(x)| < C$, using the well-known inequality $||1||_{L_{p(.)}(B(x,r))} \ll r^{\eta_p(x,r)}$(see, for example, [11]), we obtain

$$||f||_{GM_{p(.),\theta(.),w(.)}(\Omega)} = \sup_{x\in\Omega} ||w(x,r)r^{-\eta_p(x,r)}||f||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)} <$$

$$< \sup_{x\in\Omega} ||w(x,r)r^{-\eta_p(x,r)}||C||_{L_{p(.)}(\tilde{B}(x,r))}||_{L_{\theta(.)}(0,\infty)} < C \sup_{x\in\Omega} ||w(x,r)||_{L_{\theta(.)}(0,\infty)} < \infty,$$

this means that $f \in GM_{p(.),\theta(.),w(.)}(\Omega)$.

Lemma 2.1 is proved.

The following theorem was proved in [11].

Theorem 1.1 Let $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ and positive measurable functions w_1 and w_2 satisfy the condition

$$\int_r^{\infty} \frac{essinf_{t \le s < \infty} w_1(x,s)}{t^{1+\eta_p(x,t)}} dt \le C \frac{w_2(x,r)}{r^{\eta_q(x,r)}},$$

where C does not depend on x and r. Then the operator I^{α} is bounded from $M_{p(.),w_1(.)}(\Omega)$ to $M_{q(.),w_2(.)}(\Omega)$.

Let $\Omega \subset R^n$ be a bounded domain, $l = diam\Omega$. Denote by $\mathcal{W}(\delta, l)$ the set of pairs of measurable functions (θ, w) for which there exists $\delta \in (0, l)$ such that $inf_{x \in \Omega} \|w(x,.)\|_{L_{\theta(.)}(\delta,l)} > 0$.

The following theorem gives a sufficient condition for the boundedness of the Riesz Potential in Morrey-type spaces with variable exponents $p(.), \theta(.), w(.)$ over bounded domains [10].

Theorem 1.2. Assume that $p, \alpha \in \mathcal{P}^{\log}(\Omega)$ and $\alpha > 0$, $(\alpha p(.))_+ = sup_{x \in \Omega} \alpha p(x) < n$, $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, $1 < \theta_1^- \le \theta_1(t) \le \theta_1^+ < \infty$, $1 < \theta_2^- \le \theta_2(t) \le \theta_2^+ < \infty$ for any $0 < t < l$. Suppose there exists $\delta > 0$ such that $\theta_1(t) \le \theta_2(t)$, $t \in (0, \delta)$, $(\theta_1, w_1) \in \mathcal{W}(\delta, l)$. Denote $\tilde{\theta}_1(\xi) = inf_{s \in (\xi, l) \theta_1(s)}$. If

$$\sup_{x \in \Omega, 0 < t < \delta} \int_0^t (w_2(x,\xi))^{\theta_2(\xi)} \Big(\int_t^{\delta} \Big(\frac{r^{\alpha(x)-1}}{w_1(x,r)} \Big)^{[\tilde{\theta}_1(\xi)]'} dr \Big)^{\frac{\theta_2(\xi)}{[\tilde{\theta}_1(\xi)]'}} d\xi < \infty,$$

then the operator I^{α} is bounded from $M_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $M_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

We will need the following theorems on estimating the norm of the Riesz potential and its commutator over the ball, which were proved in [11], [12] respectively.

Theorem 1.3. Let $p \in \mathbb{P}_{\infty}^{log}(\Omega)$ and α satisfy the condition $0 < \alpha < n$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$. Then the following estimate holds

$$||I^{\alpha} f||_{L_{q(.)}(\tilde{B}(x,t))} \le C t^{\eta_q(x,t)} \int_t^{\infty} r^{-\eta_q(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \tag{2}$$

where C does not depend on $x \in \Omega$ and $t > 0$.

Theorem 1.4. Let $\Omega \subset R^n$ be an unbounded domain, $0 < \alpha < n$, $p \in \mathbb{P}_{\infty}^{log}(\Omega)$, $p_+ < \frac{n}{\alpha}$, $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{alpha}{n}$, $b \in BMO(\Omega)$.Then

$$||[b, I^{\alpha} f]||_{L_{q(.)}(\tilde{B}(x,t))} \le C \|b\|_* t^{\eta_q(x,t)} \int_t^{\infty} (1 + ln \frac{r}{t}) r^{-\eta_q(x,r)-1} ||f||_{L_{p(.)}(\tilde{B}(x,r))} dr, \tag{3}$$

where C does not depend on $x \in \Omega$ and $t > 0$.

Let u and v be positive measurable functions on R_+. The conjugate Hardy operator is defined by

$$\tilde{H}_{v,u} f(r) = v(x) \int_r^{\infty} f(t) u(t) dt, x \in R_+,$$

where $R_+ = (0, +\infty)$. Suppose a is a fixed positive number. Let $\theta_{1,a}(r) = essinf_{y\in[r,a)}\,\theta_1(y)$,

$$\tilde{\theta}_1(r) = \begin{cases} \theta_{1,a}(r) & \text{if } r \in [0,a]; \\ \overline{\theta}_1 = const & \text{if } r \in [a,\infty); \end{cases},$$

$\theta_1 = essinf_{r\in R_+}\,\theta_1(r)$, $\Theta_2 = essup_{r\in R_+}\,\theta_2(r)$.

The following theorem was proved in [13].

Theorem 1.5. Let $\theta_1(r)$ and $\theta_2(r)$ be positive measurable functions on R_+ and there exists a positive number a such that that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequalities $1 < \theta_1 \le \tilde{\theta}_1(r) \le \theta_2(r) \le \Theta_2 < \infty$ hold almost everywhere on R_+. If

$$G = \sup_{t>0} \int_0^t [v(r)]^{\theta_2(r)} \Big(\int_t^\infty u^{\tilde{\theta}_1'(r)}(\tau) d\tau \Big)^{\frac{\theta_2(r)}{(\tilde{\theta}_1)'(r)}} dr < \infty, \tag{4}$$

hen the operator $\tilde{H}_{v,u}$ is bounded from $L_{\theta_1(.)}(R^+)$ to $L_{\theta_2(.)}(R^+)$.

2 The main results

Theorem 2.1. Let $p(.) \in \mathbb{P}_\infty^{log}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p(.))_+ = sup_{x\in\Omega}\,\alpha p(x) < n$, $\theta_1(r)$ and $\theta_2(r)$ are positive measurable functions on R_+ and there exists a positive number a such that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequality $1 < \theta_1 \le \tilde{\theta}_1(r) \le \theta_2(r) \le \Theta_2 < \infty$ are executed almost everywhere. Suppose that the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, positive measurable functions w_1 and w_2 satisfy the condition

$$T = \sup_{x\in\Omega, t>0} \int_0^t (w_2(x,r))^{\theta_2(r)} \Big(\int_t^\infty \Big(\frac{s^{\alpha-1}}{w_1(x,s)} \Big)^{[\tilde{\theta}_1(r)]'} ds \Big)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{5}$$

Then the operator I^α is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof of Theorem 2.1. Using Theorem 1.3, we have

$$||I^\alpha||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \|w_2(x,r) r^{-\eta_{p_2}(x,r)} \|I_\alpha f\|_{L_{p_2(.)}(B(x,r))}\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le C \sup_{x\in\Omega} ||w_2(x,r) \int_r^\infty t^{-\eta_{p_2}(x,t)-1} ||f||_{L_{p_1(.)}(B(x,t))} dt||_{L_{\theta_2(.)}(0,\infty)}.$$

Denote

$$\tilde{H}_{v,u} f(r) = v(r) \int_r^\infty g(t) u(t) dt,$$

where

$$v(r) = w_2(x,r),$$

$$g(t) = \frac{w_1(x,t)}{t^{\eta_{p_1}(x,t)}} ||f||_{L_{p_1(.)}(B(x,t))},$$

$$u(t) = \frac{t^{\eta_{p_1}(x,t) - \eta_{p_2}(x,t) - 1}}{w_1(x,t)} = \frac{t^{\alpha-1}}{w_1(x,t)},$$

for every fixed $x \in \Omega$. Then condition (4) has the form (5), which, according to Theorem 1.5, implies that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|I^\alpha f\|_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} \leq CT \cdot \sup_{x\in\Omega} \|w_1(x,t)t^{-\eta_{p_1}(x,t)}\|f\|_{L_{p_1(.)}(B(x,t))}\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CT \cdot \|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)},$$

this means that the operator I^α is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)(\Omega)}$ to $GM_{p_2(.),\theta_2(.),w_2(.)(\Omega)}$. Theorem 2.1 is proved.

Theorem 2.2. Let $p(.) \in \mathbb{P}_\infty^{log}(\Omega)$ and a constant number α satisfy the conditions $\alpha > 0$, $(\alpha p(.))_+ = sup_{x\in\Omega}\, \alpha p(x) < n$, $\theta_1(r)$ and $\theta_2(r)$ are positive measurable functions on R_+ and there exists a positive number a such that $\theta_1(r) = \overline{\theta}_1 = const$, $\theta_2(r) = \overline{\theta_2} = const$ for all $r > a$, inequality $1 < \theta_1 \leq \tilde{\theta}_1(r) \leq \theta_2(r) \leq \Theta_2 < \infty$ are executed almost everywhere. Suppose that the functions $p_1(x)$ and $p_2(x)$ satisfy the equality $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\alpha}{n}$, positive measurable functions w_1 and w_2 satisfy the condition

$$B = \sup_{x\in\Omega, t>0} \int_0^t \left(\frac{w_2(x,r)}{r}\right)^{\theta_2(r)} \left(\int_t^\infty \left(\frac{s^\alpha}{w_1(x,s)}\right)^{[\tilde{\theta}_1(r)]'} ds\right)^{\frac{\theta_2(r)}{[\tilde{\theta}_1(r)]'}} dr < \infty. \tag{6}$$

Then the commutator $[b, I^\alpha]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.
Proof of Theorem 2.2. According to Theorem 1.4, we have

$$||[b, I^\alpha]f||_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} = \sup_{x\in\Omega} \|w_2(x,r)r^{-\eta_{p_2}(x,r)}\|[b, I_\alpha]f\|_{L_{p_2(.)}(B(x,r))}\|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq C \sup_{x\in\Omega} ||\frac{w_2(x,r)}{r} \int_r^\infty t^{-\eta_{p_2}(x,t)}||f||_{L_{p_1(.)}(B(x,t))}dt||_{L_{\theta_2(.)}(0,\infty)},$$

here we use the inequality $1 + ln\frac{t}{r} < \frac{t}{r}$ for $t > r > 0$. Denote

$$\tilde{H}_{v,u}f(r) = v(r)\int_r^\infty g(t)u(t)dt,$$

where

$$v(r) = \frac{w_2(x,r)}{r},$$

$$g(t) = \frac{w_1(x,t)}{t^{\eta_{p_1(x,t)}}}||f||_{L_{p_1(.)}(B(x,t))},$$

$$u(t) = \frac{t^\alpha}{w_1(x,t)},$$

for every fixed $x \in \Omega$. Then condition (4) takes the form (6), from which, according to Theorem 1.5, it follows that the operator $\tilde{H}_{v,u}f(r)$ is bounded from $L_{\theta_1(.)}(0,\infty)$ to $L_{\theta_2(.)}(0,\infty)$. Finally, we have

$$\|[b, I^\alpha]f\|_{GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)} \leq CB \cdot \sup_{x\in\Omega} \|w_1(x,t)t^{-\eta_{p_1}(x,t)}\|f\|_{L_{p_1(.)}(B(x,t))}\|_{L_{\theta_1(.)}(0,\infty)} =$$

$$= CB \cdot \|f\|_{GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)},$$

which means that the commutator $[b, I^\alpha]$ is bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

Theorem 2.2 is proved.

3 Conclusion

We have obtained the sufficient conditions for the boundedness Riesz potential and its commutator the global Morrey-type spaces with variable exponents.

We gave the conditions for variable exponents $(p_1(.), p_2(.))$, $(\theta_1(.), \theta_2(.))$ and on the functions $(w_1(.), w_2(.))$ under which the Riesz potential I^α , would be bounded from $GM_{p_1(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p_2(.),\theta_2(.),w_2(.)}(\Omega)$.

References

[1] C.B. Morrey. On the solutions of quasi-linear elliptic partial differential equations // Trans.Am.Math.Soc. –1938.– Vol.43, – Pp. 126-166.

[2] V. Burenkov, V.Guliyev. Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces // Potential Anal. – 2009. – Vol.31, – Pp.1-39.

[3] V. Burenkov, V.Guliyev. Necessary and sufficient conditions for boundedness of the Riesz potential in the local Morrey-type spaces // Doklady Ross.Akad.Nauk.Matematika. – 2007. – Vol.412, No.5, – Pp. 585-589 (in Russian).English transl. in Acad.Sci.Dokl.Math.– 2007. – Vol.76.

[4] V. Burenkov. Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces // Eurasian. Math. J. – 2012. – Vol.3, No.3, – Pp. 11-32.

[5] V. Burenkov. Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces // Eurasian. Math. J. – 2013. – Vol.4, No.1, – Pp. 21-45.

[6] L.Diening, P.Harjulehto, P.Hasto, M.Ruzicka. Lebesgue and Sobolev spaces with variable exponents // Monograpgh. – 2010. – Pp. 1-493.

[7] L.Diening Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L_{p(.)}$ and $W_{k,p(.)}$. // Math.Nachr. – 2004. – Vol.268, – Pp.31-43.

[8] A. Almeida,J.Hasanov S.Samko. Maximal and potential operators in variable exponent Morrey spaces // Georgian. Math. J. – 2008. – Vol.15, No.2, – Pp. 195-208.

[9] V. Guliyev,S.Samko, J.Hasanov Boundedness of the maximal, potential type and singular integral operators in the generalized variable exponent Morrey spaces. // Math.Scand. – 2010. – Vol.107, – Pp. 285-304.

[10] V. Guliyev,S.Samko, J.Hasanov. Boundedness of maximal, potential type,and singular integral operators in the generalized variable exponent Morrey type spaces // J.Math.Sciences. – 2010. – Vol.170, No. 4. – Pp. 423-442.

[11] V. Guliyev,S.Samko. Maximal, potential, and singular operators in the generalized variable exponent Morrey spaces on unbounded sets // J.Math.Sciences. – 2013. – Vol.193, No. 2. – Pp. 228-247.

[12] V. Guliyev, J.Hasanov, X.Badalov. Commutators of Riesz potential in the vanishing generalized Morrey spaces with variable exponent // J.Math.Sciences. – 2019. – Vol.22, No. 1. – Pp. 331-351.

[13] D. Edmunds, V.Kokilasvili, A.Meskhi. On the boundedness and compactness of weighted Hardy operators in spaces $L_{p(x)}$ // Georgian.Math.J. – 2005. – Vol.12, No. 1. – Pp. 27-44.

2-бөлім **Раздел 2** **Section 2**

Механика **Механика** **Mechanics**

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PHYSICOMATHEMATICAL MODEL OF CALCULATING CONTINUOUS BEAMS WITH ELASTIC YIELDING SUPPORTS

In this work there has been studied the rod system operation dependence on the action of external force effects. The system presents a multi-span continuous beam with elastic yielding supports. To identify the stress-strain state of the object under study, a precise analytical method of forces is used. The method of five moments is used as the resolving (canonical) equations. The final results are the parameters of the stress-strain state for a five-span continuous beam with variable compliance coefficients on 5 intermediate supports and an absolutely rigid left extreme support, deflections, bending moments, shear forces, and support reactions.Theoretical provisions and practical results can be used in the design of load-bearing beam structures in buildings and various engineering structures.
Key words: continuous beams, support, vertical displacements of beam nodes, compliance coefficient, force method, equations of five moments, deflection diagrams.

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Серпімді-икемді тіректері бар тұтас арқалықтарды есептеудің физика-математикалық моделі

Бұл жұмыста стержінді жүйеге, яғни серпімді-икемді тіректері бар көпаралықты тұтас арқалыққа сыртқы күштер әсерінің әрекеті зерттелінді. Зерттелетін объектінің кернеулі-деформацияланған күйін анықтау үшін нақты аналитикалық күштер әдісі пайдаланылды. Шешуші (канондық) теңдеулер ретінде бес моменттер әдісі қолданылды. Соңғы нәтижелер ретінде сол жағы қатаң бекітілген және бесаралық тіректерде өзгермелі коэффициенттері бар бесаралықты тұтас арқалықтың кернеулі-деформацияланған күйінің параметрлерінің иілу, иілу моменттері, көлденең күштері, тірек реакциялары келтірілген.
Алынған теориялық және практикалық нәтижелер ғимараттар мен әртүрлі инженерлік құрылыстардағы тұтас арқалықтар құрылымын жобалау кезінде қолданыла алады.
Түйін сөздер: Тұтас арқалықтар, тіректің икемділік коэффициенті, арқалықтың түйіндерінің тік жылжуы, икемділік коэффициенті, күштер әдісі, бес моменттердің теңдеулері, майысу эпюрасы.

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Физико-математическая модель расчета неразрезных балок с упруго-податливыми опорами

В данной работе выполнено исследование работы стержневой системы на действие внешних силовых воздействий, представляющей многопролетную неразрезную балку, опоры который является упруго-податливыми. Для выявления напряженного-деформированного состояния изучаемого объекта применяется точный аналитический метод сил. В качестве разрешающих (канонических) уравнений использован метод пяти моментов. В качестве конечных результатов приведены параметры напряженного-деформированного состояния для пятипролетной неразрезной балки с переменными коэффициентами податливости на 5-ти промежуточных опорах и абсолютно жесткой левой крайней опорой, прогибы, изгибающие моменты, поперечные силы, опорные реакции. Теоретические положения и практические результаты могут использоваться при проектировании несущих балочных конструкций в зданиях и различных инженерных сооружениях.

Ключевые слова: Неразрезные балки, коэффициент податливость опор, вертикальные смещения узлов балок, коэффициент податливости, метод сил, уравнения пяти моментов, эпюры прогибов.

1 Introduction

Multi-span continuous beams are widely used in bridge, industrial and civil construction [1-3] in the form of load-bearing beams, columns of multi-storey frames of high-rise buildings. They are also the base of the carriageway of pontoon bridges.

Intermediate supports of such structures can have elastic displacements at the joints of adjacent structures, which significantly affects their stress-strain state.

Continuous or multi-span beams are statically indeterminate beams that span several spans (two or more) and do not have intermediate hinges. Continuous beams constitute an important class of statically indeterminate rod systems and are often found both in construction and in other branches of present day technology. The most widely used general method of calculating statically indeterminate systems in practice is the method of forces. It is covered in detail in the scientific literature on the mechanics of structures [4-6].

The method of calculating continuous beams using the equations of five moments also has a number of significant advantages and is only a modernization of the method of force transformation [2, 7].

When calculating the main beams as a part of various beam cells, it becomes necessary to identify the parameters of their stress-strain state, that is, the values of displacements (deflections), internal forces, etc. In power engineering, such structures are used as load-bearing structures for mechanisms and machines [7, 8].

The purpose and the tasks of the study. The purpose of this work is to study the stress-strain state of rod beam systems with a high degree of static uncertainty with supports of both rigid and elastic yielding types using the precise analytical method of forces for their calculation.

At this, the following tasks are realized: developing the resolving canonical equations of the method of forces; obtaining expressions for calculating their elements taking into account the compliance coefficients of the supports; developing a methodology of calculating the

compliance coefficients of the supports of a five-span continuous beam, and their effect on the parameters of the stress-strain state.

2 Theoretical provisions and calculation methods

This paper is dealing with the stress-strain state of a five-span continuous beam with elastic yielding supports under the action of concentrated nodal forces P_1, P_2,...,P_5 (Fig. 1, a) with variable bending stiffness of the spans.

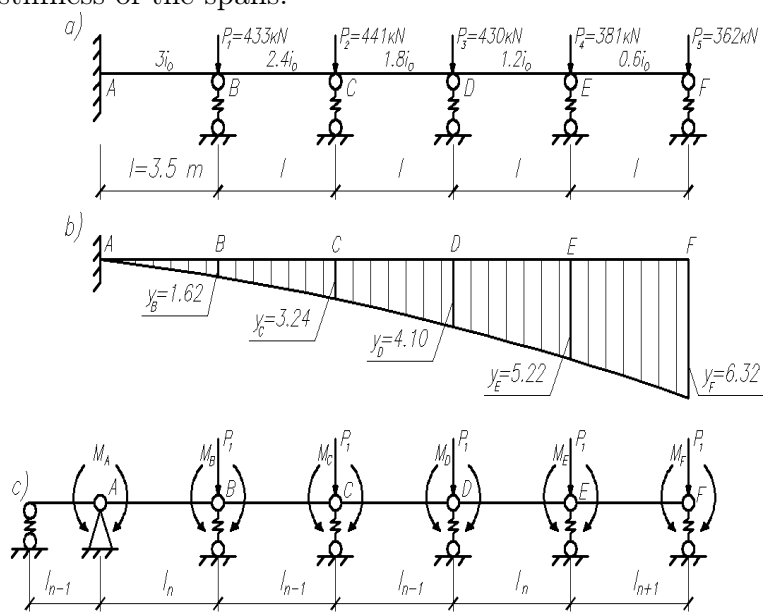

Figure 1: Towards the calculation of a continuous beam: a) preset continuous beam; b) the curve of initial (preset) deflections of the beam nodes; c) the main system of the force method

The support elasticity is determined by the coefficient of their compliance: C_i (i=1,2,...,m) is the movement of the "i–th" support caused by the action of a unit force ($\overrightarrow{P}_i = 1$) on it; the unit of measurement of C_i is cm/kg. Let the supports of the beam (except for the support "A") receive some initial displacements (Figure 1, b), then the compliance coefficients of the supports have the following values:

$$C_B = \frac{y_B}{P_1} = \frac{1.62}{43.3 \cdot 10^3} = 3.74 \cdot 10^{-6} cm/kg; \quad C_C = \frac{y_C}{P_2} = \frac{3.24}{44.1 \cdot 10^3} = 73.47 \cdot 10^{-6} cm/kg;$$

$$C_D = \frac{y_D}{P_3} = \frac{4.1}{43 \cdot 10^3} = 95.35 \cdot 10^{-6} cm/kg; \quad C_E = \frac{y_E}{P_4} = \frac{5.22}{38.1 \cdot 10^3} = 137 \cdot 10^{-6} cm/kg;$$

$$C_F = \frac{y_F}{P_5} = \frac{6.32}{36.2 \cdot 10^3} = 174.59 \cdot 10^{-6} cm/kg.$$

We take the exact analytical method of forces [9-11] as the method of calculating such continuous beams. The main system is taken a multi-span beam divided by hinges in the supports into single-span beams. The main unknowns are the support moments (Figure 1, c). The canonical equation for the n-th support will have the form of the equation of five moments [9, 10].

$$\delta_{n,n-2}M_{n-2}+\delta_{n,n-1}M_{n-1}+\delta_{n,n}M_n+\delta_{n,n+1}M_{n+1}+\delta_{n,n+2}M_{n+2}+\Delta_{np}=0,\ (i=A,B,C,D,E). \tag{1}$$

The coefficients of canonical equation (1), taking into account elastic yield of the supports will take the form:

$$\delta_{n,n-2}=\frac{C_{n-1}}{l_{n-1}l_n};\delta_{n,n-1}=\frac{l_n}{6EJ_n}-\frac{C_{n-1}}{l_n}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)-\frac{C_n}{l_n}\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right);$$

$$\delta_{n,n}=\frac{l_n}{3EJ_n}+\frac{l_{n+1}}{3EJ_nl_{n+1}}+\frac{C_{n-1}}{{l_n}^2}+C_n\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right)^2+\frac{C_{n+1}}{l_{n+1}}; \tag{2}$$

$$\delta_{n,n+1}=\frac{l_{n+1}}{6EJ_{n+1}}-\frac{C_{n+1}}{l_{n+1}}\left(\frac{1}{l_{n+1}}+\frac{1}{l_{n+2}}\right)-\frac{C_n}{l_{n+1}}\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right);\delta_{n,n+2}=\frac{C_{n+1}}{l_{n+1}l_{n+2}};$$

$$\Delta_{np}=\frac{B_n^{\Phi}}{EJ_n}+\frac{A_{n+1}^{\Phi}}{EJ_{n+1}}+\frac{C_{n-1}}{\ell_n}R_{n-1}-C_n\left(\frac{1}{\ell_n}+\frac{1}{\ell_{n+1}}\right)R_n+\frac{C_{n+1}}{\ell_{n+1}}R_{n+1},$$

where R_{n-1}, R_n, R_{n+1} are respectively the support reactions $n-1$, n, $n+1$ in the composition of the basic system; B_n^D, A_{n+1}^D are respectively fictitious reactions of the support considered in the n and $n+1$ spans of the beam (for example under the action of the uniformly distributed load q_i for a span with the length l_i these reactions are equal to $B_i^D=A_i^D=q_il_i^3/24$). The deflection on the n-th support of the beam, taking into account their elastic yield is equal to the total response of this support multiplied by C_n, that is:

$$y_n=C_n\left[\frac{M_{n-1}}{l_n}-\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right)\right]M_n+\left(\frac{M_{n+1}}{l_{n+1}}\right)+R_n. \tag{3}$$

3 Results

Let's accept the proposed theory of calculation for the preset continuous beam (Figure 1,a).

1. Support "A" ($n=A$)

$$\delta_{A,A}M_A+\delta_{A,B}M_B+\delta_{A,C}M_C+\Delta_{A,P}=0. \tag{4}$$

Let's calculate the coefficients of equation (4):

$$\delta_{A,A} = \frac{0}{3EJ_0} + \frac{10^{-8}}{3 \cdot 3 \cdot 8.06} + \frac{3.74 \cdot 10^{-6}}{(350)^2} = 0.0445 \cdot 10^{-8};$$

$$\delta_{A,B} = \frac{10^{-8}}{6 \cdot 3 \cdot 8.06} - \frac{3.74 \cdot 10^{-6}}{350}\left(\frac{1}{35} + \frac{1}{35}\right) = -0.05417 \cdot 10^{-8};$$

$$\delta_{A,C} = \frac{3.74 \cdot 10^{-6}}{(350)^2} = 0.0305 \cdot 10^{-8}; \quad \delta_{A,P} = \frac{3.74 \cdot 10^{-6}}{350} \cdot 43.3 \cdot 10^3 = 4.627 \cdot 10^{-3}.$$

2. Support "B" ($n = B$)

$$\delta_{B,A}M_A + \delta_{B,C}M_C + \delta_{B,D}M_D + \Delta_{B,P} = 0$$

$$\delta_{B,B} = \frac{10^{-8}}{3 \cdot 3 \cdot 8.06} + \frac{10^{-8}}{3 \cdot 2.4 \cdot 8.06} + \left(37.4 \cdot 10^{-6}\right) \cdot \left(\frac{1}{350} + \frac{1}{350}\right)^2 + \frac{73.47 \cdot 10^{-6}}{(350)^2} = 0.2131 \cdot 10^{-8};$$

$$\delta_{B,A} = \frac{10^{-8}}{6 \cdot 3 \cdot 8.06} - \frac{37.4 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = \delta_{A,B} = -0.05417 \cdot 10^{-8};$$

$$\delta_{B,C} = \frac{10^{-8}}{6 \cdot 2.4 \cdot 8.06} - \frac{73.47 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) - \frac{37.4 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = -0.1724 \cdot 10^{-8};$$

$$\Delta_{B,P} = -37.4 \cdot 10^{-6} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) \cdot 43.3 \cdot 10^3 + \frac{73.47 \cdot 10^{-6}}{350} \cdot 44.1 \cdot 10^3 = 0.00322 \cdot 10^{-3};$$

$$\delta_{B,D} = -\frac{73.47 \cdot 10^{-6}}{350 \cdot 350} = 0.059976 \cdot 10^{-8}.$$

3. Support "C" ($n = C$)

$$\delta_{C,A}M_A + \delta_{C,B}M_B + \delta_{C,C}M_C + \delta_{C,D}M_D + \Delta_{C,P} = 0$$

$$\delta_{C,A} = \frac{37.4 \cdot 10^{-6}}{350 \cdot 350} = 0.03053 \cdot 10^{-8};$$

$$\delta_{C,B} = \frac{10^{-8}}{6 \cdot 2.4 \cdot 8.06} - \frac{37.4 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) - \frac{73.47 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = -0.1724 \cdot 10^{-8};$$

$$\delta_{C,C} = \frac{10^{-8}}{3 \cdot 2.4 \cdot 8.06} + \frac{10^{-8}}{3 \cdot 1.8 \cdot 8.06} + \frac{37.4 \cdot 10^{-6}}{(350)^2} + 73.47 \cdot 10^{-6} \cdot \left(\frac{1}{350} + \frac{1}{350}\right)^2 +$$

$$+\frac{93.35 \cdot 10^{-6}}{(350)^2} = 0.386842 \cdot 10^{-8};$$

$$\delta_{C,D} = \frac{10^{-8}}{6 \cdot 1.8 \cdot 8.06} - \frac{93.35 \cdot 10^{-6}}{350} \frac{37.4 \cdot 10^{-6}}{(350)^2} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) -$$

$$-\frac{73.47 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = -0.26414 \cdot 10^{-8};$$

$$\delta_{C,E} = \frac{95.35 \cdot 10^{-6}}{350 \cdot 350} = 0.077837 \cdot 10^{-8};$$

$$\Delta_{C,P} = \frac{37.4 \cdot 10^{-6}}{350} \cdot 43.3 \cdot 10^3 - 73.47 \cdot 10^{-6} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) \cdot 44.1 \cdot 10^3 +$$

$$+\frac{95.35 \cdot 10^{-6}}{350} \cdot 43.3 \cdot 10^3 = -2.1731 \cdot 10^{-3}.$$

4. Support "D" ($n = D$)

$$\delta_{D,B} M_B + \delta_{D,C} M_C + \delta_{D,D} M_D + \delta_{D,E} M_E + \Delta_{D,E} = 0$$

$$\delta_{D,B} = \frac{73.47 \cdot 10^{-6}}{350 \cdot 350} = 0.059976 \cdot 10^{-8};$$

$$\delta_{D,C} = \frac{10^{-8}}{6 \cdot 1.8 \cdot 8.06} - \frac{73.47 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) - \frac{93.35 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = -0.2652 \cdot 10^{-8};$$

$$\delta_{D,D} = \frac{10^{-8}}{3 \cdot 1.8 \cdot 8.06} + \frac{10^{-8}}{3 \cdot 1.2 \cdot 8.06} + \frac{73.47 \cdot 10^{-6}}{(350)^2} + 95.35 \cdot 10^{-6} \cdot \left(\frac{1}{350} + \frac{1}{350}\right)^2 = 0.5471 \cdot 10^{-8};$$

$$\delta_{D,E} = \frac{10^{-8}}{6 \cdot 1.2 \cdot 8.06} - \frac{137 \cdot 10^{-8}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) - \frac{95.35 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right)^2 = -0.3622 \cdot 10^{-8};$$

5. Support "E" ($n = E$)

$$\delta_{E,C} M_C + \delta_{E,D} M_D + \delta_{E,E} M_E + \delta_{E,P} = 0$$

$$\delta_{E,D} = \frac{10^{-8}}{6 \cdot 1.8 \cdot 8.06} - \frac{93.35 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) - \frac{137 \cdot 10^{-6}}{350} \cdot \left(\frac{1}{350} + \frac{1}{350}\right) = -0.3621 \cdot 10^{-8};$$

$$\delta_{E,E} = \frac{10^{-8}}{3 \cdot 1.2 \cdot 8.06} + \frac{10^{-8}}{3 \cdot 0.6 \cdot 8.06} + \frac{97.35}{(350)^2} + 137 \cdot 10^{-6} \cdot \left(\frac{1}{350} + \frac{1}{350}\right)^2 +$$

$$+\frac{174.6\cdot 10^{-6}}{(350)^2}=0.77164\cdot 10^{-8};$$

$$\delta_{E,P}=\frac{95.35\cdot 10^{-6}}{350}\cdot 43\cdot 10^3-137\cdot 10^{-6}\cdot\left(\frac{1}{350}+\frac{1}{350}\right)\cdot 38.1\cdot 10^3+$$

$$+\frac{174.6\cdot 10^{-6}}{350}\cdot 36.2\cdot 10^3=-0.539\cdot 10^{-3}.$$

The system of linear algebraic equations (5) (SLAE) for the preset continuous beam will have the form:

$$A\cdot\vec{M}=\vec{P}, \tag{5}$$

where *A(5x5)* is the square matrix of the 5th order (Table 1); $\vec{M}=|\{M_A,M_B,\ M_C,M_D,M_E\}|$ is the vector of the moments at support of the basic system; $\vec{P}=|\{P_B,P_C,P_D,P_E,P_F\}|$ is the vector of free members taking into account the load preset for the beam.

Table 1– matrix of the force method for calculating 5-span continuous beam with elastic yielding supports

№	M_A	M_B	M_C	M_D	M_E	Right part
1	0.0445	-0.05417	0.03053	–	–	-4.627x10^5
2	-0.05417	0.2131	-0.1724	0.059976	–	0.00322x10^5
3	0.03053	-0.1724	0.386842	-0.2652	0.07784	2.1731x10^5
4	–	0.059976	-0.2652	0.05417	-0.3624	-0.74176x10^5
5	–	–	0.07784	-0.3624	0.77164	0.0539x10^5

By solving the system of equation (6), we obtain the moments at support of the continuous bean:

$$\vec{M}=A^{-1}\vec{P}, \tag{6}$$

where A^{-1} is the reverse matrix.

Then there is given solution for equation (6):

$$M_A=-1.513\cdot 10^7\ (kgcm);\qquad M_B=-3.434\cdot 10^6\ (kgcm);\qquad M_C=7.982\cdot 10^5\ (kgcm);$$

$$M_D=8.398\cdot 10^5\ (kgcm);\qquad M_E=3.205\cdot 10^5\ (kg\cdot cm).$$

Let's calculate the largest deflection (at the "F" point) by multiplying the curves M (Figure 2, b) and $\vec{M}_F$ (Figure 2, c) according to the Vereshchagin rule (7).

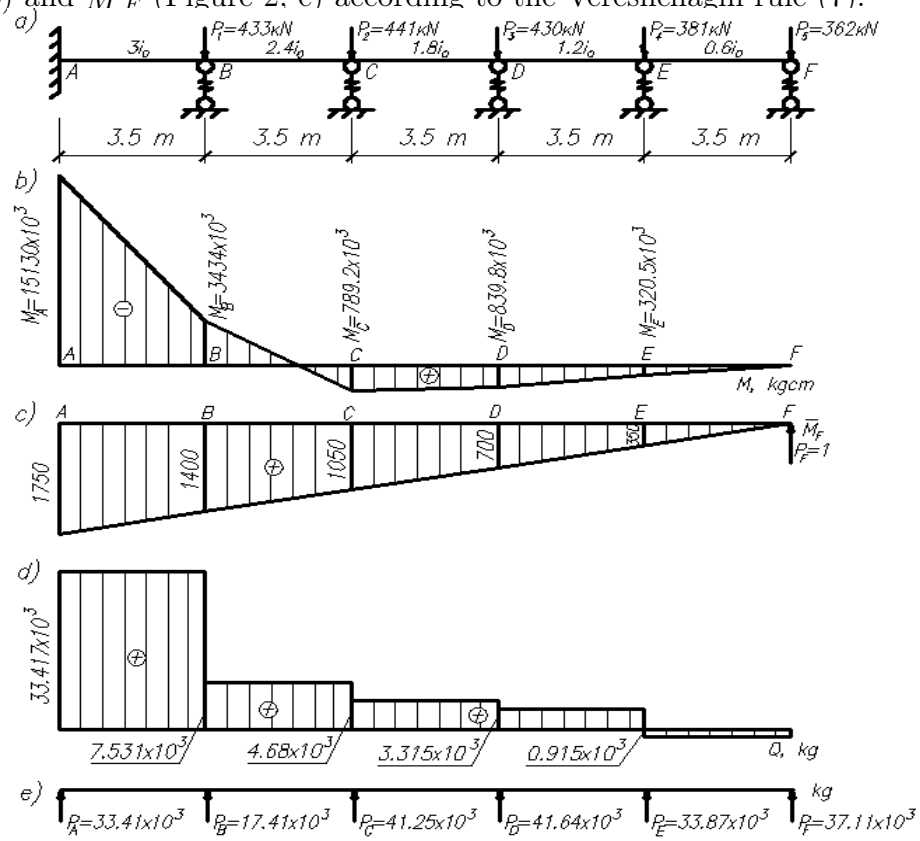

Figure 2: The results of calculating beams:a) preset continuous beam; b) the curve of moments; c) single curve of the moments; d) the curve of transverse forces; e) support reactions

$$\sum \int_0^S \frac{M \cdot \bar{M}_S}{EJ} dS = \frac{1}{EJ}(M) \times (\bar{M}_S) = y_n, \tag{7}$$

According to (7), we have

$$y_F = y_{\max} = 6.48\, cm.$$

The ($y_F = y_{\max} = 6.48\, cm$) value is close enough to the initial value (Figure 1, b; $y_F = 6.32\, cm$, the error is about 2.53 %), which confirms reliability of the theoretical provisions of this work, as well as the accuracy of practical calculations.

Using the ordinates of the obtained curves of moments (Figure 2, b), according to Zhuravsky rule, we determine the ordinates of the diagrams of transverse forces (Figure 2, d) and then the values of the support reactions (Figure 2, f).

Let's check the condition of equilibrium of the continuous beam.

$$\sum F_{k,y} = \sum P_i. \tag{8}$$

According to Figure 2, a, e, based on expression (8), we have:

$$R_A + R_B + R_C + R_D + R_E + R_F = P_1 + P_2 + P_3 + P_4 + P_5,$$

or $(33.417 + 17.14 + 41.25 + 41.64 + 33.87 + 37.116) \cdot 10^3 = 204.7 \cdot 10^3$; the error is 0.016%. The deflections are calculated by formula (3), [9].

$$y_n = \left[R_n^0 + \frac{1}{l_{n+1}} \left(M_{n+1} - M_n \right) - \frac{1}{l_n} \left(M_n - M_{n-1} \right) \right] \left(\frac{1}{\aleph_n} \right). \tag{9}$$

4 Conclusions

1. In this work, there has been studied the stress-strain state (SSS) of a five-span continuous beam with elastic yielding supports with different bending stiffness of the spans under the action of nodal forces (Figure 1, a).

2. The applied research method is the precise analytical method of forces; in this case, the unit and load coefficients are determined taking into account the coefficients of elastic compliance of the supports of the beams.

3. Based on the equation of five moments, the supporting bending moments M_A, M_B, M_C, M_D, M_E have been determined (Figure 2, b), and then, according to the Zhuravsky rule, the transverse forces on the spans (Figure 1, d) of the beam have been calculated and all the support reactions have been determined;

4. The greatest deflection has been calculated (at the "F"point); at this, sufficient proximity of its value to the initial value of the deflection in Figure 1, b has been established, which confirms the correctness of the theoretical provisions of the study with their practical implementation.

5. In the course of the study, it has been found that the presence of elastic yielding supports reduces the highest value of the corresponding parameters of the stress-strain state in comparison with the cantilever bar as follows:

   - for bending moments by 4.74 times;

   - for deflections by 14.92 times;

This suggests that in order to reduce the material consumption of a continuous beam, the presence of elastic yielding supports is advantageous.

References

[1] Umanskiy A.A. Spravochnik proektirovshchika promyshlennyh, zhilyh i obshchestvennyh zdanij i sooruzhenij [Reference-book of the designer of industrial, residential and public buildings and structures] // M.: Stroyizdat. – 1972. – 416.

[2] NuguzhinovZh.S., Akhmediyev S.K., Zholmagambetov S.R., Khabidolda O. Calculation of continuous beams taking into account the elastic compliance of supports // Bulletin of the Karaganda University - Mathematics. – 2018. – No.92(4) – Pp. 139-147.

[3] Ahmediv S.K., Filippova T.S., Oryntaeva G.ZH., Donenbaev B.S. Analiticheskie i chislennye metody raschetov mashinostroitel'nyh i transportnyh konstrukcij i sooruzhenij [Analytical and numerical methods of calculating machine-building and transport structures and structures] // Karaganda: KSTU. – 2016. – 158.

[4] Shapiro D.M., Tyutin A.P. Numerical Calculation Theory of Slab-Beam Reinforced Concrete Bridge High Structures // Russian Journal of Building Construction and Architecture. – 2019. – No.2(54). Pp 134-144.

[5] Pichugin S.F. Reliability estimation of industrial building structures // Magazine of Civil Engineering. – 2018. – No.83(7). – Pp 24-37.

[6] Neustroeva Ju.D., Ovchinnikov I.G. Aesthetics of a bridge structure as one of the ways to improve its quality // Russian journal of transport engineering. – 2020.

[7] Yakubovskiy A.Ch., Yakubovskiy Ch.A. Obshchij metod rascheta staticheski neopredelimyh sistem [General method of calculating statically indeterminate systems] // M.: Mashinostroenie. – 2005 – Issue 21.

[8] Koyankin A.A., Mitasov V.M. Stress-strain state of a precast-monolithic building // Engineering Construction Journal – 2017. – No.6. – Pp 175-184.

[9] Umansky A.A. Special'nyj kurs stroitel'noj mekhaniki. Mnogoproletnye balki na uprugih oporah [Special course in structural mechanics. H II. Multi-span beams on elastic supports] // M.: Stroyizdat. – 2000. – 212.

[10] Yevgrafov G.K., Bogdanov N.N. Proektirovanie mostov [Bridge design] // M.: Transizdat. – 2012. – 661.

[11] Vladimirsky S.R. Sovremennye metody proektirovaniya mostov [Modern methods of bridge design] // M.: Transport. – 2006. – 664.

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TEMPERATURE ANALYSIS OF A FLAT SOLAR COLLECTOR USING ALUMINUM NANOFLUIDS

In this work, the thermal characteristics of a flat solar collector were performed using a nanofluid of aluminum oxide- water. The purpose of this article is to develop a hydrodynamic model using the CFD program. The main direction of the study is that the model is confirmed by the results of the experiment conducted in this study. The model is modeled in the temperate climate of Kazakhstan. The idea of the scientific research was that with the help of the ANSYS FLUENT 19.0 CFD (Computational Fluid Dynamics) package, to calculate the presence of nanoparticles in the working fluid of a flat solar collector increases the pressure drop in a flat solar collector, but also an increase in thermal characteristics is achieved. It has been experimentally established that the optimal volume fraction of nanoparticles, which is 0.5% aluminum oxide, provides the greatest thermal efficiency of a flat solar collector. A new design of a flat solar collector has been developed, in which thermal insulation occurs in a heat-insulating transparent double-glazed window. The data on the temperature of the flat solar collector were determined using the commercial software package CFD (Computational Fluid Dynamics) ANSYS FLUENT 19.0. Numerical analysis of temperature data confirmed the accuracy of the results obtained as a result of experimental analysis. The practical significance of the results of this work suggests that the presence of nanoparticles on the upper glass of the collector increases thermal efficiency, efficiency and service life.
Key words: Flat solar collector, aluminum oxide nanoparticles, thermal model, thermal efficiency.

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Алюминий нано сұйықтықтарын қолдана отырып, жалпақ күн коллекторының температурасын талдау

Бұл жұмыста жалпақ күн коллекторының жылу сипаттамалары алюминий оксиді-судың наножидкости көмегімен жасалды. Бұл мақаланың мақсаты CFD бағдарламасын қолдана отырып гидродинамикалық модель жасау болып табылады. Зерттеудің негізгі бағыты-модель осы зерттеуде жүргізілген эксперимент нәтижелерімен расталады. Модель Қазақстанның қоңыржай климатында үлгіленген. Ғылыми зерттеудің идеясы ANSYS FLUENT 19.0 CFD пакетін (есептеу гидродинамикасы) қолдана отырып, жазық күн коллекторының жұмыс сұйықтығында нанобөлшектердің болуын есептеу жазық күн коллекторындағы қысымның төмендеуін арттырады, сонымен қатар жылу өнімділігін арттырады. 0,5% алюминий оксидін құрайтын нанобөлшектердің оңтайлы көлемдік үлесі жазық күн коллекторының ең жоғары жылу тиімділігін қамтамасыз ететіндігі тәжірибе жүзінде анықталды. Тегіс күн коллекторының жаңа дизайны жасалды, онда жылу оқшаулағыш мөлдір екі қабатты терезеде жылу оқшаулау жүреді. Жазық күн коллекторының температурасы туралы деректер ANSYS FLUENT 19.0 коммерциялық CFD (есептеу гидродинамикасы) бағдарламалық пакетін пайда-

лану арқылы анықталды. Температура деректерін сандық талдау эксперименттік талдау нәтижесінде алынған нәтижелердің дәлдігін растады. Бұл жұмыс нәтижелерінің практикалық мәні коллектордың жоғарғы әйнегінде нанобөлшектердің болуы жылу тиімділігін, тиімділігі мен қызмет ету мерзімін арттырады.
Түйін сөздер: Жазық күн коллекторы, алюминий оксидінің нанобөлшектері, жылу моделі, жылу тиімділігі.

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Анализ температуры плоского солнечного коллектора с использованием алюминиевых нано-жидкостей

В этой работе тепловые характеристики плоского солнечного коллектора были выполнены с использованием наножидкости оксида алюминия-воды. Целью данной статьи является разработка гидродинамической модели с использованием программы CFD. Основным направлением исследования является то, что модель подтверждается результатами эксперимента, проведенного в данном исследовании. Модель смоделирована в условиях умеренного климата Казахстана. Идеей научного исследования было то, что с помощью пакета CFD (Вычислительная гидродинамика) ANSYS FLUENT 19.0, рассчитать присутствие наночастиц в рабочей жидкости плоского солнечного коллектора увеличивает перепад давления в плоском солнечном коллекторе, но также достигается повышение тепловых характеристик. Экспериментально установлено, что оптимальная объемная доля наночастиц, составляющая 0,5% оксида алюминия, обеспечивает наибольшую тепловую эффективность плоского солнечного коллектора. Разработана новая конструкция плоского солнечного коллектора, в котором теплоизоляция происходит в теплоизоляционном прозрачном стеклопакете. Данные о температуре плоского солнечного коллектора были определены с использованием коммерческого программного пакета CFD (Вычислительная гидродинамика) ANSYS FLUENT 19.0. Численный анализ температурных данных подтвердил точность результатов, полученных в результате экспериментального анализа. Практическим значением итогов данной работы говорит о том, что присутствие наночастиц на верхнем стекле коллектора увеличивает тепловую эффективность, КПД и срок службы.
Ключевые слова: Плоский солнечный коллектор, наночастицы оксида алюминия, тепловая модель, тепловая эффективность.

1 Introduction

All over the world, traditional energy sources and electricity are becoming increasingly scarce resources. Solar energy can be considered the most important renewable energy source due to its sustainability, favorable environment, and vital accessibility. Therefore, the use of solar energy to meet the growing energy needs is becoming more and more relevant. The water heating sector, industrial applications, and water desalination systems consume a significant amount of energy. In articles [1-3] solar energy was used to heat water, which can save this amount of energy easy to manufacture, install and operate, as well as the cheapest solar collectors with flat plates is considered [4]. In [5], several types of studies have been developed devoted to improving the thermal characteristics of cheap solar collectors with flat plates. In the article [6], solar collectors with flat plates of nanofluid were presented. While Maxwell was the first to present a theoretical basis for predicting the effective conductivity of a suspension. Nanofluid refers to a suspension mixture between a liquid and the smallest particles of metallic

or non-metallic solids. Nanofluids are classified as a new class of liquids created by dispersing nanoscale particles in a coolant. The thermophysical properties of the nanofluid could be predicted theoretically [7]. On the one hand, the thermal conductivity of nanoparticles is high compared to the base fluid used in heat transfer applications, which leads to increased heat transfer. On the other hand, the high density of nanoparticles leads to an increase in the viscosity of nanofluids and an increase in the pressure drop and the required pumping power in forced conventional heat transfer systems [8]. The physical properties of the nanofluid are very different from the properties of the base liquid. Thermal conductivity, specific heat capacity, density, and viscosity change. The density of solids is higher than that of liquids, therefore, it is predicted that the density of the nanofluid will increase. Said et al. [4] conducted an experiment to study the effect of TiO2-water nanofluid as a working fluid on improving FPSC performance. The mass flow rates of the nanofluid varied from 0.5 to 1.5 kg/min, while the volume fraction of nanoparticles was 0.1% and 0.3%. Thermophysical properties and reduction of deposition of TiO2 nanofluid were achieved by adding polyethylene glycol (PEG 400) as a dispersant. The results showed that energy efficiency increased to 76.6%, and the highest obtained value of exergetic efficiency was 16.9%, assuming a volume fraction of 0.1% and a flow rate of 0.5 kg/min. They showed that for 0.1% and 0.3% of the volume fraction of the TiO2 nanofluid, the pumping power and pressure drop were equal to the base fluid. For more than one month, the water-based TiO2 nanofluid remained stable, the thermal conductivity is apparently affected by the volume fraction, since it increases by 6% at 0.3 vol.% TiO2. The solar collector in the case of using TiO2-H2O nanofluid has a higher exergy and energy efficiency than in the case of clean water. The use of nanofluids as an FPSC working fluid is one of the methods used to improve the thermal characteristics and performance of FPSC [9, 10]. Improving the thermal characteristics of FPSC by improving the thermal characteristics of the FPSC working fluid using nanofluids has been studied by many researchers in recent decades [7]. Dispersion of nanoparticles of highly conductive material in the base liquid increases the thermal conductivity of the liquid. High thermal conductivity and surface area of nanoparticles enhance thermal conductivity and convection in nanofluids [11,12]. Choi and Steven [6] presented the concept of increasing the thermal conductivity of nanofluids by adding nanoparticles. They reported that the addition of 1% by volume concentration of the nanoparticle can double the thermal conductivity of the liquid. Other researchers have confirmed the results of Choi and Steven [13-15]. The advent of unique technologies of the developing Solar Energy (SE), actual energy, faces economic and environmental problems. The main obstacle to the widespread use of SE is the low value of the average annual efficiency of known solar installations. In a sharply continental climate, they are exploited only in the warm season, about 6-7 months. Known combined systems, where additional conventional water heaters duplicate the operation of solar units, require additional costs for energy carriers. These disadvantages are not offered by the integrated system of SE use. In the article, the system was studied using the example of a cattle-breeding farm. The new system performs these functions; it recycles heat, organizes their movement and accumulation, and smooths out the uneven SE. The main components of the system are: Solar Power Plant (SPP), milk cooler, climate unit, Heat Pump (HP), the battery heat, automatic control system, and device heating and hot water. The main goal, i.e. lower cost of the energy produced and the elimination of the uneven SE, compared to the known SPP, is achieved through the flow of energy from the sources mentioned above [16].

The scientific novelty of this study is the development and study of a thermal model for a flat solar collector working with various nanofluids having different concentrations of nanoparticles. In this study, the developed CFD model is confirmed by experimental work.

2 Methods and Materials

Developed master control of solar thermal system is able to measure characteristics of thermal solar installation with chemical coating, which might be compared to similar features of traditional double circuit solar installation with thermosyphon circulation. Solar heat supply system with solar collector, covered with chemical etching has been constructed at the Institute of Information and Computer technologies in Almaty city, republic of Kazakhstan (latitude $45°24'5''n.l.$, longitude $9°14'58''E$). The installation has been developed without cable grooming, it is cheaper, than accessible solution and simpler in implementation to avoid the problems of communication with installation inside the building, far from a solar panel. The system anticipates installing of external heat exchanger, designed for modeling hot water consumption or dissipating the heat at temperature inside the tank exceeding the fixed value, set as a maximum threshold. Control system consists of external wireless solar power source unit with autonomous energy supply, which transfers the data on solar panel temperature (T1) to the inner control unit, which receives data and manages the system, controlling temperature values and states of two electric pumps.

Figure 2 shows the experimental installation, designed for specifying the temperature level of heated liquid and water in the reservoir, also for measuring the irradiation level on solar panels, which can be used for comparing the performances of double-circuit solar installation with thermosiphon circulation and solar installations with chemical coating.

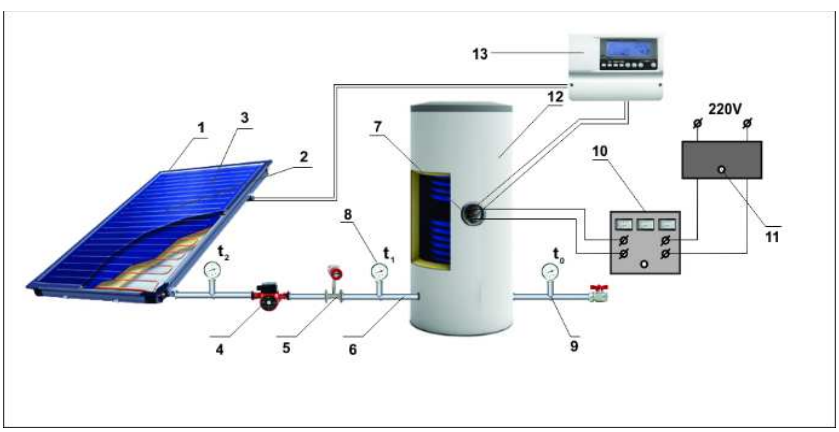


Figure 1: Experimental installation of solar heat supply system with controller. Where 1 – heat insulated body; 2 – translucent cover; 3 – tank-absorber; 4 – circulation pump; 5 – flowmeter; 6 – pipeline; 7 – THE; 8, 9 – thermometers for measuring water temperature at accumulator-tank inlet and outlet and external environment; 10 – set of electrical measuring instruments K 501; 11 – autotransformer; 12 – tank-accumulator; 13 – controller

Figures 2 a, b demonstrate the solar collector with chemical coating. It is made of hot-red glass with $1000 \times 2000\ mm^2$ dimension, 4 mm thickness. Spiral- form copper tube with 10 mm

diameter, $4,5\ m$ length is soldered up to a back side of the absorbing copper sheet. A copper sheet of 1 mm thickness and $594 \times 840\ mm^2$ size has coating, applied by means of chemical etching ($K_2S_2O_8 + NaOH + H_2O + Na_2S_2O_8$) with several microns thickness. Two-layered heating up is fulfilled with foil penofol and of 30 mm thickness foam plexus. Solar collector's reverse side is made of 2 mm thickness aluminum sheet.

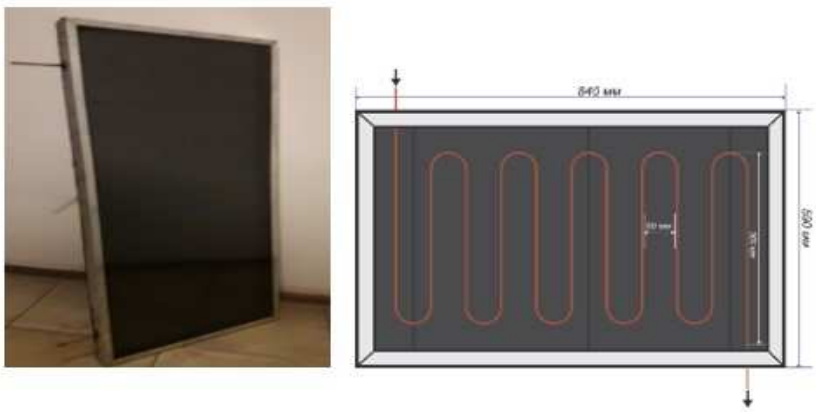


Figure 2: a) Solar collector with chemical coating; b) Solar collector's inner part with chemical coating

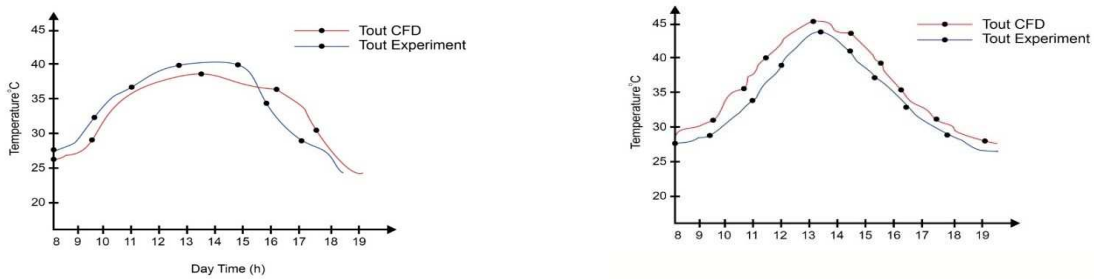
The fluid flow in a flat solar collector has a uniform velocity to the input cross-section. The uniform speed varies depending on the mass flow rate. To simulate a nanofluid, it is assumed that the nanofluid is single-phase. This means that changing the type of nanofluid and the volume fractions of nanoparticles changes the properties of the liquid. The upper half of the absorber of a flat solar collector is exposed to solar heat flux and heat loss. The two sides of a flat solar collector and the outer surface of the collector are an adiabatic process. If an adiabatic process occurs, a zero pressure gradient is used at the output boundary.

The experimental setup is shown in Figure 3.

Figure 3: Test bench of a two-circuit solar installation

3 Results

During the study of the CFD model and the experiment, graphs were constructed.



a) b)

Figure 4: Comparative results of CFD model and experiment

The outlet temperature decreases with increasing volume flow. This shows that a flat solar collector is used in a variety of applications, and can be easily controlled by the outlet temperature. A flat solar collector was investigated numerically for various volumetric flow rates of 5, 5.7 and 8 l/min for pure water.

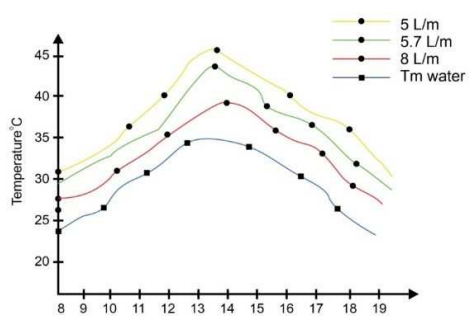


Figure 5: Temperature changes at the collector outlet for three volume flow rates under the same conditions

Figure 5 shows numerical temperature changes at the outlet of a flat solar collector for 3 volume flow rates under the same conditions. Numerical modeling was carried out for aluminum oxide nanofluids with different percentages of volume fractions. The temperature distribution of the working fluids, the pressure drop along the collectors and the thermal efficiency of the collector were evaluated for each working fluid and each percentage of the volume fraction of the nanoparticle.

Figure 6 shows the four temperature conditions at the inlet, 10°C, 30°C, 50°C, 70°C, were considered to obtain equivalent heat transfer coefficients of absorbent plates to predict the thermal characteristics of the collector.

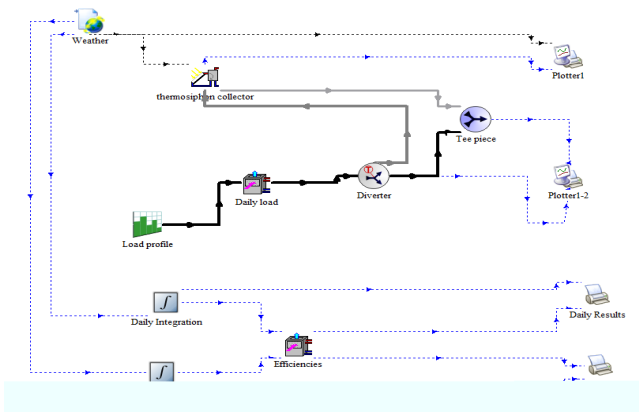


Figure 6: TRNSYS diagram of a simulated thermosiphon system of a solar collector

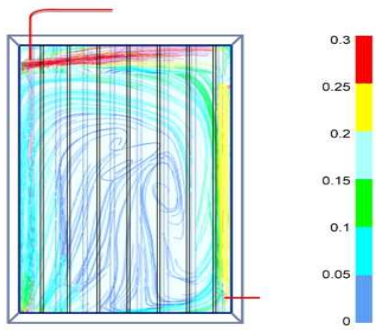


Figure 7: Temperature distribution over the absorber plate, riser pipes and collector collectors

Figure 7 shows the general behavior of the temperature of the working fluid together with the pipeline network of the absorber plate. As shown in the figure, the temperature of the liquid increases as it passes through the pipes of the risers.

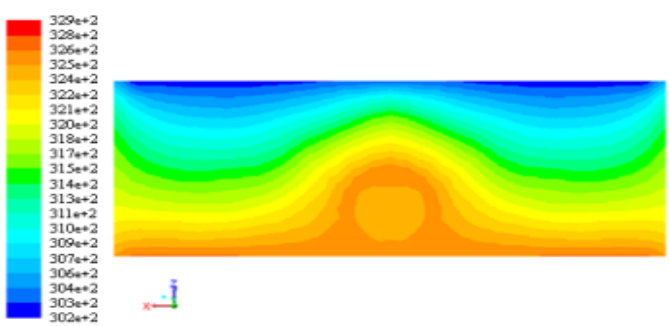


Figure 8: Contours of water temperature in a flat solar collector

Figure 8 shows the heat transfer from the absorber.to the riser with the help of convection mechanisms.

4 Discussion

The temperature characteristics of the new design of a flat solar collector presented in the article were created on the commercial software package CFD (Computational Fluid Dynamics) ANSYS FLUENT 19. 0. In comparison with previous works, analyses and conclusions, the efficiency of a flat solar collector with nanoparticles was 57.89% at a low temperature of 30 $°C$, while at a higher temperature the efficiency was 60.45%. The water temperature at the inlet of the flat solar collector was 68.56 $°C$, the consumption temperature was 31.44 $°C$, solar energy was 750.50 W/m^2. The water temperature at the outlet of the flat solar collector with nanoparticles was 75.76 $°C$, the air temperature was 31.44 $°C$ and the solar energy consumption was 750.50 W/m^2. On a clear day, the flow of solar energy reaching the Earth's surface at local noon is usually in the range of 700-1300 W/m^2, depending on latitude, longitude, altitude and time of year. In particular, for our region, Almaty, Republic of Kazakhstan, the solar energy flow is 750,50 W/m^2. Based on the results of the experimental work carried out, it can be concluded that the efficiency of a flat solar collector with nanoparticles increased by 6.5% compared to the solar collector in work [16].

5 Conclusion

In this paper, a numerical study has been developed to estimate the thermal conductivity of a flat solar collector with nanoparticles. For this purpose, CFD modeling was used, which was confirmed by comparison with previous experimental results. According to the results of the study, it was shown that nanoparticles in the working fluid slightly increases the thermal characteristics of the collector, especially in low temperature ranges, as well as an increase in the percentage of nanoparticles in nanofluids to 0.5% for aluminum oxide nanofluid. According to the thermal characteristics of aluminum oxide nanofluid, increases in pressure drop do not affect the increase in thermal characteristics of a flat solar collector. Prospects and opportunities for the implementation of the application of this development have a wide geographical location. The article [17] discusses the resources of the Republic of Kazakhstan based on solar energy. Estimates of solar systems when assessing solar energy resources on the territory of Kazakhstan are based on quantitative characteristics of direct solar radiation on a horizontal surface from which conversion from a horizontal plane to an inclined plane of any orientation can be performed. As a result, the statistical processing of the average values of direct, total exposure and duration of solar radiation was carried out, radiation was compiled, five zones were identified and a histogram was compiled characterizing the possibility of introducing solar installations in Kazakhstan.

References

[1] ASHRAE *Standard 1980* 6 (RA 89) (2003): 86244.

[2] Elghamry R., Hassan H., Hawwash A.A., "A parametric study on the impact of integrating solar cell panel at building envelope on its power, energy consumption, comfort conditions, and CO2 emissions" , *J. Cleaner Prod.,* 249 (2020): 119374.

[3] Hussein H.M.S., El-Ghetany H.H., Nada S.A., "Experimental investigation of novel indirect solar cooker with indoor PCM thermal storage and cooking unit" , *Energy Convers. Manage.,* 49 (8) (2008): 2237-2246.

[4] Said Z., Sabiha M.A., Saidur R., Hepbasli A., Rahim N.A., Mekhilef S., Ward T.A., "Performance enhancement of a flat plate solar collector using titanium dioxide nanofluid and polyethylene glycol dispersant", *J. Cleaner Prod.,* 92 (2015): 343–353.

[5] Duffie J.A., Beckman W.A., *Solar Engineering of Thermal Processes* 4th ed. (Hoboken, NJ, USA: Wiley, 2013).

[6] Choi J.A.E., Stephen U.S., "Enhancing thermal conductivity of fluids with nanoparticles", *Proc. ASME Int. Mech. Eng. Congr. Expo.,* (1995): 12–17.

[7] Gupta M., Singh V., Kumar R., Said Z., "A review on thermophysical properties of nanofluids and heat transfer applications", *Renew. Sustain. Energy Rev.,* 74 (2017): 638–670.

[8] Said Z., Sajid M.H., Alim M.A., Saidur R., Rahim N.A., "Experimental investigation of the thermophysical properties of AL2O3-nanofluid and its effect on a flat plate solar collector", *Int. Commun. Heat Mass Transf.,* 48 (2013): 99–107.

[9] Sundar L.S., Ramana E.V., Said Z., Punnaiah V., Chandra Mouli K.V.V., Sousa A.C.M., "Properties, heat transfer, energy efficiency and environmental emissions analysis of flat plate solar collector using nanodiamond nanofluids", *Diamond Rel. Mater.,* 110 (2020): 108115.

[10] Sundar L.S., Sintie Y.T., Said Z., Singh M.K., Punnaiah V., Sousa A.C.M., "Energy, efficiency, economic impact, and heat transfer aspects of solar flat plate collector with Al2O3 nanofluids and wire coil with core rod inserts", *Sustain. Energy Technol. Assessments,* 40 (2020): 100772.

[11] Zhang X., Gu H., Fujii M., "Effective thermal conductivity and thermal diffusivity of nanofluids containing spherical and cylindrical nanoparticles", *Experim. Thermal Fluid Sci.,* 31 (6) (2007): 593–599.

[12] Xie H., Lee H., Youn W., Choi M., "Nanofluids containing multiwalled carbon nanotubes and their enhanced thermal conductivities", *J. Appl. Phys.,* 94 (8) (2003): 4967–4971.

[13] Trisaksri V., Wongwises S., "Critical review of heat transfer characteristics of nanofluids", *Renew. Sustain. Energy Rev.,* 11 (3) (2007): 512–523.

[14] Wen D., Lin G., Vafaei S., Zhang K., "Review of nanofluids for heat transfer applications", *Particuology,* 7 (2) (2009): 141–150.

[15] Murshed S.M.S., Leong K.C., Yang C., "Thermophysical and electrokinetic properties of nanofluids-A critical review", *Appl. Thermal Eng.,* 28 (17-18) (2008): 2109-2125.

[16] Omarov R., Abdygaliyeva S., Omar D., Kunelbayev M., "Integrated system for the use of solar energy in the animal farm", *Scientia Iranica,* 24 (6) (2017): 3213–3222.

[17] Amirgaliyev Y.N., "Solar-Driven Resources of the Republic of Kazakhstan", *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences,* 3 (430) (2018): 18–27.

3-бөлім **Раздел 3** **Section 3**

Информатика **Информатика** **Computer Science**

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A MACHINE LEARNING MODEL BASED ON HETEROGENEOUS DATA

Big data is widely used in many areas of business. The information between organizations is systematically reproduced and processed by data, and the collected data differs significantly in attributes. By composing heterogeneous data sets, they complement each other, therefore, data exchange between organizations is necessary. In a machine learning collaborative learning process based on heterogeneous data, the current schema has many challenges, including efficiency, security, and availability in real-world situations. In this paper, we propose a secure SVM learning mechanism based on the consortium blockchain and a threshold homomorphic encryption algorithm. By implementing the consortium's blockchain, it is possible to build a decentralized data exchange platform, and also to develop a secure algorithm for the support-vector machine classifier based on threshold homomorphic encryption.
Key words: Blockchain, heterogeneous data, SVM, secure scheme.

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Гетерогенді деректерге негізделген машиналық оқыту моделі

Үлкен деректер бизнестің көптеген салаларында кеңінен қолданыс тапқан. Ұйымдар арасындағы ақпараттарды жүйелі түрде шығаруға және мәліметтерді өңдейді, жинаған мәліметтер атрибуттарда айтарлықтай ерекшеленеді. Гетерогенді мәліметтер жиынтығын құра отырып, олар бір-бірін толықтырады, сондықтан ұйымдар арасында мәліметтер алмасу қажет. Гетерогенді деректерге негізделген машиналық оқытуды бірлесіп оқыту процесінде қазіргі сызба көптеген мәселелерге ие, соның ішінде нақты жағдайларда тиімділік, қауіпсіздік және қол жетімділік. Бұл мақалада біз консорциум блокчейніне және шекті гомоморфты шифрлау алгоритміне негізделген қауіпсіз ТВӘ оқыту механизмін ұсынамыз. Консорциум блокчейнін енгізу арқылы орталықтандырылмаған деректер алмасу платформасын құруға, сонымен қатар шекті гомоморфты шифрлау негізінде тірек-векторлар әдісі классификаторының қауіпсіз алгоритмін құруға болады.
Түйін сөздер: Блокчейн, гетерогенді деректер, ТВӘ, қауіпсіздік сызбасы.

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Модель машинного обучения, основанная на гетерогенных данных

Большие данные широко используются во многих областях бизнеса. Информация между организациями систематически воспроизводится и обрабатывается данными, а собранные данные существенно различаются по атрибутам. Составляя разнородные наборы данных, они дополняют друг друга, следовательно, необходим обмен данными между организациями.

В процессе совместного обучения машинного обучения на основе разнородных данных текущая схема имеет множество проблем, включая эффективность, безопасность и доступность в реальных ситуациях. В этой статье мы предлагаем безопасный механизм обучения МОВ, основанный на блокчейне консорциума и пороговом гомоморфном алгоритме шифрования. Путем внедрения блокчейна консорциума можно построить децентрализованную платформу обмена данными, а также разработать безопасный алгоритм классификатора опорно-векторные машины на основе порогового гомоморфного шифрования.
Ключевые слова: Блокчейн, гетерогенные данные, МОВ, безопасная схема.

1 Introduction

Intelligent automotive technology is developing very rapidly, and recent advances suggest that autonomous car navigation will be possible in the near future. One of the new trends of protecting data is the blockchain technology that today is used in different areas. In this regard, we believe that absolutely all vehicles will have a full-fledged on-board computer with the ability to install secure applications with access to navigation and other sensors in reading mode. Therefore, the implementation of blockchain solutions will be quite affordable without additional hardware modifications [1].

The development of cloud computing and edge computing has led to a proliferation of data, such as the large amount of data generated everyday in vehicular social networks, which can be used to optimize the security, convenience and entertainment of applications in vehicular social networks [2]. Effective data analysis methods need to be used in such scenarios, among which machine learning and deep learning are particularly important [3]. Among the commonly used machine learning methods, support vector machine (SVM) model has significant advantages in performance and robustness, so it has a wide range of applications [4].

Take the vehicular social network as an example. There are various organizations within transport networks, such as a vehicle manufacturer, a vehicle management agency, and a provider of vehicle social media application services. These entities have different data sources, and differences in the data sources cause the data to complement each other in terms of attributes [5]. We call the scenario data heterogeneous data.

However, for a single organization, its dataset cannot cover the multidimension, which has great limitations in the use process. Especially in the training process of SVM classifier, the classification effect of the final model is highly correlated with the quality of the data set, so it is difficult for a single organization to train an ideal classifier through its own data. Therefore, it is necessary to share heterogeneous data among multiple institutions. Through data sharing, a dataset covering multiple attributes can be combined to improve the effectiveness of the classifier. From another perspective, the dataset obtained after the fusion of these heterogeneous data can be vertically partitioned into sub-data sets provided by each unit according to the attributes. However, in the process of data sharing, data privacy is facing serious challenges. First of all, the heterogeneous data to be shared contains users' privacy information. With the increasing attention of the government and individuals to users' privacy issues, more and more regulations restrict the sharing of users' data by enterprises. As a result, direct data sharing is subject to increasingly stringent regulations. In addition, for the data owners, the high value of heterogeneous data is mainly reflected in the privacy of the data, that is, the data is only owned by itself, or a small number of institutions. So

if the data is shared directly, it becomes less private and less valuable and data owners are unwilling to reduce the value of their data [6].

For a long time, privacy disclosure issues raised in diverse scenarios has been highly concerned [7]. Among those scenarios, many researches pay attention to train a machine learning classifier securely over both horizontally and vertically partitioned datasets. Many existing solutions adopt secure multi-party computation (SMC) to prevent privacy disclosure. Firstly, in those schemes, how to balance security and efficiency issues still faces big challenges [8]. Then, one or more aided servers are essential with the assumption that they are trusted or semi-trusted during the training process. Obviously, in a real-world scenario, it is impractical to provide such aided servers for the participants. To deal with the two challenges of applying the privacy protection scheme to real-world scenarios, we propose an efficient and secure SVM classifier training scheme based on consortium blockchain where no third party is introduced [9].

In this paper, we propose a security SVM training mechanism based on consortium blockchain for multi-source heterogeneous data sharing scenario, which solves the above two problems. First, because the differential privacy protection scheme introduces noise to the training results and the training process is not secure, we adopt the scheme based on homomorphic encryption [10,11].

We introduce block chain to establish a decentralized data sharing platform for sharing secret data. When each participant shares data, they simply upload the data to the data sharing platform. The access control and permission mechanism of the consortium blockchain fully ensures the unknowability of the external data and the openness and transparency of the internal data.

We propose a SVM training scheme that contributes more secure and efficient heterogeneous data sharing. First, an open, reliable and transparent data sharing platform was built based on blockchain technology. The operation of the platform does not rely on trusted third parties. The data on the platform is visible to members in the blockchain and not visible to the outside. After that, most of the training work was completed locally by each participant based on clear text data. We introduce threshold homomorphic encryption scheme to ensure a data privacy protection scheme in a decentralized environment. All data that needs to be shared can be fully protected by this scheme and maintain its homomorphic property. Our scheme guarantees a controllable degree of privacy protection by setting the size of the threshold. A large number of experiments based on real datasets prove the feasibility and efficiency of the scheme.

2 Secure Machine Learning over Heterogeneous Data

Consider a dataset D is combined with several participants who have its own dataset $D^p p \in A, B, \ldots, N$, where x_i^p represent the i-th instance in D^p, and y_i is shared as a data label between all related i-th instance x_i^X. When training a SVM classifier, we define w as the model parameters, Δ_t as gradient in the t iteration, λ as the learning rate. Meanwhile, we assume that $[[m]]$ as the encryption of message m under Paillier. Table 1 shows the notations used in this paper.

Table 1. Notations

Notations	Description
D^A	The dataset of participant
d^A	The dimension of dataset D^A
x_i^A	The i-th data instance of dataset D
y_i	Number

3 System Model

We divide our system into three components based on their relationship with the data. As shown in Fig. 1, they are data device (DD), data provider (DP) and blockchain service platform (BSP).

- Data Device: Refer to devices capable of generating data, including sensors, mobile devices, and so on. Because the data directly collected from these devices contains high-value information, these data are collected, processed, and then used for data analysis.

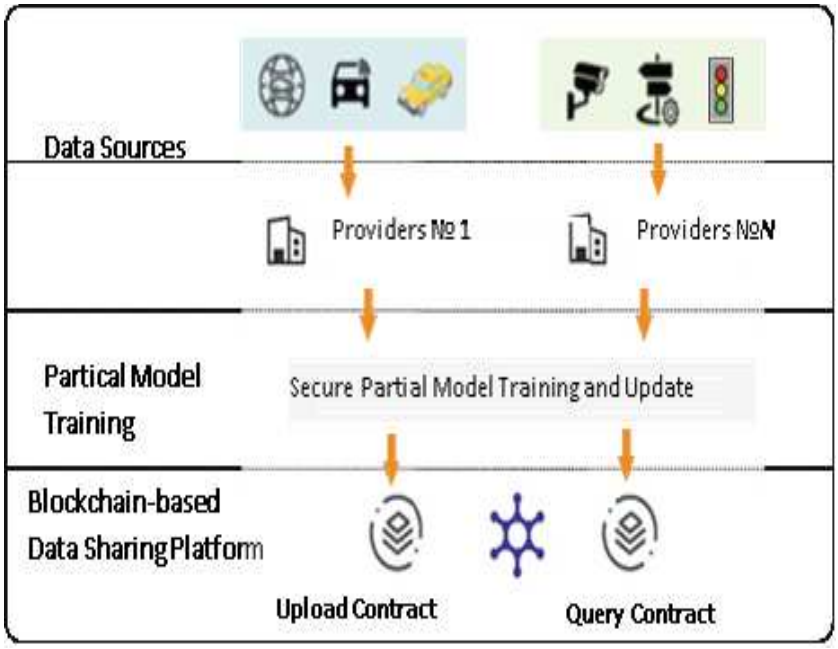

Figure 1: Overview of secure SVM training scheme over heterogeneous data

- Data Provider: The equipment that generates the data is collected, stored, and used by different parties. These participants are called participants and act as data providers in our solution. Due to the different equipment, the collected data is different, and due to the different data processing methods, the available data after processing has different

attributes and complement each other. In addition to serving as data providers, these participants also act as model trainers to train machine learning models in collaboration. According to the scheme in this paper, most of the training work is done locally in participant.

- Blockchain Service Platform: This is a service platform that runs on the consortium blockchain. On one hand, it provides a transparent data sharing platform distributed in participant, allowing participant to retrieve all the data recorded in the BSP. At the same time, no one captured the data recorded on the BSP for changes. On the other hand, BSP has strong security protections, making data outside of participant invisible to entities. In addition, communication data between the BSP and participant is also encrypted, preventing data leakage.

4 Threat Model

In the scheme, there is only one role of the data provider. We treat participants honest but curious when it comes to the security model, that is, all participants are curious about the data of other participants, but they will execute the scheme according to the rules. In addition, due to the large number of interactions between participant and BSP, potential threats in the interaction process are also considered.

- Known Ciphertext Model. BSP is a common and transparent data sharing platform for all participants. The data shared by each participant is visible to other participants. These data include the dense intermediate value and the decrypted calculation results.

- Known Background Model. We assume that multiple participants can conspire and collaborate to analyze shared data. Compared with the above threat model, this model can obtain more information. Under the above system model and threat model, we established the following three system design goals to meet the system's requirements for security, accuracy and performance.

- Data privacy is fully protected. Under the two threat models, during the entire training process, the privacy of the original data and the shared intermediate value will not be leaked, and the participants cannot infer valuable information from the shared data. Second, the data in the data sharing platform is guaranteed to be invisible to the outside world.

- High accuracy of training results. Generally speaking, the introduction of privacy protection schemes may introduce noise into the calculation process and cause inaccurate calculation results. Our design goal is to obtain a classifier that is not significantly different from conventional training conditions.

- Low training overhead. Similarly, the introduction of privacy protection schemes will increase training overhead. These overheads are mainly caused by additional computing operations such as encryption and decryption, and additional communication overhead. Therefore, our solution needs to ensure low training overhead while ensuring security

5 Secure SVM Training Scheme over Heterogeneous Datasets

In this section, in order to clearly introduce the work of each participant in the training process, we assume that three participants participate in the SVM model training. The respective training sets are complementary in attributes. As shown in Fig. 3, the entire training process mainly includes three parts: local training, gradient update judgment, and model update. In these three steps, two data sharing and one decryption operation are involved. Finally, after multiple iterations, each participant gets its own partial model and uploads it to the blockchain to form a complete model together.

The data privacy protection method of this solution is based on a threshold homomorphic encryption algorithm. Before training the model, a pair of public and private keys needs to be generated for each participant. The public key is the same and the private key is different. Through the secret sharing scheme combined with the existing threshold key management scheme, such a key pair is negotiated and distributed. In addition, the three participants join the consortium blockchain data sharing platform as nodes, and they need to pass identity authentication before joining. Finally, all participants need to initialize the model parameters and preprocess the data set, including unified labeling and sample order.

6 Local Training Process

In order to ensure the efficient training of the model, this solution puts most of the work locally on three participants. During one iteration, all training work can be done locally before the gradient update judgment. This section will introduce how each participant can be trained locally based on its own heterogeneous data. SVM optimization algorithm based on stochastic gradient descent (SGD) is easy to perform. SVM based on stochastic gradient descent can be expressed in the following form:

$$f(w) = \frac{1}{2}w^T w + C\sum_{i=1}^{m}\max(0, 1 - y_i w^T x_i) \tag{1}$$

Algorithm 1. SVM based on SGD

Require: Training set D, learning rate λ, maxIters T.
Ensure: Trained model w^*.
1: **for** $t = 1$ to T **do**
2: Select it from D randomly.
3: Update $\Delta t + 1$ by Eq. (1).
4: Update $wt + 1$ by Eq. (4).
5: **end for**
6: return w^*.

The right part of the equation is the hinge-loss function, where C is the misclassification penalty and we take $\frac{1}{m}$ as its value.

At each iteration, we use Eq. (2) to calculate the gradient.

$$\Delta_t = \lambda w_t - I[(wx_i < 1)]x_i y_i \tag{2}$$

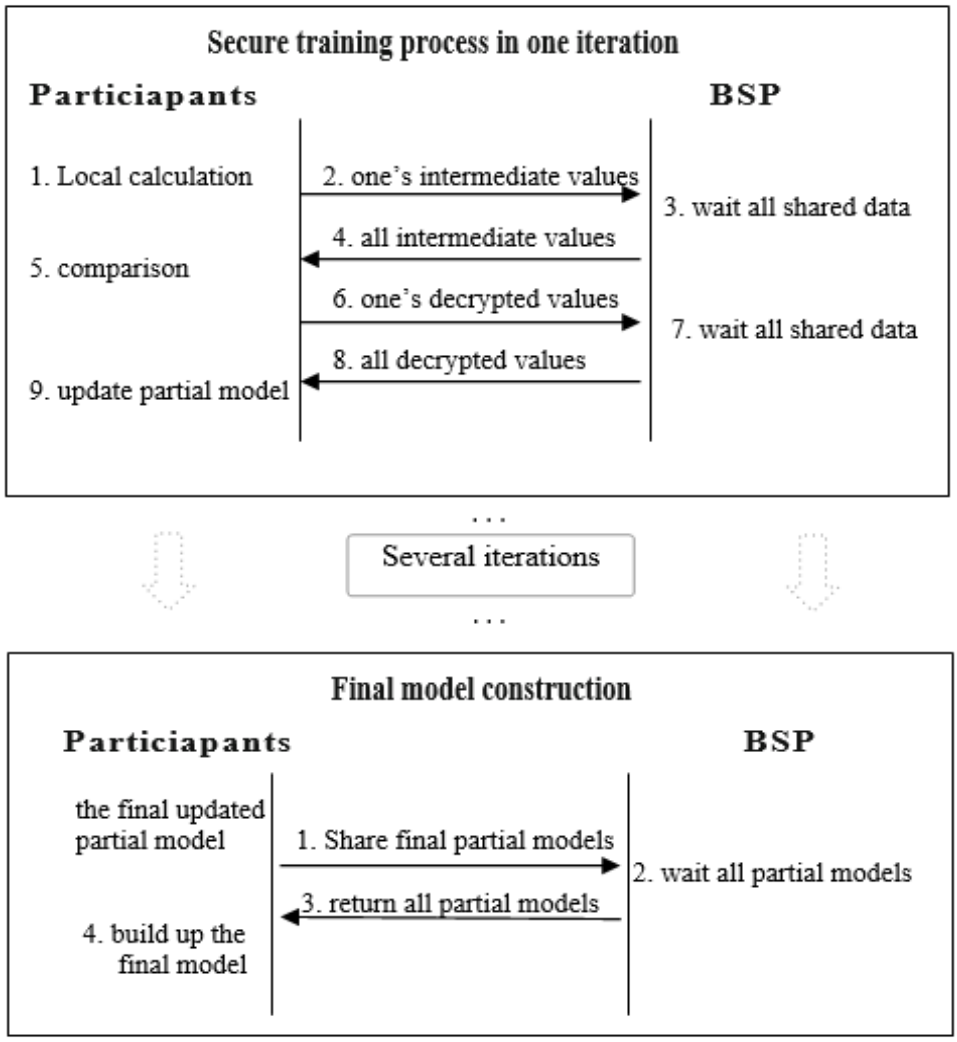

Figure 2: Workflow of secure training over heterogeneous datasets

If $I|(wx_i) < 1)|$ is true which means $(wx_i < 1)$, $I[wx_i < 1)] = 1$; Otherwise, $I[(wx_i < 1)] = 0$. Then we can update the w by Eq. (4).

$$w_(t+1) = w_t - \lambda \Delta_t \tag{3}$$

Through one iteration of the training process over several heterogeneous datasets, only when calculating I, data exchange between multiple participants is required. The rest of the training operations are performed locally. We represent wx_i by a in the following sections.

$$I = \begin{cases} 1 & y_1(w^A x_i^A + w^B x_i^B + w^C x_i^C) < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The compete algorithm is described in Algorithm 2.

The three participants need to share the calculated median value to the BSP during the training model. This solution treats the shared data with a threshold homomorphic encryption

scheme to ensure data security and ensure that the gradient can be calculated correctly. To judge how to update the gradients, here we use additive homomorphic encryption to construct Eqs. (5), (6) and (7)

Algorithm 2. Partial model training process

Require: Training set D^A, D^B, D^C, learning rate λ, maxIters T.
Ensure: Trained model w^*.
1: All participants perform the following operations simultaneously.Take participant A to describe in detail.
2: **for** $t = 1$ to T **do**
3: Select it i_t randomly.
4: Calculatey $^{-d^A}_{i=1} wx_i \; _i$
5: Cooperate with other participants to judge how to update gradient by Eq. (5).
6: Update Δ_{t+1} by Eq. (2).
7: Update $wt + 1$ by Eq. (4).
8: **end for**
9: Get several partial model parameters and combine them.
10: return w_{t+1}.

$$[[a]] = [[\sum_{i=1}^{n} a^i]] = \prod_{i=1}^{n} [[a^i]] \tag{5}$$

$$[[r_2]] = [[\sum_{i=1}^{n} r_2^i]] = \prod_{i=1}^{n} [[r_2^i]] \tag{6}$$

$$[[ar_1 + r_2]] = [[ar_1]][[r_2]] = [[\sum_{i=1}^{r_1} a]][[r_2]] = \prod_{i=1}^{r_1} [[a]][[r_2]] = [[a]]^{r_1}[[r_2]] \tag{7}$$

In order to determine the update method of the gradient, the method adopted in this solution is to compare the encrypted calculation result with the constant 1. In Algorithm 3, the security comparison algorithm in three participant scenarios is introduced in detail. It is obvious that for integer a, if $(ar_1 + r_2) > (r_1 + r_3)$, we can derive that $a > 1$, otherwise $a < 1$.

Algorithm 3. Privacy-preserving gradient update judge

Input A: $[[a^i]]$ from participant i.
Input B: r_1^i, $[[r_2^i]]$, r_3^i from participant i.
Ensure: $a > 1$ or $a < 1$.
1: Each participant i picks three positive integers r_1^i, r_2^i, r_3^i, where $|r_3^i - r_2^i| < r_1^i$, and encrypts r_2^i to get $[[r_2^i]]$.
2: Each participant i uploads $[[ai]]$, r_1^i, $[[r_2^i]]$, r_3^i.
3: Each participant i downloads all the other participants' $[[ai]]$, r_1^i, $[[r_2^i]]$, r_3^i.
4: Each participant i calculates $[[a]]$, $[[r2]]$ by Eq. (5) and Eq. (6), and calculates r_1 and r_3 where $r_1 = \sum_{i=1}^{n} r_1^i$ and $r_3 = \sum_{i=1}^{n} r_3^i$.
5: Each participant i calculates $[[ar1 + r2]]$ by Eq. (7).
6: Each participant i decrypts $[[ar1 + r2]]$ by sub-private key SK^i and uploads it to BSP.
7: Each participant i downloads all other decrypted values from participants to recover $(ar1 + r2)$, and compares $(ar1 + r2)$ with $(r1 + r3)$.
8: If $(ar1 + r2) > (r1 + r3)$, $a > 1$; Else $a < 1$.
9: return $a > 1$ or $a < 1$.

7 Data Sharing on BSP and Security Analysis

Participant relies on BSPs to securely calculate intermediate values. BSP simplifies complex point-to-point communication between participants. Participant completes data on-chain and data query by calling smart contracts. During the iteration process, each participant uploads data twice: calculating the intermediate value (IV) and the decrypted value (DV), respectively. These two data are also read twice.

1. The Format of IVs
   Iteration Round: When multiple data providers train the model collaboratively, some data needs to be exchanged in each iteration. Therefore, in order to represent the data exchanged in each round and to distinguish it from other rounds of data, a field is required to indicate the training round. Iteration Round is maintained by smart contracts.
   DP ID: A field that identifies the owner of the data. When a node calls a contract to upload data, its address will be automatically recorded in this field.
   Training Intermediate Value: The intermediate value of the encrypted state during model training. The values provided by each participant will be summed and compared to the magnitude of 1 in the encrypted state.
   r1: An unencrypted random positive integer which is used to compare.
   r2: An encrypted random positive integer which is used to compare.
   r3: An unencrypted random positive integer which is used to compare. Random Positive Integer: It is generated randomly by each participant and its value is between 1 and m, the sum of which determines the data instances selected in the next iteration.

2. The Format of DVs
   Iteration Round: Similar function described in IVs. DP ID: Similar function described

in IVs. Decrypted Value: Each participant decrypts the result obtained based on his own private key. By combining all these values, each participant can obtain the final decryption result.

The definition of computing security for a secure multiparty computing protocol is given below.

Definition 1 *The multi-party computation protocol with n participants under the cryptography model is considered to be computation security, if for any attacker A, there exists a corresponding simulator S in the ideal model interacting with A, and satisfying the following conditions:*

(1) The running time of S is the polynomial of A's running time.

(2) For any input set, the n+1 outputs produced by the multi-party computation protocol are computationally indistinguishable from the n+1 outputs produced by the ideal model.

We conducted a security analysis based on the above idea. Thus we acquire the information which an attacker can get from the ideal model and the real protocol. Then we compare them and prove they are indistinguishable. In this scheme, n participants are involved to share their encrypted intermediate values to calculate

$$F : F([[a]]^1, \ldots, [[a]]^n, 1, r_1^1, [[r_2^1]], r_3^1, \ldots, r_1^n, [[r_2^n]], r_3^n).$$

Assume that the attacker has corrupted a set of participants $A = P_i 1, \ldots, P_i|A|$. Then all the data the attacker obtained in the ideal model is the output F of the participants and the input:

$$([[a]]^{i1}, \ldots, [[a]]^{i|A|}, 1, r_1^{i1}, [[r_2^{i1}]], r_3^{i1}, \ldots, r_1^{i|A|}, [[r_2^{i|A|}]], r_3^{i|A|}).$$

We construct a simulator S that simulates all the data the attacker gets in the real model based on the data the attacker obtained in the ideal model. Firstly, we analyze the information that the attacker can get in the real protocol.

Input Phase. Since all the participants share their encrypted input:

$$([[a]]^1, \ldots, [[a]]^n, 1, r_1^1, [[r_2^1]], r_3^1, \ldots, r_1^n, [[r_2^n]], r_3^n)$$

The attacker is able to get all of them. Especially for the corrupted participants, the attacker also gets $a^i 1, \ldots, a^{i|A|}$, $r_2^1, \ldots, r_2^{|A|}$.

Computation Phase. At each step of the calculation phase, the attacker obtains data $[[x + y]]$ based on $[[x]]$ and $[[y]]$.

Output Phase. In the output phase, the attacker gets the result:

$$F([[a]]^1, \ldots, [[a]]^n, 1, r_1^1, [[r_2^1]], r_3^1, \ldots, r_1^n, [[r_2^n]], r_3^n).$$

Then we construct the simulator S of the polynomial time. S takes $a^{i1}, \ldots, a^{i|A|}$ and $r_2^1, \ldots, r_2^n$ and F as the input. The following step S0 simulates the information calculated based on the input.

Step $S0$. S generates the encrypted data $[[a]]^{i1}, \ldots, [[a]]^{i|A|}$, $[[r_2^{i1}]], \ldots, [[r_2^{i|A|}]]$ and $[[F]]$ based on $a^{i1}, \ldots, a^{i|A|}$, $r_2^1, \ldots, r_2^n$ and F. Then, S can simulate the calculations based on those encrypted data such as $[[a^{i1} + a^i 2]]$ and $[[r_2^{i1}, \ldots, r_2^{i2}]]$.

Step $S1$. After step $S0$, we can simulate part of the calculated intermediate values which are defined as $[[a]]^{j1}, \ldots, [[a]]^{j|r|}, [[r_2^{j1}]], \ldots, [[r_2^{j|r|}]]$. Then for the remaining intermediate values that cannot be directly simulated by $S0, S$ simulates them by selecting the random numbers to generate the corresponding ciphertext. According to the threshold cryptosystem's security, these simulations are successful.

Step S2 Based on steps $S0$ and $S1$, we can simulate all the values calculated in the computation phase.

Step S3. F is one of S's input, so S can easily get a simulation of F. From the above simulation process, the information obtained by the attacker from the ideal model and the information obtained from the real model are computationally indistinguishable. You can prove the security of the solution.

8 Conclusion

In this section, we propose an effective and secure SVM training scheme that helps multiple data providers train SVM classifiers on vertically partitioned datasets. The target of this chapter is to combine consortium blockchain technology and threshold Paillier to create a decentralized and secure SVM training platform. To achieve high performance, most training operations are performed locally on raw data, so there are only a few intermediate values that need to be shared across platforms.

References

[1] Narbayeva S., Bakibayev T., Abeshev K., Makarova I., Shubenkova K., Pashkevich A., "Blockchain Technology on the Way of Autonomous Vehicles Development" , *Transportation Research Procedia* 44 (2020): 168-175.

[2] Cheng N. , Lyu F. , Chen J., Xu W., Zhou H., Zhang S., Shen X.S., "Big data driven vehicular networks" , *IEEE Netw.* 32(6) (2018): 160-167.

[3] Fadlullah Z.M., Tang F., Mao B., Kato N., Akashi O., Inoue T., Mizutani K., "State-of-theart deep learning: evolving machine intelligence toward tomorrow's intelligent network traffic control systems" , *IEEE Commun. Surv. Tutorials* 19(4) (2017): 2432-2455.

[4] Lv L., Zhang Y., Li Y., Xu K., Wang D., Wang W., Li M., Cao X., Liang Q., "Communicationaware container placement and reassignment in large-scale Internet data centers" , *IEEE J. Sel. Areas Commun.* 37(3) (2019): 540-555.

[5] Song D.X., Wagner D., Perrig A., "Practical techniques for searches on encrypted data" , *IEEE S&P* (2000): 44-55.

[6] Li H., Zhu L., Shen M., Gao F., Tao X., Liu S., "Blockchain-based data preservation system for medical data" , *J. Med. Syst.* 42(8) (2018): 141.

[7] Shen M., Ma B., Zhu L., Du X., Xu K., "Secure phrase search for intelligent processing of encrypted data in cloud-based IoT" , *IEEE Internet Things J.* 6(2), (2019): 1998-2008.

[8] Xu K., Yue H., Guo L., Guo Y., Fang Y., "Privacy-preserving machine learning algorithms for big data systems" , *IEEE 35th International Conference on Distributed Computing Systems (IEEE, Piscataway* (2015): 318-327.

[9] Mohassel P., Rindal P., "ABY 3: a mixed protocol framework for machine learning," , *Proceedings of the 2018 ACM SIGSAC Conference on Computer and Communications Security, ACM, New York* (2018): 35-52.

[10] Katz J., Lindell Y., " , *Introduction to Modern Cryptography* (Chapman and Hall/CRC, Boca Raton, 2014)

[11] Mangasarian O.L., Wolberg W.H., " , *Cancer diagnosis via linear programming* Technical report (University of Wisconsin-Madison Department of Computer Sciences, 1990).

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QURMA: A TABLE EXTRACTION PIPELINE FOR KNOWLEDGE BASE POPULATION

This paper is proposed a pipeline aimed at automatically extracting tables from heterogeneous Web sources, such as HTML pages, pdf files and images. Table extraction is one of the actively developing areas of Information Extraction, for which many applications, libraries and frameworks are currently being developed. Nevertheless, most of these tools are focused on solving some specific tasks, for example, only on recognizing tables presented in the form of images. We propose to combine these tasks into a single pipeline that will support the full cycle of table processing – from the stages of their search, recognition and extraction to the stages of semantic analysis and interpretation, that is, the understanding of tables. Understanding tables and population of knowledge bases (knowledge graphs) with meaningful information contained in these tables is the ultimate goal of our design. The first part of the work presents methods for detecting tables on web pages, in pdf documents, as well as methods for automatically detecting attributes and values of objects. The second part presents the conceptual architecture of the Qurma system, designed to extract tables from heterogeneous sources on the Internet. The results section provides an example of a parser that parses the input resource type and passes control to one of the table lookup modules. Next, an operation is performed to determine the main column and link the entities contained in the main column with the corresponding categories in the external knowledge base.

Key words: web tables, table extraction, table recognition, table understanding, knowledge base population.

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Qurma: білім базасын толтыруға арналған кестені шығарып алу құбыры

Бұл мақалада HTML парақтары, PDF файлдары және суреттер сияқты, веб парақтардың гетерогенді деректер көздерінен кестелерді автоматты түрде шығарып алу құбыры ұсынылады. Кестелерді шығару – қазіргі уақытта көптеген қосымшалар, кітапханалар мен жақтаулар құрастырылып жатқан ақпарат алудың белсенді дамып келе жатқан бағыттарының бірі. Алайда, бұл құралдардың көпшілігі кейбір нақты мәселелерді шешуге бағытталған, мысалы, тек суреттер түрінде ұсынылған кестелерді тануға бағытталған. Тексттік кестелерді танитын, оқитын, оларды топтарға жіктейтін бағдарламалар кең танылмаған, және дайын кітапханалар неемесе құралдар жоқ. Біз бұл тапсырмаларды кестелерді өңдеудің толық циклын қолдайтын біртұтас құбырға біріктіруді ұсынамыз – оларды іздеу, тану және шығарып алу кезеңдерінен бастап, семантикалық талдау мен түсіндіру кезеңдерімен аяқтайды, яғни кестелерді түсінеді. Кестелерді түсіну және білім базасын (білім бағандары) осы кестелердегі маңызды ақпаратпен толтыру біздің жобамыздың түпкі мақсаты болып табылады. Жұмыстың бірінші бөлімінде веб-беттерде, pdf құжаттарында кестелерді анықтау, сонымен қатар атрибуттар мен объектілердің мәнін автоматты түрде анықтау әдістері берілген.

Екінші бөлімде ғаламтордың гетерогенді көздерінен кестелерді шығарып алуға арналған, Qurma жүйесінің концептуалды архитектурасы көрсетілген. Нәтижелер бөлімінде кіріс ресурстардың типін талдайтын және басқаруды кесте іздеудің бір модуліне тапсыратын парсердің жұмыс жасау мысалы келтірілген. Ары қарай, басты бағанды анықтау мен осы басты бағанда орналасқан мәндерді сыртқы білім базасындағы сәйкес категорияларымен байланыстыру операциясы орындалады.
Түйін сөздер: веб-кестелер, кестелер шығару, кестелерді тану, кестелерді түсіну, білім базасын толтыру.

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Qurma: конвейер извлечения таблиц для пополнения баз знаний

В данной работе предлагается конвейер, направленный на автоматическое извлечение таблиц из гетерогенных источников Веба, так как HTML-страницы, pdf-файлы и изображения. Извлечение таблиц – одно из активно развивающихся направлений извлечения информации, для которого в настоящее время разрабатывается множество приложений, библиотек и фреймворков. Тем не менее, большинство этих инструментов ориентировано на решение каких-то конкретных задач, например, только на распознавание таблиц, представленных в виде изображений. Мы предлагаем объединить эти задачи в единый конвейер, который будет поддерживать полный цикл обработки таблиц – начиная с этапов их поиска, распознавания и извлечения и заканчивая этапами семантического анализа и интерпретации, то есть пониманием таблиц. Понимание таблиц и пополнение баз знаний (графов знаний) значимой информацией, содержащейся в этих таблицах, является конечной целью нашего проектирования. В первой части работы представлены методы обнаружения таблиц на веб-страницах, в pdf документах, также методы автоматического выявления атрибутов и значений объектов. Во второй части представлена концептуальная архитектура системы Qurma, предназначенной для извлечения таблиц из гетерогенных источников в сети Интернет. В разделе результатов представлен пример работы парсера, который анализирует тип входного ресурса и передает управление одному из модулей поиска таблиц. Далее выполняется операция по определению главного столбца и связыванию сущностей, содержащихся в главном столбце, с соответствующими категориями во внешней базе знаний.

Ключевые слова: веб-таблицы, извлечение таблиц, распознавание таблиц, понимание таблиц, пополнение базы знаний.

1 Introduction

Most of the significant and useful data available on the Internet is published in the form of tables. A person can easily identify, interpret and link the contents of these tables with the information available to him, while the methods of automatic analysis of web tables hardly cope with their task due to the wide variety of table presentation formats. In order to extract useful data from web tables, it is necessary to first correctly determine the boundaries and types of cells containing this data, and then match the identified cells to the corresponding headers. Thus, the process of automatic analysis of web tables is divided into 2 stages: 1) extracting tables, which implies defining the boundaries and structure of the cells of each table; 2) understanding tables, which implies linking the contents of cells with semantic information both inside and outside the tables. As a rule, the understanding of tables in

automatic streaming mode is used for the purpose of forming and filling the knowledge base population in any subject area.

Extracting tables involves two subtasks: 1) detecting a table on a web page or in a document; 2) directly extracting information from the detected table [1]. The subtask of detecting a table on a web page or in a document only looks trivial at first glance. Firstly, it is connected with the problem of classification, since tables are not only meaningful, but also layout tables. Mock-up tables do not contain meaningful information, but are used on a web page or in a document for formatting, for example, to align text or drawings. Secondly, some tables are not highlighted on the page or in the document with TABLE tags, i.e. other signs have to be used to search for them. Thirdly, long tables can be located on different pages of a website or document or hidden using special drop-down elements in order to save space, respectively, connecting individual fragments of the table into a single structure requires additional parsing operations. After detecting and verifying the table, it is necessary to correctly extract data from it for transmission to the next stage - the stage of understanding the table. The correct extraction of information involves such operations as the definition of headers (attribute names), the separation of combined data (when two different attributes are recorded in one cell, for example, address and phone number, or the cell contains list data, for example, several phones for one contact), etc.

In turn, understanding the table for the purposes of knowledge base population includes solving the following subtasks: 1) linking the contents of tables obtained from the Internet with the knowledge base; 2) building hypotheses about the structure and content of tables; 3) extracting new information from tables; 4) adding this new information to the knowledge base [2]. At the same time, a class of tables in which entities are described is of particular interest to the knowledge base population, i.e. tables in which one column, called the main or key, contains the name of the entity, and all their other attributes [3]. Such tables are easier to extract and interpret, so a large number of processing methods have been developed for them, unlike more complex tables expressing n-dimensional relationships, i.e. relationships between several entities.

The authors [3] call this four-step method of extracting data from tables in order to fill the knowledge base with the interpretation of tables. Interpretation concerns the rows and columns of the table, and allows you to determine which entities from the knowledge base are mentioned in the table, what are the types of these entities and what relationships are expressed in the columns. After the interpretation is completed, this information can be used to fill the knowledge base slots. In this paper, we are implementing a pipeline that includes a full cycle of table extraction plus the first stage of understanding tables, namely the identification of entities in the knowledge base.

2 Related works

As noted above, many applications and tools have been developed to extract tables from heterogeneous sources: from the contents of web pages, from PDF documents, from files representing images. In this section we will consider the following applications and tools: Tabula [4], TableSeer [5], TAO [6], TaKCO [7], TableLab [8], TableNet [9], TabbyPDF [10] and Camelot [11].

Among these applications, the oldest application is the TableSeer table search engine [5].

TableSeer scans scientific documents from electronic libraries, finds those that contain tables, then extracts information from each table, saving it to a table metadata file, indexes tables according to metadata, and provides the user with an interface for searching tables. The TableSeer architecture consists of five main components: 1) table crawler; 2) table metadata extractor; 3) table metadata indexer; 4) the TableRank algorithm for ranking tables according to the search query; 5) the interface for supporting search queries to tables. The extraction of tables is based on a statistical analysis of the templates for the design of articles used in the proceedings of the conference or in the journal, based on these templates, a set of heuristic rules is formed that compare different blocks of the document with various logical components (titles, list of authors, abstract, list of references), and physical components (tables, figures, etc.). The TableRank table ranking algorithm deserves special mention, which adapts the traditional model of the vector space <query, document> to the <query, table> pair, replacing the document vectors with table vectors. To determine the weight of each term in the vector space, the authors propose a new weighing scheme: The tabular frequency of the term is the inverted tabular frequency (TTF-ITTF).

Another Table Organization (WTO) table extraction system [6] generates an extended representation of data also extracted from tables in PDF documents. This representation includes the page number on which the table was found, the table number, and metadata for each cell: cell contents, column number, coordinates, font, size, data type, title, or data label. TAO transmits this data as an annotated document in JSON format. Directly to detect tables in a document and extract information about tables, TAO uses the k-nearest neighbors method and heuristic layout rules.

Another application for automatic detection and extraction of tables from PDF files Tabula [4] can both automatically detect tables and allows users to manually select them. The application uses two different algorithms to extract data from selected tables: the first algorithm (Lattice) is based on searching for control rows in the table and identifies cells in the table as separate if they are separated by a line, the second algorithm (Stream) processes text as separate cells if the text fragments are far from each other. The extracted data is output in several formats, including CSV format. Architecturally, Tabula consists of two separate modules: Tabula-Java and Tabula-Ruby. Tabula-Ruby is responsible for the graphical user interface for Tabula-Java, a module that, as the name suggests, is written in Java and is the server part of the application. Although it is intended to be used as a library for Tabula-Ruby, it can also be run separately as a command-line application.

[7] presents a new large-scale TAKCO platform designed to extract facts from tables that can be added to knowledge graphs (KG), such as, for example, WikiData. Takco works with both tables describing entities and tables describing n-dimensional relationships. For entity tables, the system first identifies a pool of candidate entities from the knowledge graph, then calculates an a priori probability distribution by comparing the attributes of the candidate object in the knowledge graph with other cell values in the same row, and then re-weights these matches by the significance of the relationships in the table. Then the system connects entities by constructing a probabilistic graphical model and collectively eliminating the ambiguity of all cells using Loopy Belief Propagation [11]. To interpret n-dimensional tables, the system applies several heuristic rules to transform the table into a "normal"form. Then the schema is compared and functional dependencies are detected to calculate the first elementary interpretation of the table.

Finally, similar tables are grouped using schema and mapping components to improve the quality of interpretation.

PyTabby [10] is another tool for extracting text from PDF tables with a text layer. The system uses a set of customizable special heuristics to detect tables and reconstruct the structure of cells based on the features of the text and lines presented in PDF documents. Most of them, such as horizontal and vertical distances, fonts and rulers, are well known and used in existing methods. Additionally, the system allows you to use the ability to display instructions for printing text in PDF files.

TableNet [8] is a system for extracting information from scanned tables via mobile phones or cameras. The system is based on deep learning models and allows you to accurately detect tabular areas in an image, and then extract information from the rows and columns of the detected table. The TableNet architecture includes neural networks working together to: 1) generate feature maps from low-level text rectangles (in fact, column names); 2) determine the border of the table, if it is in the image; 4) identify rows and identify columns and related canonical data (description, quantity, unit price, etc.).

The TableLab system proposed in [9] provides user interaction with data extraction models, which allows you to quickly train models on several labeled examples. Having received a collection of documents as input, TableLab first detects tables with a similar structure (templates) by clustering embeddings from the extraction model. Document collections often contain tables created using a limited set of templates or similar structures. The system then selects several representative examples of tables already extracted using a pre-trained basic deep learning model. Through an easy-to-use user interface, users provide feedback on these options without necessarily identifying every bug. TableLab then applies such feedback to fine-tune the pre-trained model and returns the fine-tuning results back to the user. The user can choose to repeat this process iteratively until a customized model with satisfactory performance is obtained.

The Camelot library [11] was created to offer users full control over table extraction. Despite the fact that there are both open source systems (for example, Tabula) and closed source systems (for example, PDFTables) that are widely used to extract tables from PDF files, they all have their strengths and weaknesses that do not allow us to talk about their versatility. The Camelot library contrasts versatility with flexibility of configuration due to which it achieves high accuracy and completeness of information extraction. Like Tabula, Camelot uses two methods of syntactic parsing when extracting tables: 1) Stream, which looks for spaces between words to identify the table; 2) Lattice, which looks for lines on the page to identify the table. Another interesting feature of Camelot is that it has a web interface called Excalibur for users who do not want to develop the code themselves, but at the same time want to use the library functions to extract data from tables.

Initially, work with the interpretation of web tables was presented in the work Annotating and Searching Web Tables Using Entities, Types and Relationships. Limaye et al. [12]. In this work, a system is developed that uses a probabilistic graphical model that makes controlled predictions based on a large number of attributes. Subsequent work approached the task-specific knowledge graph problem [13,14] and accelerated predictions by limiting the feature set [15] or using distributed processing [16].

In work [17] presents a semantic analysis for extracting attribute value pairs from product specifications on the Internet. Here are used HTML tables and HTML lists inside web page

as product specification. Supervised learning is used to extract attribute-value pairs from the HTML fragments identified by the specification detector columns as attribute column or value column.

Other successful feature extraction models based on named entity recognition have been developed in [18, 19, 20]. All approaches use similar models to extract attributes. In [18], an approach to annotating product descriptions based on NLP text fragmentation was proposed. Specifically, the authors train a linear chain conditional random field model on a hand-annotated training dataset to identify only eight generic term classes. However, this approach does not allow explicit attribute-value pairs to be extracted. Ristosky and Mika [20] corrected this shortcoming by applying a method with a full set of discrete features derived from the standard distribution of the NER3 mode. Ortona et al. [19] propose a triple approach that performs the following functions: checking the values of sentences, blocking to reduce the number of compared sentences, and evaluation of paired sentences. For verification, an annotator is used that performs NER extraction (places, locations, names, organizations) and an ontology that contains some domain-specific constraints. At the blocking stage, all pairs of products that violate some ontology constraints are grouped into different clusters.

3 Materials and methods

In this paper, we propose our own solution – a system called Qurma, which is based on the Camelot library. The Qurma system receives an input document from the user with its URL and then searches for the tables contained in this document. HTML pages, pdf documents, images can be used as a document. At the output, the system outputs a flat dataset, which is the result of extracting information from tables found in the document. Next, this data set can be exported in any way convenient for the user: to a CSV file, to a json format, or to an attribute-value format. The conceptual architecture of the Qurma system is based on the Clean Architecture concept [21]. The essence of the concept boils down to the fact that it is necessary to clearly understand the needs and limit the software interfaces in order not to lose control of the entire system. To do this, the system is divided into layers, and the interaction between layers is regulated by the boundary rule, which states that only data can be transferred between layers (see Figure 1). Layers are not equal, the main thing in the system is not the platform or technology used, but the layer containing the business logic or business model. Accordingly, two more rules are generated. The inner layer priority rule states that it is the inner layer that determines the interface through which it will interact with the user or the rest of the system. The dependency rule specifies that dependencies should be directed from the inner layer to the outer one.

As follows from the diagram, the core of the architecture are entity models, in this case, these are TableModel and User Model. TableModel is a model in which all data types from different table parsing packages are serialized, which allows you to have a single standard table object and process only this object. Socket Service provides the transfer of commands from the server to the client, while Table Service provides the processing of tables, searching for the main column, determining the types of input cells, etc.

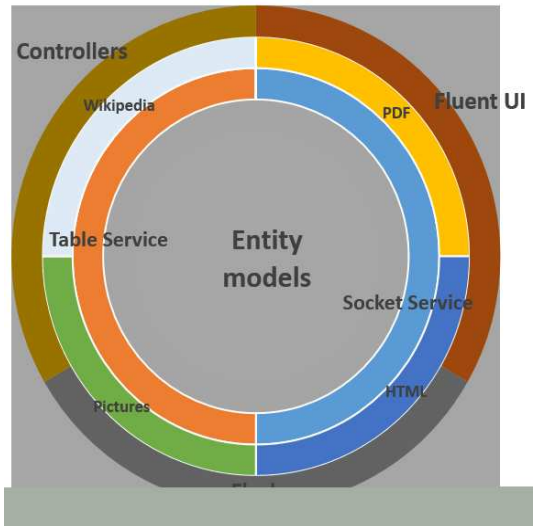


Figure 1: Software "clean" architecture of the Qurma system

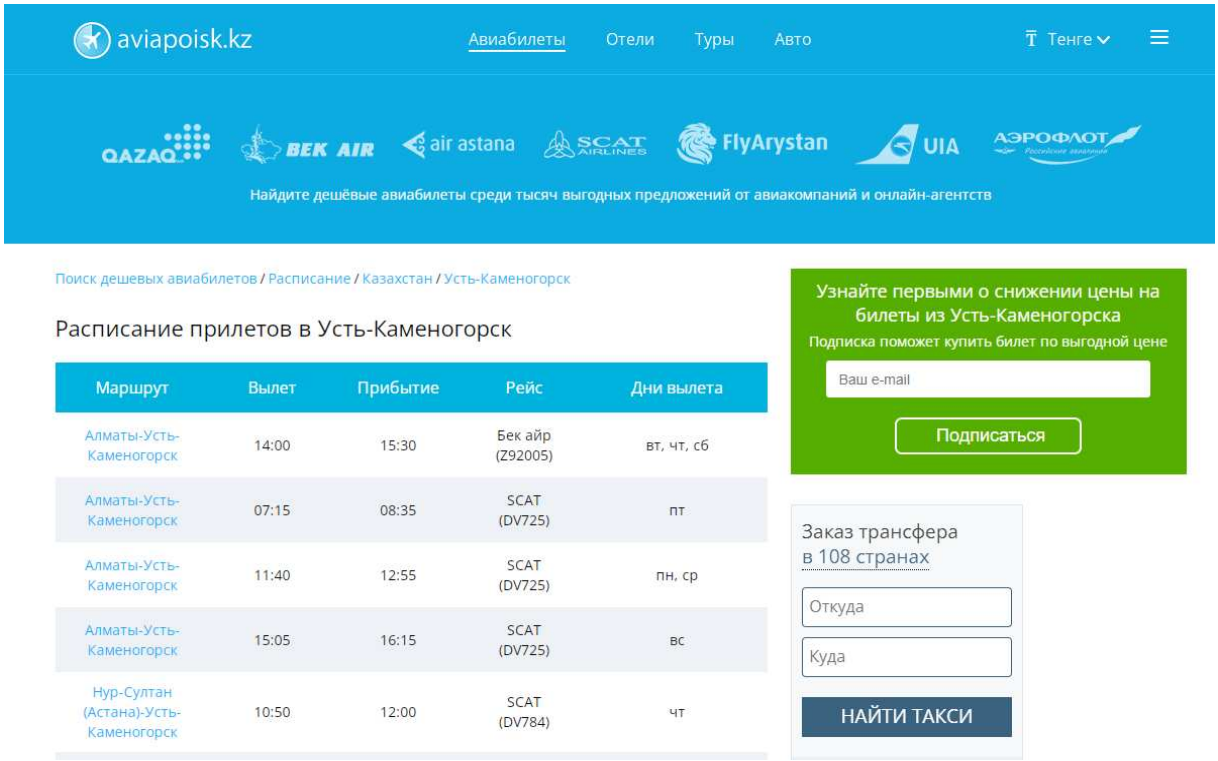
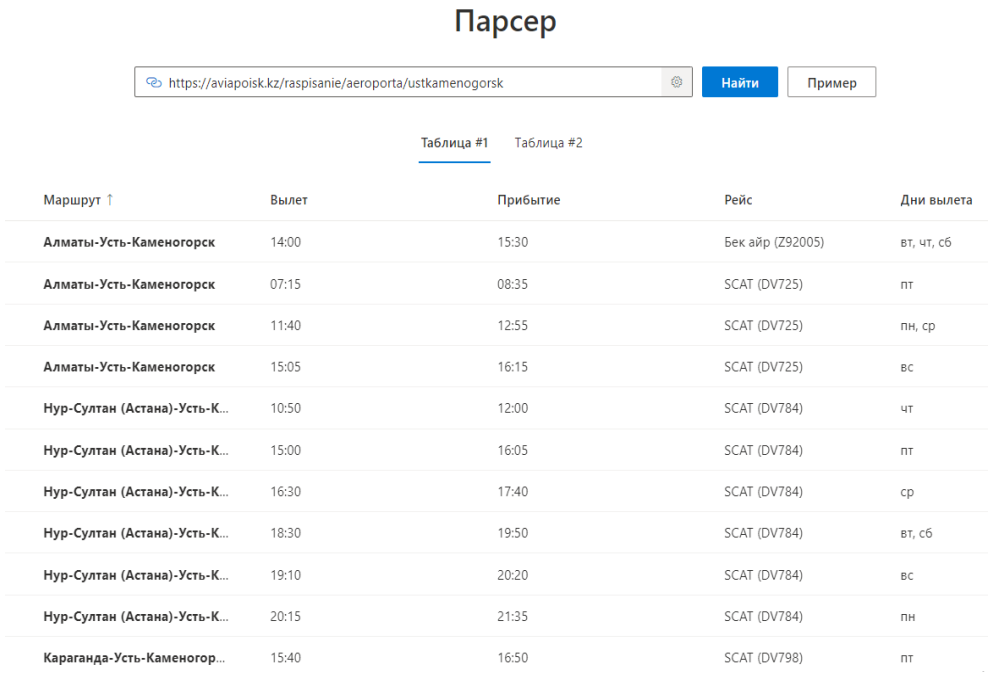


Figure 2: Web page of website aviapoisk.kz with tabular data

4 Results and discussions

The system interface is implemented using a set of Fluent Design elements from Microsoft. The user specifies the URL of the document from which the tables are extracted, for example,



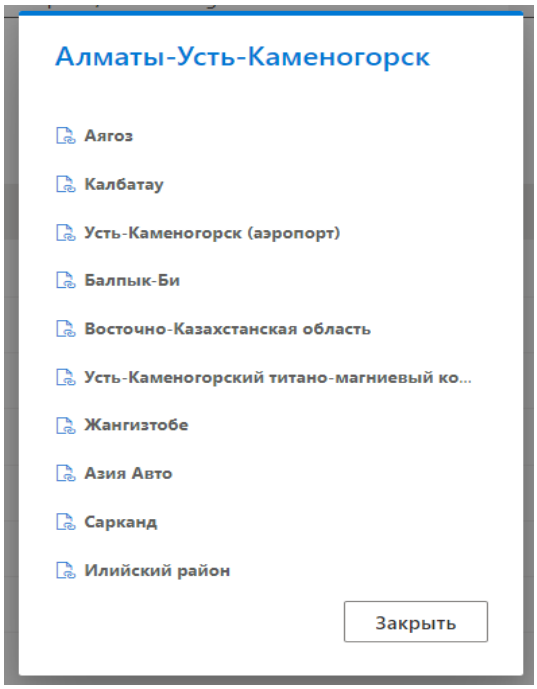
Парсер

https://aviapoisk.kz/raspisanie/aeroporta/ustkamenogorsk Найти Пример

Таблица #1 Таблица #2

Маршрут ↑	Вылет	Прибытие	Рейс	Дни вылета
Алматы-Усть-Каменогорск	14:00	15:30	Бек айр (Z92005)	вт, чт, сб
Алматы-Усть-Каменогорск	07:15	08:35	SCAT (DV725)	пт
Алматы-Усть-Каменогорск	11:40	12:55	SCAT (DV725)	пн, ср
Алматы-Усть-Каменогорск	15:05	16:15	SCAT (DV725)	вс
Нур-Султан (Астана)-Усть-К...	10:50	12:00	SCAT (DV784)	чт
Нур-Султан (Астана)-Усть-К...	15:00	16:05	SCAT (DV784)	пт
Нур-Султан (Астана)-Усть-К...	16:30	17:40	SCAT (DV784)	ср
Нур-Султан (Астана)-Усть-К...	18:30	19:50	SCAT (DV784)	вт, сб
Нур-Султан (Астана)-Усть-К...	19:10	20:20	SCAT (DV784)	вс
Нур-Султан (Астана)-Усть-К...	20:15	21:35	SCAT (DV784)	пн
Караганда-Усть-Каменогор...	15:40	16:50	SCAT (DV798)	пт

Figure 3: The result of tabular parsing in the source document, presented as a web page



Алматы-Усть-Каменогорск

Аягоз

Калбатау

Усть-Каменогорск (аэропорт)

Балпык-Би

Восточно-Казахстанская область

Усть-Каменогорский титано-магниевый ко...

Жангизтобе

Азия Авто

Сарканд

Илийский район

Закрыть

Figure 4: Comparison of the entity in the main column with the categories in the external knowledge base (Wikipedia)

a regular web page address can be used as the URL, as shown in Figure 2 [22]. The parser analyzes the type of input resource and transfers control to one of the three modules in which the table search is already implemented. In this case, control is transferred to the HTML parser, which finds two tables, passes them to the table parser, and the final result is returned as a data set, as shown in Figure 3. In addition to parsing, an operation is performed to determine the main column and associate the entities contained in the main column with the corresponding categories in the external knowledge base (see Figure 4).

5 Conclusion

In this paper, we presented our Qurma system, designed to extract tables from heterogeneous sources on the Internet. The system is a pipeline for searching, extracting and interpreting tables, the ultimate goal of which will be to replenish the knowledge graph on the subject area of interest to the user. Despite the fact that the subject area has not yet been selected and the basic principles of knowledge graph design have not been defined, the presented pipeline already allows solving the problems of semantic analysis of tables contained in web resources. The conceptual architecture of the proposed pipeline, based on the Clean Architecture metaphor, provides a hassle-free increase in the capacity of the designed system. Our future work involves fine-tuning the understanding of web tables using deep learning models. This will allow us to scale the proposed solution using comparatively small sets of training data. Accordingly, further work will be aimed at connecting data annotation modules and precision learning modules to the pipeline.

References

[1] Embley D.W., Tao C., Liddle S.W., "Automating the extraction of data from HTML tables with unknown structure", *Data & Knowledge Engineering,* 54 (1) (2005): 3–28.

[2] Ell B., Hakimov S., Braukmann P., et al., "Towards a Large Corpus of Richly Annotated Web Tables for Knowledge Base Population", *15th International workshop on Linked Data for Information Extraction (LD4IE) at ISWC2017,* Vienna.

[3] Kruit B., Boncz P., Urbani J., "Extracting novel facts from tables for knowledge graph completion", *International Semantic Web conference. Springer. Cham.,* (2019): 364–381.

[4] Rosén G., *Analysis of Tabula: A PDF-Table extraction tool.* (2019).

[5] Liu Y., *TableSeer: automatic table extraction, search, and understanding.* (2009).

[6] Perez-Arriaga M.O., Estrada T., Abad-Mota S., "TAO: system for table detection and extraction from PDF documents", *The Twenty-Ninth International Flairs Conference,* (2016).

[7] Kruit B., Boncz P., Urbani J., "TAKCO: A Platform for Extracting Novel Facts from Tables", *Companion Proceedings of the Web Conference,* (2021): 705–707.

[8] Paliwal S.S. et al., "TableNet: Deep learning model for end-to-end table detection and tabular data extraction from scanned document images", *International Conference on Document Analysis and Recognition (ICDAR). IEEE,* (2019): 128–133.

[9] Wang N.X.R., Burdick D., Li Y., "TableLab: An Interactive Table Extraction System with Adaptive Deep Learning", *26th International Conference on Intelligent User Interfaces,* (2021): 87–89.

[10] Mikhailov A.A., Shigarov A., Kozlov I.S., *PyTabby: a Docreader's module for extracting text and tables from PDF with a text layer.* (2021).

[11] Camelot: PDF Table Extraction for Humans. Camelot 0.8.2 documentation (Jan. 29, 2021). URL: https://camelot-py.readthedocs.io/en/master/ (visited on 10.10.2021).

[12] Limaye G., Sarawagi S., Chakrabarti S., "Annotating and Searching Web Tables Using Entities, Types and Relationships" , *PVLDB,* 3 (1-2) (2010): 1338–1347.

[13] Venetis P., Halevy A., Madhavan J., Paca M., Shen W., Wu F., Miao G., Wu C., "Recovering Semantics of Tables on the Web" , *PVLDB,* 4 (2011): 528–538.

[14] Wang J., Shao B., Wang H., "Understanding Tables on the Web" , *In: ER,* 1 (2010): 141–155

[15] Mulwad V., Finin T., Joshi A., "Semantic Message Passing for Generating Linked Data from Tables" , *In: Proceedings of ISWC,* (2013): 363–378.

[16] Hassanzadeh O., Ward M.J., Rodriguez-Muro M., Srinivas K., "Understanding a Large Corpus of Web Tables Through Matching with Knowledge Bases: an Empirical Study" , *In: Proceedings of OM at ISWC,* (2015): 25–34.

[17] Petar Petrovski, Christian Bizer, "Extracting Atribute-Value Pairs from Product Specifications on theWeb" , *Web Intelligence (WI'17). Leipzig, Germany. ACM,* (2017) 978-1-4503-4951-2/17/08. DOI: 10.1145/3106426.3106449.

[18] Gabor Melli, "Shallow Semantic Parsing of Product Offering Titles (for better automatic hyperlink insertion)" , *In Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining. ACM,* (2014): 1670–1678.

[19] Stefano Ortona, "An analysis of duplicate on web extracted objects" , *In Proceedings of the companion publication of the 23rd international conference on World wide web companion,* (2014): 1279–1284.

[20] Petar Ristoski and Peter Mika, "Enriching Product Ads with Metadata from HTML Annotations" , *In Proceedings of the 13th Extended SemanticWeb Conference (To Appear),* (2016).

[21] Ihler A.T. et al., "Loopy belief propagation: convergence and effects of message errors" , *Journal of Machine Learning Research,* 6 (5) (2005).

[22] https://aviapoisk.kz/raspisanie/aeroporta/ustkamenogorsk.

4-бөлім **Раздел 4** **Section 4**

Қолданбалы математика **Прикладная математика** **Applied Mathematics**

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ERROR ESTIMATES OF THE NUMERICAL METHOD FOR THE FILTRATION PROBLEM WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES

This paper investigates a model of fluid flow in a fractured porous medium under the assumption of a uniform distribution of fractures throughout the volume. This model is based on the use of a fractional differential analogue of Darcy's law, as well as on the assumption that the properties of rock and fluid depend on pressure and its fractional derivative. Unlike previous studies, this study uses a fractional derivative in the Caputo-Fabrizio sense with a non-singular kernel. In this paper, we propose a numerical method for solving this initial boundary value problem and theoretically investigate the order of its convergence. The formulation of a fully discrete scheme is based on application of the finite difference approximation for integer and fractional time derivatives, and the Galerkin method in the spatial variable. A second-order formula is used to approximate both integer derivative and the fractional derivative in the sense of Caputo-Fabrizio. A priori estimates are obtained for both semi-discrete and fully discrete schemes, which imply their second-order convergence in time and space variables. A number of computational experiments were carried out on the example of a model problem to validate the accuracy of the scheme. The results of the numerical tests fully confirm the outcome of the theoretical analysis.
Key words: Finite element method, fractional derivative of Caputo-Fabrizio, convergence, filtration problem, fractured porous medium.

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Капуто-Фабрицио бөлшек туындылы фильтрация есебі үшін сандық әдістің қателігін бағалау

Бұл мақалада жарықшалары көлемі бойынша біртекті таралуы болжамында кеуекті ортада сұйықтықтың қозғалыс үлгісі зерттеледі. Бұл үлгі Дарси заңының бөлшек-дифференциалды баламасын қолдануға, сонымен қатар тау жынысы мен сұйықтықтың қасиеттері қысымнан және оның бөлшек туындысынан тәуелділік болжамына негізделген. Алдыңғы зерттеулерге қарағанда, бұл мақалада сингулярлық емес ядросы бар Капуто-Фабрицио мағынасындағы бөлшек туынды қолданылады. Бұл мақалада осы бастапқы шекаралық есепті шешудің сандық әдісі ұсынылған және оның жинақтылық реті теориялық тұрғыдан зерттелген. Толық дискретті сұлбаның құрылуы уақыт бойынша бүтін және бөлшек туындыларына ақырлы айырымдық жуықтауды, ал кеңістіктік айнымалысы бойынша Галеркин әдісін қолдануға негізделген. Бүтін туынды және Капуто-Фабрицио мағынасындағы бөлшек туындыны жуықтау үшін екінші ретті формула қолданылды. Априорлық бағалау жартылай дискретті және толық дискретті сұлбалар үшін алынды, олардан уақыт және кеңістіктік айнымалылары бойынша екінші ретті жинақтылық шығады. Сұлбаның дәлдігін тексеру үшін үлгі есептің мысалында бірқатар есептеу тәжірибелері жүргізілді. Сандық тәжірибелердің нәтижелері теориялық талдау нәтижелерін толық растайды.

Түйін сөздер: Ақырлы элементтер әдісі, Капуто-Фабрицио бөлшек ретті туындысы, жинақтылық, фильтрация есебі, жарықшалы кеуекті орта.

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Оценки погрешности численного метода для задачи фильтрации с дробными производными Капуто-Фабрицио

В данной статье изучается модель движения жидкости в трещиноватой пористой среде в предположении равномерного распределения трещин по объему. Данная модель основана на использовании дробно-дифференциального аналога закона Дарси и построена в предположении, что свойства породы и жидкости зависят от давления и его дробной производной. В отличие от предыдущих исследований, в настоящей статье используется дробная производная в смысле Капуто-Фабрицио с несингулярным ядром. В статье предлагается численный метод решения данной начально-краевой задачи и теоретически исследуется порядок его сходимости. Формулировка полностью дискретной схемы основана на применении конечно-разностной аппроксимации для целых и дробных производных по времени и метода Галеркина по пространственной переменной. Для аппроксимации целочисленной производной и дробной производной в смысле Капуто-Фабрицио используется формула второго порядка. Получены априорные оценки как для полудискретной, так и для полностью дискретной схем, из которых следует их сходимость со вторым порядком по временной и пространственной переменным. На примере модельной задачи проведен ряд вычислительных экспериментов для проверки точности схемы. Результаты численных тестов полностью подтверждают результаты теоретического анализа.

Ключевые слова: Метод конечных элементов, дробная производная Капуто-Фабрицио, сходимость, задача фильтрации, трещиновато-пористая среда.

1 Introduction

Fractional equations play an important role in modern science due to their extensive applications in natural and technical sciences. Interest in these equations is primarily due to their ability to describe power-law long-term memory and spatial nonlocality of complex environments and processes. Many authors confirm that models containing equations with fractional derivatives more adequately describe a particular physical process. Many studies are devoted to the study of various equations of fractional order.

This paper discusses the initial boundary value problem for the fractional differential equation

$$\frac{\partial u}{\partial t} + \bar{c}_{\phi\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} + \bar{c}_{f\beta}\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}} - \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}\right)\right)_x = \bar{f}_0, \quad t>0, \ \ x\in\Omega \tag{1}$$

in a one-dimensional domain Ω, where $\alpha,\beta \in (-1,0)$, $\gamma \in (0,1)$, $\frac{\partial^{\nu}}{\partial t^{\nu}}$ is the fractional differentiation operator in the sense of the Caputo-Fabrizio definition [1]:

$$\frac{\partial^{\nu}u}{\partial t^{\nu}} = \frac{M(\nu)}{1-\nu}\int_0^t \frac{\partial u}{\partial \theta}\exp\left(-\frac{\nu}{1-\nu}(t-\theta)\right)d\theta, \quad 0<\nu<1, \quad t>0, \tag{2}$$

where $M = M(\nu)$ is a function such that $M(0) = M(1) = 0$. The important application examples of equations of the form (1) include the processes of anomalous diffusion in

heterogeneous media [2–4], the flow of multiphase fluid in fractured porous formations [5,6]. In particular, in [6] an equation of the form (1) was derived to describe the pressure distribution during the flow of a single-phase fluid in a fractured porous medium, provided that the fractures are uniformly distributed over the volume. Unlike other known fluid flow models with fractional derivatives [7–9], the peculiarity of the model under consideration is that the model retains the structure of classical integer order filtration equations when the fractional differentiation order is replaced by an integer order.

Despite the fact that there are many analytical methods [10, 11] for solving problems for fractional differential equations, such equations are difficult to solve using these methods in many cases. Therefore the development of numerical methods based on the features of fractional derivatives and fractional equations is relevant. There are many numerical methods for solving fractional differential equations arising in fluid mechanics, and these methods differ mainly in the approach in which integer and fractional derivatives are discretized. These methods include the finite difference methods [12–14], compact difference scheme [15–17], finite element methods [18–20], finite volume schemes [21], mixed finite element schemes [22] and others. However, it is rather difficult to obtain a high-order approximation in time due to the peculiarities of the fractional derivatives.

In [23, 24], the authors considered the issues of the numerical solution of fractional differential equations, to which the filtration equations are reduced, using the methods of the theory of difference schemes, and they carried out a rigorous theoretical study of the convergence order of the proposed schemes. In the previous work [25], two finite element schemes of the convergence order $O\left(\tau^{2-\nu}\right)$, $\nu = \max\{\alpha, \beta, \gamma\}$, $\alpha, \beta, \gamma \in (0, 1)$ were constructed for solving the initial boundary value problem for an equation of the form (1) with a fractional Caputo derivative. In this paper, we continue this endeavor, but unlike [25], we use a fractional derivative in the sense of Caputo-Fabrizio and assume that its use provides a more realistic description of the fluid flow process and helps to better capture the dynamic behavior of real phenomena as discussed in works [26, 27]. In addition, the use of the Caputo-Fabrizio derivative eliminates the difficulty of a degenerate singular kernel, which makes it difficult to apply approximate methods of its discretization. When constructing numerical methods for solving fractional-order equations, approximation formulas are used. With regard to the derivative in the sense of Caputo-Fabrizio, for example, the $L1$ formula of order $O\left(\tau^2\right)$, the $L1-2$ formula of order $O\left(\tau^3\right)$ [28] are known, where τ is a time step.

The purpose of this paper is to construct and study a finite element method for solving an initial boundary value problem for the equation with a fractional derivative in the sense of Caputo-Fabrizio, describing the pressure distribution during fluid flow through a fractured porous medium with a uniform distribution of fractures over the volume [6]. The paper defines a semi-discrete formulation of the problem with respect to time, obtained using the approximation of the fractional order derivative, and a fully discrete formulation of the problem. Theoretical a priori estimates are obtained for the convergence order of both semi-discrete and fully discrete schemes. Finally, the results of numerical tests are presented to verify the results of theoretical analysis.

2 Materials and methods

2.1 Formulation of the problem

In $Q_T = \overline{\Omega} \times [0,T]$, where $\Omega = (0,1)$, the following initial boundary value problem is considered:

$$\frac{\partial u}{\partial t} + \bar{c}_{\phi\alpha}\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} + \bar{c}_{f\beta}\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}} - \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}\right)\right)_x = \bar{f}_0, \quad 0<t<T, \;\; x\in\Omega, \tag{3}$$

$$u(0,t) = u(1,t) = 0, \quad 0<t<T, \tag{4}$$

$$u(x,0) = u_0(x), \quad x\in\overline{\Omega}, \tag{5}$$

where $\alpha,\beta\in(-1,0)$, $\gamma\in(0,1)$, $\bar{c}_{\phi\alpha}$, $\bar{c}_{f\beta}$, $\bar{f}_0$ are some positive constants, and the fractional differentiation operator $\frac{\partial^{\nu}}{\partial t^{\nu}}$, $0<\nu<1$ is defined in (2). Let us assume that:

(A1) F is a differentiable function defined on Ω such that

$$F(u) = \mu u + \varphi_0(x,t), \tag{6}$$

where φ_0 is a given function, and μ is a positive constant.

(A2) Suppose that the problem (3)-(5) has a unique solution that has sufficient number of derivatives required for conducting the theoretical analysis.

Definition 1 *A weak solution to the problem (3)-(5) is the function $u\in H^1(0,T;H_0^1(\Omega))$, $u(x,0)=u_0(x)$, satisfying the identity*

$$\left(\frac{\partial u}{\partial t},v\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}},v\right) + \bar{c}_{f\beta}\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}},v\right) + \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}\right),v_x\right) = \left(\bar{f}_0,v\right) \tag{7}$$

for any $v\in H_0^1(\Omega)$, where $\alpha,\beta\in(-1,0)$, $\gamma\in(0,1)$.

2.2 Discretization of the problem

First, let us discretize the problem (3)-(5) with respect to the temporal variable. To this end, we divide the time interval $[0,T]$ by points $t_n = n\tau$, $n=0,1,...,N_t$, $N_t\tau = T$ and let u^n denote the semi-discrete approximation of u with respect to the temporal variable. We use the following approximation formula for the Caputo-Fabrizio fractional derivative.

Lemma 1 *The Caputo-Fabrizio fractional derivative $\dfrac{\partial^{\nu}u}{\partial t^{\nu}}$ of order ν, $0<\nu<1$ at $t=t_n$ is approximated by [28]*

$$\left.\frac{\partial^{\nu}u}{\partial t^{\nu}}\right|_{t=t_n} = \Delta^{\nu}u^n + r_n^{\nu}, \tag{8}$$

$$\Delta^{\nu}u^n = \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}\right)$$

where

$$d_{n,s}^{\nu} = \frac{M(\nu)}{\tau\nu}\left(\exp\left(-\sigma_\nu\left(t_n - t_s\right)\right) - \exp\left(-\sigma_\nu\left(t_n - t_{s-1}\right)\right)\right), \quad \sigma_\nu = \frac{\nu}{1-\nu},$$

and the following relation holds for the approximation error r_n^{ν}*:*

$$|r_n^{\nu}| \le \frac{(1-\nu)\,M(\nu)}{2\nu^2}\max_{0\le t\le t_n}\left|\frac{\partial^2 u}{\partial t^2}\right|\tau^2.$$

It is easy to show that the coefficients $d_{n,s}^{\nu}$ satisfy the following properties:
a) $d_{n,s}^{\nu}$ are strictly positive for all $1 \le s \le n$;
b) The sequence $\left\{d_{n,s}^{\nu}\right\}_{s=1}^{n}$ is increasing;
c) $d_{n,n}^{\nu} = O(1)$.
Approximate the first-order derivative at $t = t_n$ in the following form:

$$\frac{\partial u}{\partial t}(t_n) = \begin{cases} \dfrac{1}{2\tau}\left(3u^n - 4u^{n-1} + u^{n-2}\right) + O\left(\tau^2\right), & n \ge 2 \\ \dfrac{1}{\tau}\left(u^1 - u^0\right) + O(\tau), & n = 1. \end{cases}$$

Let us define a semi-discrete formulation of the problem (3)-(5):

Problem 1 *Let* $u^i \in H_0^1(\Omega)$*,* $i = 0, 1, ..., n-1$ *be known,* $u^0 = u_0(x)$*. Find* $u^n \in H_0^1(\Omega)$ *satisfying the identity:*
a) when $n = 1$*:*

$$\frac{1}{\tau}\left(u^1 - u^0, v\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}u^1, v\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1}u^1, v\right) + \left(F\left(\Delta^{\gamma}u_x^1\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{9}$$

b) when $n \ge 2$*:*

$$\frac{1}{2\tau}\left(3u^n - 4u^{n-1} + u^{n-2}, v\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}u^n, v\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1}u^n, v\right) +$$
$$+ \left(F\left(\Delta^{\gamma}u_x^n\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{10}$$

for all $v \in H_0^1(\Omega)$*, where* $\alpha, \beta \in (-1, 0)$*,* $\gamma \in (0, 1)$*.*

To formulate a fully discrete scheme, we define a discrete space $V_h \subset H_0^1$:

$$V_h = \left\{v_h \in H_0^1(\Omega) \cap C^0\left(\overline{\Omega}\right) \;\middle|\; v_h\Big|_e \in P_1(e), \;\; \forall e \in \mathcal{K}_h\right\},$$

where $\mathcal{K}_h$ is a quasi-uniform domain triangulation in Ω.
Define the projection operator $Q_h : H_0^1(\Omega) \to V_h$, satisfying

$$\left(\left(Q_h u - u\right)_x, u_{h,x}\right) = 0 \quad \forall u \in H_0^1(\Omega), \;\; u_h \in V_h.$$

The projection operator has the following properties:

$$\|u - Q_h u\|_0 + h\|u - Q_h u\|_1 \le Ch^2\|u\|_2 \quad \forall u \in H_0^1(\Omega) \cap H^2(\Omega), \tag{11}$$

where $\|\cdot\|_q$ denotes the norm in $H^q(\Omega)$.
Let us define the fully discrete scheme for the problem (3)-(5) as follows.

Problem 2 *Let $u_h^i \in V_h$, $i = 0, 1, ..., n-1$ be given, $u_h^0 = Q_h u_0$. Find $u_h^n \in V_h$ satisfying the following identities:*

a) when $n = 1$:

$$\frac{1}{\tau}\left(u_h^1 - u_h^0, v_h\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1} u_h^1, v_h\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1} u_h^1, v_h\right) + \left(F\left(\Delta^{\gamma} u_{h,x}^1\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right), \quad (12)$$

b) when $n \geq 2$:

$$\frac{1}{2\tau}\left(3u_h^n - 4u_h^{n-1} + u_h^{n-2}, v_h\right) + \bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1} u_h^n, v_h\right) + \bar{c}_{f\beta}\left(\Delta^{\beta+1} u_h^n, v_h\right) +$$

$$+ \left(F\left(\Delta^{\gamma} u_{h,x}^n\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right) \quad (13)$$

for any $v_h \in V_h$, where $\alpha, \beta \in (-1, 0)$, $\gamma \in (0, 1)$.

2.3 Study of convergence of the discrete schemes

Lemma 2 *Let $\{u^i\}_{i=0}^{N_t}$, $u^i \in L^2(\Omega)$ be the sequence of functions. For any $u^n \in L^2(\Omega)$, $n \geq 1$,*

$$\left(\Delta^{\nu} u^n, u^n\right) \geq \Phi_n - \Phi_{n-1} - \frac{1}{2} d_{n,1}^{\nu} \left\|u^0\right\|_0^2, \quad (14)$$

where $\Phi_n = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} d_{n,i}^{\nu} \left\|u^i\right\|_0^2$, $n \geq 1$, $\Phi_0 = 0$.

Proof. First, let us show that

$$\left(\Delta^{\nu} u^n, u^n\right) \geq \frac{1}{2}\Delta^{\nu} \left\|u^n\right\|_0^2. \quad (15)$$

Consider the difference $A = \left(\Delta^{\nu} u^n, u^n\right) - \dfrac{1}{2}\Delta^{\nu} \left\|u^n\right\|_0^2$. Using the definition of the discrete analogue of the Caputo-Fabrizio fractional derivative (8), we obtain the chain of equalities

$$\begin{aligned} A &= \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, u^n\right) - \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, \frac{u^s + u^{s-1}}{2}\right) = \\ &= \sum_{s=1}^{n} d_{n,s}^{\nu}\left(u^s - u^{s-1}, \frac{1}{2}\left(u^s - u^{s-1}\right) + \sum_{k=s+1}^{n}\left(u^k - u^{k-1}\right)\right) = \\ &= \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\nu}\left(\left(u^s - u^{s-1}\right)^2, 1\right) + \sum_{k=2}^{n} d_{n,s}^{\nu}\left(u^k - u^{k-1}, \sum_{s=1}^{k-1}\left(u^s - u^{s-1}\right)\right). \end{aligned} \quad (16)$$

Further, it is easy to show that

$$u^k - u^{k-1} = \frac{\zeta^k - \zeta^{k-1}}{d_{n,k}^{\nu}}, \quad k = 1, 2, ..., n,$$

where $\sum_{s=1}^{k} d_{n,s}^{\nu}\left(u_i^s - u_i^{s-1}\right) = \zeta^k$. Then from (16) we get

$$A = \frac{1}{2d_{n,1}^{\nu}}\left\|\zeta^1\right\|_0^2 + \sum_{k=2}^{n}\frac{1}{2d_{n,k}^{\nu}}\left(\left\|\zeta^k\right\|_0^2 - \left\|\zeta^{k-1}\right\|_0^2\right) =$$
$$= \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{1}{d_{n,k}^{\nu}} - \frac{1}{d_{n,k+1}^{\nu}}\right)\left\|\zeta^k\right\|_0^2 + \frac{1}{2d_{n,n}^{\nu}}\left\|\zeta^n\right\|_0^2 \geq 0,$$

whence the validity of the inequality (15) follows.

Let us now prove the inequality (14). Transform the right-hand side of (15) using the definition of a discrete analog of the derivative:

$$\frac{1}{2}\Delta^{\nu}\left\|u^n\right\|_0^2 = \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\nu}\left\|u^s\right\|_0^2 - \frac{1}{2}\sum_{s=1}^{n} d_{n-1,s-1}^{\nu}\left\|u^{s-1}\right\|_0^2 =$$
$$= \Phi_n - \Phi_{n-1} - \frac{1}{2}d_{n,1}^{\nu}\left\|u^0\right\|_0^2.$$

The lemma is proved.

Let us turn to the study of the question of the convergence of the method. Below we sometimes use the notation $u(t) = u(\cdot, t)$.

Theorem 1 *Under the assumptions (A1)-(A2) the solution u^n of Problem 1 converges to the solution of the problem (3)-(5) and the following inequality holds:*

$$\left\|u(t_n) - u^n\right\|_0 + \tau\sqrt{2c_0T^{-1}}\left\|u(t_n) - u^n\right\|_1 \leq C\tau^2,$$

where $c_0 = \min\left\{\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1}, \bar{c}_{f\beta}d_{n,1}^{\beta+1}, \mu d_{n,1}^{\gamma}\right\}$.

Proof. Denote $w^n = u(t_n) - u^n$. Consider the difference of identity (7) at $t = t_n$ and identities (9), (10) and choose $v = w^n$:

a) when $n = 1$:

$$\left(\frac{\partial u}{\partial t}(t_1) - \frac{u^1 - u^0}{\tau}, w^n\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_1) - \Delta^{\alpha+1}u^1, w^n\right) +$$
$$+\bar{c}_{f\beta}\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_1) - \Delta^{\beta+1}u^1, w^n\right) + \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}(t_1)\right) - F\left(\Delta^{\gamma}u_x^1\right), w_x^n\right) = 0; \tag{17}$$

b) when $n \geq 2$:

$$\left(\frac{\partial u}{\partial t}(t_n) - \frac{3u^n - 4u^{n-1} + u^{n-2}}{2\tau}, w^n\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}}(t_n) - \Delta^{\alpha+1}u^n, w^n\right) +$$
$$+\bar{c}_{f\beta}\left(\frac{\partial^{\beta+1}u}{\partial t^{\beta+1}}(t_n) - \Delta^{\beta+1}u^n, w^n\right) + \left(F\left(\frac{\partial^{\gamma}u_x}{\partial t^{\gamma}}(t_n)\right) - F\left(\Delta^{\gamma}u_x^n\right), w_x^n\right) = 0. \tag{18}$$

Let us estimate the terms in (17) and (18):

$$\left(\frac{\partial u}{\partial t}(t_1) - \frac{u^1 - u^0}{\tau}, w^n\right) \geq \frac{1}{2\tau}\left\|w^1\right\|_0^2 - \frac{1}{2\tau}\left\|w^0\right\|_0^2 + \frac{\tau}{2}\left(\frac{\partial^2 u}{\partial t^2}(\zeta_1), w^1\right),$$

$$\left(\frac{\partial u}{\partial t}(t_n) - \frac{3u^n - 4u^{n-1} + u^{n-2}}{2\tau}, w^n\right) \geq$$

$$\geq \frac{1}{4\tau}\left(\|w^n\|_0^2 + \left\|2w^n - w^{n-1}\right\|_0^2 + \left\|w^n - 2w^{n-1} + w^{n-2}\right\|_0^2\right) -$$

$$-\frac{1}{4\tau}\left(\left\|w^{n-1}\right\|_0^2 + \left\|2w^{n-1} - w^{n-2}\right\|_0^2\right) + \frac{\tau^2}{3}\left(\frac{\partial^3 u}{\partial t^3}(\zeta_n), w^n\right),$$

$$\left(\frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}}(t_n) - \Delta^{\alpha+1} u^n, w^n\right) \geq \left(r_n^{\alpha+1}, w^n\right) + \left(\Phi_n^{\alpha+1} - \Phi_{n-1}^{\alpha+1}\right) - \frac{1}{2} d_{n,1}^{\alpha+1}\left\|w^0\right\|_0^2,$$

$$\left(\frac{\partial^{\beta+1} u}{\partial t^{\beta+1}}(t_n) - \Delta^{\beta+1} u^n, w^n\right) \geq \left(r_n^{\beta+1}, w^n\right) + \left(\Phi_n^{\beta+1} - \Phi_{n-1}^{\beta+1}\right) - \frac{1}{2} d_{n,1}^{\beta+1}\left\|w^0\right\|_0^2,$$

$$\left(F\left(\frac{\partial^\gamma u_x}{\partial t^\gamma}(t_n)\right) - F\left(\Delta^\gamma u_x^n\right), w_x^n\right) \geq \mu\left(r_n^\gamma, w_x^n\right) + \mu\left(\Phi_n^\gamma - \Phi_{n-1}^\gamma\right) - \frac{1}{2}\mu d_{n,1}^\gamma\left\|w_x^0\right\|_0^2,$$

where

$$\Phi_n^{\alpha+1} = \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\alpha+1}\|w^s\|_0^2, \qquad r_n^{\alpha+1} = \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}}(t_n) - \Delta^{\alpha+1} u(t_n),$$

$$\Phi_n^{\beta+1} = \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\beta+1}\|w^s\|_0^2, \qquad r_n^{\beta+1} = \frac{\partial^{\beta+1} u}{\partial t^{\beta+1}}(t_n) - \Delta^{\beta+1} u(t_n),$$

$$\Phi_n^{\gamma+1} = \frac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\gamma}\|w_x^s\|_0^2, \qquad r_n^{\gamma+1} = \frac{\partial^\gamma u_x}{\partial t^\gamma}(t_n) - \Delta^\gamma u_x(t_n),$$

and $\Phi_0^r = 0$. Taking into account the obtained estimates in (17) and (18), we arrive at the following inequalities:

$$\left\|w^1\right\|_0^2 + 2\tau\Phi_1 \leq \left\|w^0\right\|_0^2 + 2\tau\Phi_0 + 2\tau^2\left|\left(\frac{\partial^2 u}{\partial t^2}(\zeta_1), w^1\right)\right| +$$

$$+2\tau\bar{c}_{\phi\alpha}\left|\left(r_1^{\alpha+1}, w^1\right)\right| + 2\tau\bar{c}_{f\beta}\left|\left(r_1^{\beta+1}, w^1\right)\right| + 2\tau\mu\left|\left(r_1^\gamma, w_x^1\right)\right|, \tag{19}$$

$$\|w^n\|_0^2 + \left\|2w^n - w^{n-1}\right\|_0^2 + 4\tau\Phi_n + \left\|w^n - 2w^{n-1} + w^{n-2}\right\|_0^2 \leq$$

$$\leq \left\|w^{n-1}\right\|_0^2 + \left\|2w^{n-1} - w^{n-2}\right\|_0^2 + 4\tau\Phi_{n-1} + \frac{4\tau^3}{3}\left|\left(\frac{\partial^3 u}{\partial t^3}(\zeta_n), w^n\right)\right| + 4\tau\bar{c}_{\phi\alpha}\left|\left(r_n^{\alpha+1}, w^n\right)\right| +$$

$$+4\tau\bar{c}_{f\beta}\left|\left(r_n^{\beta+1}, w^n\right)\right| + 4\tau\mu\left|\left(r_n^\gamma, w_x^n\right)\right|, \tag{20}$$

where the notation

$$\Phi_n = \bar{c}_{\phi\alpha}\Phi_n^{\alpha+1} + \bar{c}_{f\beta}\Phi_n^{\beta+1} + \mu\Phi_n^{\gamma+1}$$

is used. By estimating the last four terms on the right-hand side of (20), applying the Cauchy inequality, we obtain

$$\|w^n\|_0^2 + \left\|2w^n - w^{n-1}\right\|_0^2 + 4\tau\Phi_n \le \left\|w^{n-1}\right\|_0^2 + \left\|2w^{n-1} - w^{n-2}\right\|_0^2 + 4\tau\Phi_{n-1} +$$
$$+\frac{4\tau^3}{3}\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_n)\right\|_0 \|w^n\|_0 + 4\tau\bar{c}_{\phi\alpha}\left\|r_n^{\alpha+1}\right\|_0 \|w^n\|_0 +$$
$$+4\tau\bar{c}_{f\beta}\left\|r_n^{\beta+1}\right\|_0 \|w^n\|_0 + 4\tau\mu\|r_n^{\gamma}\|_0 \|w_x^n\|_0 . \tag{21}$$

Sum the inequality (21) for n from 2 to n to get

$$\|w^n\|_0^2 + 4\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 +$$
$$+\frac{C}{\varepsilon_1}\left(\tau^{5/2}\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_n)\right\|_0 + \tau\left\|r_n^{\alpha+1}\right\|_0 + \tau\left\|r_n^{\beta+1}\right\|_0\right)^2 + \varepsilon_1\tau\|w^n\|_0^2 + \frac{C}{\varepsilon_2}\tau\|r_n^{\gamma}\|_0^2 + \varepsilon_2\tau\|w_x^n\|_0^2 +$$
$$+\frac{C}{\varepsilon_3}\sum_{i=2}^{n-1}\left(\tau^2\left\|\frac{\partial^3 u}{\partial t^3}(\zeta_i)\right\|_0 + \bar{c}_{\phi\alpha}\left\|r_i^{\alpha+1}\right\|_0 + \bar{c}_{f\beta}\left\|r_i^{\beta+1}\right\|_0\right)^2 +$$
$$+\varepsilon_3\tau^2\sum_{i=2}^{n-1}\left\|w^i\right\|_0^2 + \frac{C}{\varepsilon_4}\sum_{i=2}^{n}\|r_i^{\gamma}\|_0^2 + \varepsilon_4\tau^2\sum_{i=2}^{n}\left\|w_x^i\right\|_0^2$$

or

$$\|w^n\|_0^2 + 4\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 +$$
$$+2\tau\left(\left(\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1} + \bar{c}_{f\beta}d_{n,1}^{\beta+1}\right)\|w^n\|_0^2 + \mu d_{n,1}^{\gamma}\|w_x^n\|_0^2\right) +$$
$$+\tau^2\sum_{i=2}^{n-1}\left(\left(\bar{c}_{\phi\alpha}d_{i,1}^{\alpha+1} + \bar{c}_{f\beta}d_{i,1}^{\beta+1}\right)\left\|w^i\right\|_0^2 + \mu d_{i,1}^{\gamma}\left\|w_x^i\right\|_0^2\right) + C\tau^4.$$

Considering that $\left(\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1} + \bar{c}_{f\beta}d_{n,1}^{\beta+1}\right)\|w^n\|_0^2 + \mu d_{n,1}^{\gamma}\|w_x^n\|_0^2 \le \Phi_n$, it follows that

$$\|w^n\|_0^2 + 2\tau\Phi_n \le 5\left\|w^1\right\|_0^2 + 4\tau\Phi_1 + \tau^2\sum_{i=2}^{n-1}\Phi_i + C\tau^4.$$

Applying the discrete Gronwall's lemma, we obtain

$$\|w^n\|_0^2 + 2\tau\Phi_n \le C\left(\left\|w^1\right\|_0^2 + \tau\Phi_1 + \tau^4\right). \tag{22}$$

Let us now evaluate terms in (19):

$$\left\|w^1\right\|_0^2 + 2\tau\Phi_1 \le \left\|w^0\right\|_0^2 + 2\tau\Phi_0 + 2\tau^2\left\|\frac{\partial^2 u}{\partial t^2}(\zeta_1)\right\|_0 \left\|w^1\right\|_0 + 2\tau\bar{c}_{\phi\alpha}\left\|r_1^{\alpha+1}\right\|_0 \left\|w^1\right\|_0 +$$
$$+ 2\tau\bar{c}_{f\beta}\left\|r_1^{\beta+1}\right\|_0 \left\|w^1\right\|_0 + 2\mu\|r_1^{\gamma}\|_0 \cdot \tau\left\|w_x^1\right\|,$$

or

$$\left\|w^1\right\|_0^2 + 4\tau\Phi_1 \leq \frac{\tau^2}{2}\mu d_{1,1}^{\gamma}\left\|w_x^1\right\|_0^2 + C\tau^4.$$

Noticing that $\frac{1}{2}\mu d_{1,1}^{\gamma}\tau\left\|w_x^1\right\|_0^2 \leq \Phi_1$, we get

$$\left\|w^1\right\|_0^2 + 3\tau\Phi_1 \leq C\tau^4. \tag{23}$$

By substituting (23) into (22), and applying elementary transformations, we arrive at the assertion of the theorem.

Theorem 2 *Under the assumptions (A1)-(A2) there exists $\tau_0 > 0$ such that for all $\tau \leq \tau_0$ the solution u_h^n of Problem 2 converges to the solution of Problem (3)-(5) and the following inequality holds:*

$$\left\|u\left(t_n\right) - u_h^n\right\|_0 + 2\tau\sqrt{c_0 T^{-1}}\left\|u\left(t_n\right) - u_h^n\right\|_1 \leq C\left(\tau^2 + h^2\right),$$

where $c_0 = \min\left\{\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1}, \bar{c}_{f\beta}d_{n,1}^{\beta+1}, \mu d_{n,1}^{\gamma}\right\}$.

Proof. Let $u^n - u_h^n = (u^n - Q_h u^n) + (Q_h u^n - u_h^n) = \vartheta^n + \eta^n$.
Consider the difference of (10) and (13) and choose $v_h = \eta^n$:

$$\begin{aligned}
&\left\|\eta^n\right\|_0^2 + \left\|2\eta^n - \eta^{n-1}\right\|_0^2 - \left\|\eta^{n-1}\right\|_0^2 - \left\|2\eta^{n-1} - \eta^{n-2}\right\|_0^2 + \left\|\eta^n - 2\eta^{n-1} + \eta^{n-2}\right\|_0^2 + \\
&+4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) + 4\tau\bar{c}_{f\beta}\left(\Delta^{\beta+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) + \\
&+4\tau\left(F\left(\Delta^{\gamma}u_x^n\right), \eta_x^n\right) - 4\tau\left(F\left(\Delta^{\gamma}u_{h,x}^n\right), \eta_x^n\right) + 4\tau\left(\frac{3\vartheta^n - 4\vartheta^{n-1} + \vartheta^{n-2}}{2\tau}, \eta^n\right) = 0.
\end{aligned} \tag{24}$$

Consider the term

$$4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\left(\vartheta^n + \eta^n\right), \eta^n\right) = 4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\eta^n, \eta^n\right) + 4\tau\bar{c}_{\phi\alpha}\left(\Delta^{\alpha+1}\vartheta^n, \eta^n\right) = K_1 + K_2.$$

Using Lemma 2, we get:

$$K_1 \geq 4\tau\bar{c}_{\phi\alpha}\left(\Phi_n^{\alpha+1} - \Phi_{n-1}^{\alpha+1}\right) - 2\tau\bar{c}_{\phi\alpha}d_{n,1}^{\alpha+1}\left\|\eta^0\right\|_0^2,$$

$$\begin{aligned}
K_2 &\leq 4\tau\bar{c}_{\phi\alpha}\left\|\Delta^{\alpha+1}\vartheta^n\right\|_0\left\|\eta^n\right\|_0 \leq \\
&\leq 4\tau\bar{c}_{\phi\alpha}^2\left\|\sum_{s=1}^{n} d_{n,s}^{\alpha+1}\left(\vartheta^s - \vartheta^{s-1}\right)\right\|_0^2 + 2\tau\left\|\eta^n\right\|_0^2 = \\
&= 4\tau\bar{c}_{\phi\alpha}^2\int_{\Omega}\left(\sum_{s=1}^{n} d_{n,s}^{\alpha+1}\int_{t_{s-1}}^{t_s}\vartheta_t d\theta\right)^2 dx + 2\tau\left\|\eta^n\right\|_0^2 \leq \\
&\leq 4T\tau\left(\bar{c}_{\phi\alpha}d_{n,n}^{\alpha+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2 d\theta + 2\tau\left\|\eta^n\right\|_0^2,
\end{aligned}$$

where $\Phi_n^{\alpha+1} = \dfrac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\alpha+1} \left\|\eta^s\right\|_0^2$. Similarly,

$$4\tau\bar{c}_{f\beta}\left(\Delta^{\beta+1}\left(\vartheta^n+\eta^n\right),\eta^n\right) \geq 4\tau\bar{c}_{f\beta}\left(\Phi_n^{\beta+1}-\Phi_{n-1}^{\beta+1}\right) - 2\tau\bar{c}_{f\beta}d_{n,1}^{\beta+1}\left\|\eta^0\right\|_0^2 -$$

$$-4T\tau^2\left(\bar{c}_{f\beta}d_{n,n}^{\beta+1}\right)^2\int_0^T \left\|\vartheta_t\right\|_0^2 d\theta - 2\tau\left\|\eta^n\right\|_0^2,$$

where $\Phi_n^{\beta+1} = \dfrac{1}{2}\sum_{s=1}^{n} d_{n,s}^{\beta+1} \left\|\eta^s\right\|_0^2$. Estimate the remaining terms as follows:

$$4\tau\left(F\left(\Delta^\gamma u_x^n\right),\eta_x^n\right) - 4\tau\left(F\left(\Delta^\gamma u_{h,x}^n\right),\eta_x^n\right) = 4\tau\mu\left(\Delta^\gamma\left(\vartheta_x^n+\eta_x^n\right),\eta_x^n\right) =$$
$$= 4\tau\mu\left(\Phi_n^\gamma-\Phi_{n-1}^\gamma\right) - 2\tau\mu d_{n,1}^\gamma\left\|\eta_x^0\right\|_0^2,$$

$$K_6 \equiv 4\tau\left(\frac{3\vartheta^n-4\vartheta^{n-1}+\vartheta^{n-2}}{2\tau},\eta^n\right) \leq$$
$$\leq 2\tau\left\|\frac{\int_{t_{n-1}}^{t_n}\vartheta_t d\theta - \int_{t_{n-2}}^{t_{n-1}}\vartheta_t d\theta}{2\tau}\right\|_0^2 + 2\tau\left\|\eta^n\right\|_0^2 =$$
$$= \frac{1}{2}\left(\int_{t_{n-1}}^{t_n}\int_\Omega \vartheta_t^2 dx d\theta + \int_{t_{n-2}}^{t_{n-1}}\int_\Omega \vartheta_t^2 dx d\theta\right) + 2\tau\left\|\eta^n\right\|_0^2 =$$
$$= \frac{1}{2}\int_{t_{n-2}}^{t_n}\left\|\vartheta_t\right\|_0^2 d\theta + 2\tau\left\|\eta^n\right\|_0^2,$$

where $\Phi_n^\gamma = \dfrac{1}{2}\sum_{s=1}^{n} d_{n,s}^\gamma \left\|\eta_x^s\right\|_0^2$. Then it follows from (24) that

$$\left\|\eta^n\right\|_0^2 + \left\|2\eta^n-\eta^{n-1}\right\|_0^2 + \left\|\eta^n-2\eta^{n-1}+\eta^{n-2}\right\|_0^2 + 4\tau\Phi_n \leq$$

$$\leq \left\|\eta^{n-1}\right\|_0^2 + \left\|2\eta^{n-1}-\eta^{n-2}\right\|_0^2 + 4\tau\Phi_{n-1} + 4T\tau\left(\bar{c}_{\phi\alpha}d_{n,n}^{\alpha+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2 d\theta +$$

$$+4T\tau^2\left(\bar{c}_{f\beta}d_{n,n}^{\beta+1}\right)^2\int_0^T\left\|\vartheta_t\right\|_0^2 d\theta + 6\tau\left\|\eta^n\right\|_0^2 + \frac{1}{2}\int_{t_{n-2}}^{t_n}\left\|\vartheta_t\right\|_0^2 d\theta, \tag{25}$$

where

$$\Phi_n = \bar{c}_{\phi\alpha}\Phi_n^{\alpha+1} + \bar{c}_{f\beta}\Phi_n^{\beta+1} + \mu\Phi_n^\gamma.$$

Sum the inequality (25) for n from 2 to n to obtain

$$\left\|\eta^n\right\|_0^2 + 4\tau\Phi_n \leq 5\left\|\eta^1\right\|_0^2 + 4\tau\Phi_1 + 6\tau\left\|\eta^n\right\|_0^2 + 6\tau\sum_{i=2}^{n-1}\left\|\eta^i\right\|_0^2 + Ch^4,$$

whence, for sufficiently small τ, we obtain

$$\|\eta^n\|_0^2 + 4\tau\Phi_n \le 5\left\|\eta^1\right\|_0^2 + 4\tau\Phi_1 + 6\tau\sum_{i=2}^{n-1}\left\|\eta^i\right\|_0^2 + Ch^4.$$

By applying the discrete Gronwall's lemma, we get

$$\|\eta^n\|_0^2 + 4\tau\Phi_n \le C\left(\left\|\eta^1\right\|_0^2 + \tau\Phi_1 + h^4\right). \tag{26}$$

Considering the difference of (9) and (12), choosing $v_h = \eta^1$ and using a similar technique for estimating the terms in the resulting identity, we arrive at

$$\left\|\eta^1\right\|_0^2 + \tau\Phi_1 \le C\tau^2\left\|\vartheta^1\right\|_0^2 + \frac{1}{2}\left\|\eta^1\right\|_0^2 + \frac{3\tau^2}{2}\left\|\frac{\vartheta^1 - \vartheta^0}{\tau}\right\|_0^2,$$

therefore,

$$\frac{1}{2}\left\|\eta^1\right\|_0^2 + \tau\Phi_1 \le C\left(\tau^4 + h^4\right). \tag{27}$$

Combining (26) and (27), we obtain

$$\|\eta^n\|_0^2 + 4c_0\tau\sum_{s=1}^{n}\|\eta^s\|_1^2 \le C\left(\tau^4 + h^4\right),$$

whence the assertion of the theorem follows.

3 Results

To check the accuracy of the scheme, computational experiments were carried out using a model problem as an example.

Example 1 *In $Q_1 = \overline{\Omega} \times [0,1]$, where $\Omega = (0,1)$, consider the problem*

$$\frac{\partial u}{\partial t} + \frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} + \frac{\partial^{\beta+1} u}{\partial t^{\beta+1}} - \frac{\partial^{\gamma} u_{xx}}{\partial t^{\gamma}} = \bar{f}_0, \quad 0 < t < 1, \quad x \in \Omega, \tag{28}$$

$$\bar{f}_0(x,t) = -\frac{2}{\gamma}\cdot\left(\exp\left(t\left(\gamma-1\right)/\left(\gamma-2\right)\right) - \exp\left(t/2\right)\right) +$$

$$+x\left(x-1\right)\left[\frac{\exp\left(\alpha t/\left(\alpha-1\right)\right) - \exp\left(t/2\right)}{\alpha+1} + \frac{\exp\left(\beta t/\left(\beta-1\right)\right) - \exp\left(t/2\right)}{\beta+1} - \frac{\exp\left(t/2\right)}{2}\right]$$

$$u(x,0) = x(1-x), \quad x \in \overline{\Omega}, \tag{29}$$

$$u(0,t) = u(1,t) = 0, \quad 0 < t < 1, \tag{30}$$

where $\alpha, \beta \in (-1,0)$, $\gamma \in (0,1)$.

The exact solution to the problem is $u(x,t) = x(1-x)\exp(t/2)$.

The first series of computational experiments was carried out to compare the convergence order of the scheme with respect to the time step with a fixed value of the spatial step, $h = 1/20000$. For this, the time step value was gradually halved from 1/10 to 1/640, and the convergence order was evaluated as $(\ln(R_{2\tau}/R_\tau))/\ln 2$, where R_τ is the L^2-error of the approximate solution calculated with the use of the time step τ. Tables 1-3 outline the results of the analysis for different values of the fractional derivative orders, $\alpha = \beta \in \{-0.9, -0.5, -0.1\}$ and $\gamma \in \{0.1, 0.5, 0.9\}$. It can be clearly seen from the presented values that the convergence order does not depend on the fractional derivative orders for all considered cases, and its value approaches 2. This behavior agrees well with the theoretically predicted order with respect to the time step obtained in Theorem 2.

Similarly, the second series of computational experiments was conducted in order to compare the convergence order with respect to the spatial step with a fixed temporal step, $\tau = 1/20000$. The corresponding L^2-errors and convergence orders are presented in Tables 4-6. As it follows from numerical experiments, the actual convergence order for all considered cases is close to 2. Hence, the results obtained fully confirm the theoretically predicted order obtained in Theorem 2.

Table 1: Error analysis with respect to the temporal step, $\gamma = 0.1$

τ	$\alpha=\beta=-0.9$		$\alpha=\beta=-0.5$		$\alpha=\beta=-0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.5578×10^{-4}	-	1.9224×10^{-4}	-	3.2447×10^{-3}	-
1/20	3.7507×10^{-5}	2.05	4.4459×10^{-5}	2.11	7.6783×10^{-4}	2.08
1/40	9.1597×10^{-6}	2.03	1.0491×10^{-5}	2.08	1.8135×10^{-4}	2.08
1/80	2.2580×10^{-6}	2.02	2.5154×10^{-6}	2.06	4.2878×10^{-5}	2.08
1/160	5.5977×10^{-7}	2.01	6.1017×10^{-7}	2.04	1.0196×10^{-5}	2.07
1/320	1.3920×10^{-7}	2.01	1.4951×10^{-7}	2.03	2.4359×10^{-6}	2.07
1/640	3.4467×10^{-8}	2.01	3.7320×10^{-8}	2.00	5.9093×10^{-7}	2.04

Table 2: Error analysis with respect to the temporal step, $\gamma = 0.5$

τ	$\alpha=\beta=-0.9$		$\alpha=\beta=-0.5$		$\alpha=\beta=-0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	3.1783×10^{-4}	-	3.4405×10^{-4}	-	3.2313×10^{-3}	-
1/20	6.7777×10^{-5}	2.23	7.2527×10^{-5}	2.25	7.6007×10^{-4}	2.09
1/40	1.4763×10^{-5}	2.20	1.5600×10^{-5}	2.22	1.7861×10^{-4}	2.09
1/80	3.2883×10^{-6}	2.17	3.4378×10^{-6}	2.18	4.1985×10^{-5}	2.09
1/160	7.4652×10^{-7}	2.14	7.7628×10^{-7}	2.15	9.8740×10^{-6}	2.09
1/320	1.7842×10^{-7}	2.06	1.7728×10^{-7}	2.13	2.3550×10^{-6}	2.07
1/640	4.2906×10^{-8}	2.06	4.1230×10^{-8}	2.10	5.7194×10^{-7}	2.04

Table 3: Error analysis with respect to the temporal step, $\gamma = 0.9$

τ	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.0618×10^{-3}	-	1.0576×10^{-3}	-	1.0644×10^{-3}	-
1/20	2.5121×10^{-4}	2.08	2.4874×10^{-4}	2.09	2.5491×10^{-4}	2.06
1/40	5.9316×10^{-5}	2.08	5.8445×10^{-5}	2.09	6.1094×10^{-5}	2.06
1/80	1.4018×10^{-5}	2.08	1.3736×10^{-5}	2.09	1.4812×10^{-5}	2.04
1/160	3.3405×10^{-6}	2.07	3.2921×10^{-6}	2.06	3.6240×10^{-6}	2.03
1/320	7.9946×10^{-7}	2.06	7.8824×10^{-7}	2.06	8.9512×10^{-7}	2.02
1/640	1.9589×10^{-7}	2.03	1.9291×10^{-7}	2.03	2.2394×10^{-7}	2.00

Table 4: Error analysis with respect to the spatial step, $\gamma = 0.1$

h	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	1.0955×10^{-6}	-	1.2410×10^{-6}	-	1.3720×10^{-6}	-
1/20	2.7388×10^{-7}	2.00	3.1026×10^{-7}	2.00	3.4538×10^{-7}	1.99
1/40	6.8469×10^{-8}	2.00	7.7565×10^{-8}	2.00	8.6946×10^{-8}	1.99
1/80	1.7117×10^{-8}	2.00	1.9257×10^{-8}	2.01	2.1737×10^{-8}	2.00
1/160	4.2498×10^{-9}	2.01	4.7810×10^{-9}	2.01	5.4341×10^{-9}	2.00
1/320	1.0551×10^{-9}	2.01	1.1788×10^{-9}	2.02	1.3491×10^{-9}	2.01
1/640	2.6014×10^{-10}	2.02	2.9064×10^{-10}	2.02	3.3496×10^{-10}	2.01

Table 5: Error analysis with respect to the spatial step, $\gamma = 0.5$

h	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	9.3002×10^{-7}	-	1.0559×10^{-6}	-	1.1687×10^{-6}	-
1/20	2.3251×10^{-7}	2.00	2.6581×10^{-7}	1.99	2.9421×10^{-7}	1.99
1/40	$5,8126 \times 10^{-8}$	2.00	6.6453×10^{-8}	2.00	7.4065×10^{-8}	1.99
1/80	1.4532×10^{-8}	2.00	1.6499×10^{-8}	2.01	1.8516×10^{-8}	2.00
1/160	3.6078×10^{-9}	2.01	4.0961×10^{-9}	2.01	4.6291×10^{-9}	2.00
1/320	8.9572×10^{-10}	2.01	1.0099×10^{-9}	2.02	1.1493×10^{-9}	2.01
1/640	2.2085×10^{-10}	2.02	2.4901×10^{-10}	2.02	2.8533×10^{-10}	2.01

Table 6: Error analysis with respect to the spatial step, $\gamma = 0.9$

h	$\alpha = \beta = -0.9$		$\alpha = \beta = -0.5$		$\alpha = \beta = -0.1$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	7.7391×10^{-7}	-	8.8972×10^{-7}	-	1.0233×10^{-6}	-
1/20	1.9482×10^{-7}	1.99	2.2243×10^{-7}	2.00	2.5761×10^{-7}	1.99
1/40	4.8706×10^{-8}	2.00	5.5607×10^{-8}	2.00	6.4849×10^{-8}	1.99
1/80	1.2176×10^{-8}	2.00	1.3806×10^{-8}	2.01	1.6212×10^{-8}	2.00
1/160	3.0231×10^{-9}	2.01	3.4276×10^{-9}	2.01	4.0531×10^{-9}	2.00
1/320	7.5055×10^{-10}	2.01	8.4510×10^{-10}	2.02	1.0063×10^{-9}	2.01
1/640	1.8505×10^{-10}	2.02	2.0837×10^{-10}	2.02	2.4983×10^{-10}	2.01

4 Conclusion

Thus, the constructed numerical method allows obtaining an approximate solution to the problem of fluid flow in a fractured porous medium with the second order in both time and spatial variable. The results of computational experiments carried out for various orders of fractional derivatives and grid configurations fully confirm the results of theoretical analysis. The methods used and the conclusions drawn, described in the work, can be used to solve other classes of fractional differential equations.

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References

[1] Caputo M. and Fabrizio M., "A new definition of fractional derivative without singular kernel", *Progress in Fractional Differentiation and Applications,* 2 (2015): 73–85.

[2] Chechkin A. V., Gorenflo R. and Sokolov I. M., "Fractional diffusion in inhomogeneous media", *Journal of Physics A: Mathematical And General,* 38 (2005): 679–984.

[3] Esen A., Ucar Y., Yagmurlu M. and Tasbozan O., "Solving fractional diffusion and fractional diffusionwave equations by Petrov Galerkin finite element method", *Turkish World Mathematical Society Journal of Applied and Engineering Mathematics,* 4, no. 2 (2014): 155–168.

[4] Zhu A., Wang Y. and Xu Q., "A weak Galerkin finite element approximation of two-dimensional subdiffusion equation with time fractional derivative", *AIMS Mathematics,* 5, no. 5 (2020): 4297–4310.

[5] Abiola O. D., Enamul H. M., Kaseem M. and Sidqi A. A., "A modified memory based mathematical model describing fluid flow in porous media", *Computers and Mathematics with Applications,* 73, no. 6 (2017): 1385–1402.

[6] Gazizov R. K. and Lukashchuk S. Yu., "Drobno-differentsialnyi podkhod k modelirovaniyu protsessov filtratsii v slozhnykh neodnorodnykh poristykh sredakh [Fractional differential approach to modeling filtration processes in complex inhomogeneous porous media]", *Vestnik UGATU,* 21, no. 4 (2017): 104–112 (in Russian).

[7] Caffarelli L. and Vazquez J. L., "Nonlinear porous medium flow with fractional potential pressure", *Archive for Rational Mechanics and Analysis,* 202, no. 2 (2011): 537–565.

[8] Gazizov R. K., Kasatkin A. A. and Lukashchuk S. Yu., "Symmetries and exact solutions of fractional filtration equations" , *AIP Conference Proceedings,* 1907, no. 020010 (2017): 1–9.

[9] Meilanov R. R., Akhmedov E. N. and Beybalaev V. D., "To the theory of nonlocal nonisothermal filtration in porous medium" , *IOP Conf. Series: Journal of Physics: Conf. Series,* 946, no. 012076 (2018): 1–8.

[10] El Amin M. F., Radwan A. G. and Sun S., "Analytical solution for fractional derivative gasflow equation in porous media" , *Results in Physics,* 7 (2017): 2432–2438.

[11] Ray S. S., "Exact solutions for time fractional diffusion wave equations by decomposition method" , *Phys. Scr.,* 75 (2007): 53–61.

[12] Zhang Y. N., Sun Z. Z. and Liao H. L., "Finite difference methods for the time fractional diffusion equation on nonuniform meshes" , *Journal of Computational Physics,* 265 (2014): 195–210.

[13] Qiao H. L., Liu Z. G. and Cheng A. J., "Two unconditionally stable difference schemes for time distributed order differential equation based on Caputo–Fabrizio fractional derivative" , *Advances in Difference Equations,* 2020 (2020): 1–17.

[14] Alikhanov A. A., "A new difference scheme for the time fractional diffusion equation" , *Journal of Computational Physics,* 208 (2015): 424–438.

[15] Du R., Cao W. R. and Sun Z. Z., "A compact difference scheme for the fractional diffusion wave equation" , *Applied Mathematical Modelling,* 34 (2010): 2998–3007.

[16] Xu T., Lu S., Chen W. and Chen H., "Finite difference scheme for multiterm variable order fractional diffusion equation" , *Advances in Difference Equations,* 1 (2018): 1-13.

[17] Huang J., Tang Y., Wang W. and Yang J. "A compact difference scheme for time fractional diffusion equation with Neumann boundary conditions" , *Communications in Computer and Information Science,* 323 (2012): 273–284.

[18] Zhang C., Liu H. and Zhou Z. J., "A priori error analysis for timestepping discontinuous Galerkin finite element approximation of time fractional optimal control problem" , *Journal of Scientific Computing,* 80, no. 2 (2019): 993–1018.

[19] Liu J. and Zhou Z., "Finite element approximation of time fractional optimal control problem with integral state constraint" , *AIMS Mathematics,* 6, no. 1 (2020): 979–997.

[20] Liu K., Feckan M., O'Regan D. and Wang J. R., "Hyers–Ulam stability and existence of solutions for differential equations with Caputo–Fabrizio fractional derivative" , *Mathematics,* 333, no. 7 (2019): 1-14.

[21] Coronel Escamilla A., Gomez Aguilar J.F., Torres L. and Escobar Jimenez R.F., "A numerical solution for a variable order reaction diffusion model by usingfractional derivatives with nonlocal and nonsingular kernel" , *Physica A: Statistical Mechanics and its Applications,* 491 (2017): 406–424.

[22] Liu Y., Du Y. and Li H., "A twogrid mixed finite element method for a nonlinear fourthorder reaction diffusion problem with time fractional derivative" , *Computers and Mathematics with Application,* 70 (2015): 2474–2492.

[23] Alimbekova N. B., Baigereyev D. R. and Madiyarov M. N., "Study of a numerical method for solving a boundary value problem for a differential equation with a fractional time derivative" , *Izvestiya of Altai State University,* 114, no. 4 (2020): 64–69.

[24] Alimbekova N. B., Berdyshev A. S. and Baigereyev D. R., "Study of initial boundary value problem for two dimensional differential equation with fractional time derivative in the sense of Caputo" , *Third International Conference on Material Science, Smart Structures and Applications: (ICMSS 2020)* (2021): 1–6.

[25] Baigereyev D., Alimbekova N., Berdyshev A. and Madiyarov M., "Convergence analysis of a numerical method for a fractional model of fluid flow in fractured porous media" , *Mathematics,* 9, no. 2179 (2021): 1-24.

[26] Caputo M., "Models of flux in porous media with memory" , *Water Resources Research,* 36, no. 3 (2000): 693– 705.

[27] Agarwal R., Yadav M. P., Baleanu D. and Purohit S. D., "Existence and uniqueness of miscible flow equation through porous media with a non singular fractional derivative" , *AIMS Mathematics,* 5, no. 2 (2020): 1062–1073.

[28] Akman T., Yildiz B. and Baleanu D., "New discretization of Caputo-Fabrizio derivative" , *Computational and Applied Mathematics,* 37, no. 3 (2017): 3307-3333.

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ANALYSIS OF A FINITE VOLUME ELEMENT SCHEME FOR SOLVING THE MODEL TWO-PHASE NONEQUILIBRIUM FLOW PROBLEM

The paper proposes a hybrid numerical method for solving a model problem of two-phase nonequilibrium flow of an incompressible fluid in a porous medium. This problem is relevant in the modern theory of the motion of multiphase fluids in porous media and has many applications. The studied model is based on the assumption that the relative phase permeabilities and capillary pressure depend not only on saturation, but also on its time derivative. The saturation equation in this problem refers to the type of convection-diffusion with a predominance of convection, which also includes a third-order term to account for the nonequilibrium effects. Due to the hyperbolic nature of the equation, its solution is accompanied by a number of difficulties that lead to the need for an appropriate choice of the solution method. In contrast to previous works, this paper uses a finite volume element method for solving the problem, the construction of which is based on integral balance equations, and an approximate solution is chosen from the finite element space. To discretize the problem, two different dual grids are used based on the main triangulation. In this paper, a number of a priori estimates are obtained which yields the unconditional stability of the scheme as well as its convergence with the second order. The advantages of the approach used include the local conservatism of the scheme, as well as the comparative simplicity of the software implementation of the method. These results are confirmed by a numerical test carried out on the example of a model problem.

Key words: Finite volume element method, nonequilibrium fluid flow, dynamic capillary pressure, dual mesh, a priori estimate, convergence, stability, computational experiment.

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Екі фазалы теңөлшемсіз фильтрацияның модельді есебін шешудің ақырлы көлемді элементті сұлбасын талдау

Бұл жұмыста екі фазалы сығылмайтын сұйықтықтың теңөлшемсіз фильтрациясының модельдік есебін шешудің гибридті сандық әдісі ұсынылған. Бұл есеп көпфазалы сұйықтықтардың кеуекті ортадағы қозғалысының заманауи теориясында өзекті болып табылады және көптеген қолданбаларға ие. Зерттелетін модель салыстырмалы фазалық өткізгіштіктер мен капиллярлық қысымның қанықтықтан ғана емес, сонымен қатар оның уақыт бойынша туындыларынан да тәуелді деген болжамға негізделген. Бұл есептегі қанықтық теңдеуі конвекциясы басым болатын конвекция-диффузия түріне жатады, сонымен қатар оның құрамына теңөлшемсіздік әсерлерін ескеретін үшінші ретті қосылғыш кіреді. Теңдеудің гиперболалық сипатына байланысты оның шешімі бірқатар қиындықтарға ие болады, сондықтан оны шешу әдісін лайықты таңдау қажет етіледі. Алдыңғы жұмыстарға қарағанда бұл жұмыста есепті шешудің ақырлы көлемді-элементтік әдісі қолданылады. Бұл әдіс интегралдық баланс теңдеулері негізінде құрастырылған, ал жуық шешім ақырлы элементтер кеңістігінен таңдалады. Бұл жағдайда есепті дискретизациялау үшін негізгі триангуляция негізінде екі түрлі қосарланған тор қолданылады. Бұл жұмыста бірқатар априорлық бағалаулар алынған, олардан сұлбаның шартсыз орнықтылығы, сондай-ақ екінші ретпен жинақталуы шығады. Қолданылатын тәсілдің артықшылығына сұлбаның локальды консервативтілігі, сонымен қатар әдісті бағдарламалық жүзеге асырудың салыстырмалы қарапайымдылығы жатады. Бұл нәтижелер модельдік есеп мысалында жүргізілген сандық тәжірибемен расталады.

Түйін сөздер: Ақырлы көлемді-элементті әдіс, теңөлшемсіз фильтрация, динамикалық капиллярлық қысым, қосарланған тор, априорлық бағалау, жинақтылық, орнықтылық, есептеу тәжірибесі.

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Анализ конечно-объемно-элементной схемы решения модельной задачи двухфазной неравновесной фильтрации

В работе предлагается гибридный численный метод решения модельной задачи двухфазной неравновесной фильтрации несжимаемой жидкости. Данная задача является актуальной в современной теории движения многофазных жидкостей в пористых средах и имеет множество приложений. Изучаемая модель основана на предположении, что относительные фазовые проницаемости и капиллярное давление зависят не только от насыщенности, но также от ее временной производной. Уравнение для насыщенности в данной задаче относится к типу конвекции-диффузии с преобладанием конвекции, которое также содержит слагаемое третьего порядка для учета эффектов неравновесности. В силу гиперболического характера уравнения его решение сопровождается рядом трудностей, которые приводят к необходимости надлежащего выбора метода решения. В отличие от предыдущих работ, в данной работе применяется конечно-объемно-элементный метод решения задачи, построение которого основывается на интегральных уравнениях баланса, а приближенное решение выбирается из конечно-элементного пространства. При этом для дискретизации задачи используются две различные двойственные сетки на базе основной триангуляции. В работе получен ряд априорных оценок, из которых следует безусловная устойчивость схемы, а также ее сходимость со вторым порядком. К числу преимуществ используемого подхода можно отнести локальную консервативность схемы, а также сравнительную простоту программной реализации метода. Данные результаты подтверждаются численным тестом, проведенным на примере модельной задачи.

Ключевые слова: Конечно-объемно-элементный метод, неравновесная фильтрация, динамическое капиллярное давление, двойственная сетка, априорная оценка, сходимость, устойчивость, вычислительный эксперимент.

1 Introduction

Modeling the flow of a multiphase fluid in porous media is of great economic importance in the petroleum engineering, hydrology, carbon sequestration, and nuclear waste management [1–3]. These models form the basis of fluid dynamics simulators used in the development of oil fields, allowing predictive calculations of development indicators. Most simulators contain descriptions of the so-called classical fluid flow models in porous media that do not take into account a number of important factors. One of these factors is the phenomenon of a delay in the establishment of saturations, which is observed in microheterogeneous fractured rocks.

There are several approaches to modeling nonequilibrium effects. The first approach [4] is based on thermodynamic arguments and volume averaging of the microscopic equations of conservation of mass and momentum, as a result of which the authors of [4] came to the conclusion that it is necessary to add additional terms to the macroscopic equations. [4] introduced the concept of dynamic capillary pressure, i.e. instantaneous local difference between phase pressures. Dynamic capillary pressure has been the subject of many experimental [5] and theoretical [6, 7] studies.

The second approach [8] is based on the assumption that relative phase permeabilities and the capillary pressure are considered as functions not only of saturation, but also of the derivative of the saturation with respect to time $\frac{\partial s_w}{\partial t}$. Thus, a characteristic feature of nonequilibrium flows, i.e. the dependence on the rate of the process is taken into account.

Many works [9–12] are devoted to the numerical implementation of the two-phase fluid flow model with the nonequilibrium law from [4]. For example, in [9], a second-order numerical scheme for both spatial and temporal variables is proposed using a mixed finite element method with the lowest order Thomas-Raviar elements and an implicit Euler scheme. To show the convergence of the scheme, the error estimates for saturation, fluxes and phase pressures are obtained in $L^\infty(0,T;L^2(\Omega))$ norm for temporal and spatial triangulation. The authors of [10] present an a posteriori error estimate for (piecewise linear) approximation of finite elements, which corresponds to some linear Sobolev equations using the implicit Euler scheme.

Also, a class of quasiparabolic equations is considered in [11]. Such equations simulate the two-phase flow in porous media, where dynamic effects are included in capillary pressure. The existence and uniqueness of the weak solution were proved, and the error estimates for the implicit Euler time discretization were obtained.

The paper [12] analyzes the convergence of a "two-point flow" finite volume scheme to approximate the flow of two incompressible phases with dynamic capillary pressure in porous media. In that work, a fully implicit scheme is based on a non-standard approximation of mobility and capillary pressure on a double grid is proposed. A discrete variational formulation was derived and a new result of convergence in a two-dimensional and three-dimensional porous medium was presented. Compared to static capillary pressure, the nonequilibrium capillary model requires more powerful methods, especially not standard discrete energy estimates.

This work is devoted to the construction of a numerical method for solving the problem of two-phase fluid flow with the inequality law proposed in [4]. In contrast to the above works, we use the finite volume element method (FVEM). The essence of this method consists in constructing dual grids based on the main grid and generating control volumes on the dual grid. Compared to finite difference and finite element methods, the finite volume element method is simple to implement and provides flexibility in handling complex geometric domains, as well as automatically provides local mass conservation. The last property is most important in problems of fluid flow in porous media.

The FVEM has been successfully applied to problems of non-stationary equations of an incompressible fluid in the Boussinesq approximation [13], for problems of fluid flow of an incompressible fluid [14], for a non-stationary equation of convection-diffusion [15] and many others. In the cited works, the accuracy theoretical estimates of the proposed schemes are obtained, a comparative analysis of the estimates obtained with the results of numerical tests is carried out, and the implementation advantages of the method are shown.

In the present paper, two dual grids are constructed on the base of the main triangulation. The first dual grid is used for the velocity and pressure equations, and the second one is used for the saturation equation. Theoretical estimates are obtained which show the stability of the scheme, as well as the convergence of the scheme with the second order.

2 Materials and Methods

2.1 Statement of the Problem

In $Q_T = \Omega \times [0,T]$, where $\Omega \subset \mathbb{R}^2$ is a convex bounded domain with a Lipschitz-continuous boundary Γ, $T > 0$, the following model problem of two-phase nonequilibrium flow is considered under the assumption of incompressibility of phases and the absence of gravitational forces:

$$\nabla \cdot \vec{u} = 0, \quad (x,t) \in Q_T, \tag{1}$$

$$(k\lambda(s))^{-1} \vec{u} + \nabla p = 0, \quad (x,t) \in Q_T, \tag{2}$$

$$\phi s_t + f_w \vec{u} \cdot \nabla s - \nabla \cdot (\gamma \nabla s) - \nabla \cdot (\gamma_1 \nabla (L s_t)) = 0, \quad (x,t) \in Q_T, \tag{3}$$

$$s(x,0) = s_0(x), \quad x \in \Omega, \tag{4}$$

$$\vec{u} \cdot \vec{n} = 0, \quad \nabla s \cdot \vec{n} = 0, \quad (x,t) \in \Gamma \times (0,T], \tag{5}$$

where $\vec{u}$ is the total velocity vector, p is pressure, $s = s(x,t)$ is the water saturation, ϕ is porosity, k is the absolute permeability, L is the replacement time; f_w, γ, γ_1 are some positive constants; $\lambda(s) = \lambda_w(s) + \lambda_o(s)$, $\lambda_\alpha(s) = k_\alpha(s)\mu_\alpha^{-1}$, $k_\alpha(s)$ and μ_α are relative permeability and viscosity of the phase α; $\vec{n}$ is the outer unit normal to the boundary Γ.

Introduce the following functional spaces:

$$U = \{\vec{v} \in H(\mathrm{div};\Omega) \ : \ \vec{v} \cdot \vec{n} = 0 \text{ on } \Gamma\}, \quad M = L^2(\Omega)/\mathbb{R}, \quad W = H_0^1(\Omega).$$

The mixed variational formulation of Problem (1)-(5) is as follows: find $(\vec{u}, p, s) \in U \times M \times H^1(0,T;W)$ such that the following identities hold for all $\vec{v} \in U$, $w \in M$, $\varphi \in W$, $t \in (0,T)$

$$(\nabla \cdot \vec{u}, w) = 0, \tag{6}$$

$$\left((K\lambda(s))^{-1} \vec{u}, \vec{v}\right) - (p, \nabla \cdot \vec{v}) = 0, \tag{7}$$

$$(s_t, \varphi) + a(\vec{u}, s, \varphi) + d(s, \varphi) + d_1(s_t, \varphi) = 0, \tag{8}$$

$$s(x,0) = s_0(x), \quad x \in \Omega, \tag{9}$$

where

$$a(\vec{u}, \eta, \varphi) = \frac{f_w}{2} \int_\Omega \vec{u} \cdot (\varphi \nabla \eta - \eta \nabla \varphi)\, dx,$$

$$d(s, \varphi) = \int_\Omega \gamma \nabla s \cdot \nabla \varphi\, dx, \quad d_1(s, \varphi) = \int_\Omega \gamma_1 \nabla (Ls) \cdot \nabla \varphi\, dx$$

for all $(\vec{u}, \eta, \varphi) \in U \times W \times W$. It is known that [16]

$$a(\vec{u}, \eta, \varphi) = -a(\vec{u}, \varphi, \eta), \quad a(\vec{u}, \varphi, \varphi) = 0. \tag{10}$$

2.2 The Finite Volume Element Method

Let us first discretize Problem (1)-(5) with respect to time. Let $\{t_n\}_{n=0}^N$ be a uniform partitioning introduced in the time interval $[0,T]$. Further, let $(\vec{u}^n, p^n, s^n)$ denote the semi-discrete approximation of $(\vec{u}, p, s)$ at $t = t_n$.

Introduce the notations $\Delta_t s^{n-1/2} = \dfrac{s^n - s^{n-1}}{\tau}$, $s^{n-1/2} = \dfrac{s^n + s^{n-1}}{2}$.

The semi-discrete formulation of Problem (6)-(9) reads: find $(\vec{u}^n, p^n, s^n) \in U \times M \times W$, $n = 1, 2, ..., N$ such that for all $\vec{v} \in U$, $w \in M$, $\varphi \in W$:

$$(\nabla \cdot \vec{u}^n,\, w) = 0, \tag{11}$$

$$\left((k\lambda)^{-1}\, \vec{u}^n, \vec{v}\right) - (p^n, \nabla \cdot \vec{v}) = 0, \tag{12}$$

$$(s^n, \varphi) + \tau a\left(\vec{u}^{n-1/2}, s^{n-1/2}, \varphi\right) + \tau d\left(s^{n-1/2}, \varphi\right) + d_1\left(s^n - s^{n-1}, \varphi\right) = \left(s^{n-1}, \varphi\right), \tag{13}$$

$$s^0 = s_0(x), \quad x \in \Omega. \tag{14}$$

To solve Problem (1)-(5), we use the finite volume element method. In $\overline{\Omega}$ introduce a quasi-uniform triangulation $\mathfrak{T}_h$ and let h be its diameter. Let us construct two dual partitions on the basis of the basic partition $\mathfrak{T}_h$.

Let $\{V_i\}_{i=1}^{N_h}$, $\{E_i\}_{i=1}^{N_e}$ and $\{M_i\}_{i=1}^{N_m}$ denote the sets of vertices, edges and midpoints of the triangles in $\mathfrak{T}_h$, respectively. Consider two adjacent triangles $T_i \in \mathfrak{T}_h$ and $T_j \in \mathfrak{T}_h$ and let E_k be their common edge, and M_k be the midpoint of E_k. We form a quadriterial Q_k^* by connecting the barycenters of T_i and T_j with the ends of E_k. In the case when T_i is a boundary element and E_k is its edge lying on the domain boundary, Γ, we form a triangle T_l^* by connecting the barycenter T_i with the ends of the edge E_k. The set of internal elements Q_k^* and boundary triangles T_l^* is called the dual partition for the pressure and velocity equation and is denoted by $\mathfrak{T}_h^*$.

To construct the second dual partition, $\mathfrak{V}_h^*$, we connect the barycenter C_i of the triangle $T_i \in \mathfrak{T}_h$ with the midpoints of its edges by straight lines. This leads to the partitioning of T_i into three quadrilaterals. By combining them, we obtain a control volume $T_{V_i}^*$ which surround the vertex V_i. A set of control volumes cover Ω, and is called the dual partition of Ω of the barycentric type corresponding to the triangulation $\mathfrak{T}_h$.

Let us define the functional spaces U_h, W_h and M_h of trial functions as

$$U_h = \left\{\vec{v}_h \in U \cap \left(C\left(\overline{\Omega}\right)\right)^2 : \quad \vec{v}_h|_K \in (P_1(K))^2 \quad \forall K \in \mathfrak{T}_h\right\},$$

$$M_h = \left\{w_h \in W \cap C\left(\overline{\Omega}\right) : \quad w_h|_K \in P_0(K) \quad \forall K \in \mathfrak{T}_h\right\},$$

$$W_h = \left\{q_h \in M : \quad q_h|_K \in P_1(K) \quad \forall K \in \mathfrak{T}_h\right\},$$

where $P_l(K)$ is the space of polynomial functions of degree not greater than l on K.

Define the spaces of test functions $\tilde{U}_h$ and $\tilde{W}_h$ in the following form:

$$\tilde{U}_h = \left\{\vec{v}_h \in \left(L^2(\Omega)\right)^2 : \quad \vec{v}_h|_V \in (P_0(V))^2, \quad \vec{v}_h \cdot \vec{n}|_V = 0 \quad \forall V \in \mathfrak{T}_h^*\right\},$$

$$\tilde{W}_h = \left\{w_h \in L^2(\Omega) : \quad w_h|_V \in P_0(V), \quad w_h|_V = 0 \quad \forall V \in \mathfrak{V}_h^*\right\}. \tag{15}$$

Let $\Pi_h^*\vec{u}$ and $\rho_h^* w$ be the interpolation projections of $\vec{u} \in U$ and $w \in W$ into the spaces of trial functions $\tilde{U}_h$ and $\tilde{W}_h$ defined as

$$\Pi_h^*\vec{u}_h(x) = \sum_{i=1}^{N_m} \vec{u}_h(M_i)\,\zeta_i(x), \quad \rho_h^* w_h(x) = \sum_{i=1}^{N_h} w_h(P_i)\,\chi_i(x)$$

for all $x \in \Omega$, where ζ_i is the characteristic function of Q_i^* and $\chi_i(x)$ is the characteristic function of $T_{V_i}^*$.

Now we define a fully discrete scheme corresponding to Problem (1)-(5): find $(\vec{u}_h^n, p_h^n, s_h^n) \in U_h \times M_h \times W_h$ $(1 \le n \le N)$, such that the following identities hold for all $\vec{v}_h \in \tilde{U}_h$, $w_h \in \tilde{M}_h$:

$$\left((k\lambda)^{-1}\vec{u}_h^n, \Pi_h^*\vec{v}_h\right) + l_h\left(\Pi_h^*\vec{v}_h, p_h^n\right) = 0, \tag{16}$$

$$(\nabla \cdot \vec{u}_h^n,\ w_h) = 0, \tag{17}$$

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^*\varphi_h\right) + a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\varphi_h\right) + d_h\left(s_h^{n-1/2}, \rho_h^*\varphi_h\right) +$$
$$+ d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^*\varphi_h\right) = 0, \tag{18}$$

$$s_h^0 = \rho_h\varphi(x), \tag{19}$$

where

$$a_h\left(\vec{u}_h^n, s_h^n, \rho_h^*\varphi_h\right) = \sum_{V_z \in \mathfrak{V}_h^*} f_w\varphi_h(z) \int_{\partial V_z} s_h^n \vec{u}_h^n \cdot \vec{n}\, ds,$$

$$d_h\left(s_h^n, \rho_h^*\varphi_h\right) = \sum_{V_z \in \mathfrak{V}_h^*} \varphi_h(z) \int_{\partial V_z} \gamma \nabla s_h^n \cdot \vec{n}\, ds,$$

$$d_{1h}\left(s_h^n, \rho_h^*\varphi_h\right) = L \sum_{V_z \in \mathfrak{V}_h^*} \varphi_h(z) \int_{\partial V_z} \gamma_1 \nabla s_h^n \cdot \vec{n}\, ds,$$

$$l_h\left(\Pi_h^*\vec{v}_h, p_h^n\right) = -\sum_{V_z \in \mathfrak{T}_h^*} \vec{v}_h(z) \int_{\partial V_z} p_h^n \cdot \vec{n}\, ds.$$

Let us formulate the following lemmas from [13, 14] without proof.

Lemma 1 *The following results hold [13]:*

$$d_h\left(s_h, \rho_h^*\varphi_h\right) = d\left(s_h, \varphi_h\right), \quad a_h\left(\vec{u}_h, s_h, \rho_h^*\varphi_h\right) = a\left(\vec{u}_h, s_h, \varphi_h\right),$$

$$d_{1h}\left(s_h, \rho_h^*\varphi_h\right) = d_1\left(s_h, \varphi_h\right),$$

$$a_h\left(\vec{u}_h, s_h, \rho_h^* s_h\right) = 0, \quad \forall s_h, \varphi_h \in W_h, \quad \forall \vec{u}_h \in U_h.$$

Moreover, $d_h\left(s_h, \rho_h^* w_h\right)$ is a symmetric, bounded and positive definite form, i.e.

$$d_h\left(s_h, \rho_h^* w_h\right) = d_h\left(w_h, \rho_h^* s_h\right), \quad \forall s_h, w_h \in W_h$$

and there are constants h_0, C_0, such that for $0 < h \le h_0$,

$$d_h\left(s_h, \rho_h^* s_h\right) \ge \gamma_0 |s_h|_1^2, \quad \left|d_h\left(s_h, \rho_h^* w_h\right)\right| \le C_0 \|s_h\|_1 \|w_h\|_1, \quad \forall s_h, w_h \in W_h.$$

Lemma 2 *The following result is valid [13]:*

$$(\vec{u}_h, \Pi_h^* \vec{v}_h) = (\vec{v}_h, \Pi_h^* \vec{u}_h), \quad \forall \vec{u}_h, v_h \in U_h.$$

For every $\vec{u} \in H^m(\Omega)^2$, $m = 0, 1$ *and* $\vec{v}_h \in U_h$,

$$|(\vec{u}, \vec{v}_h) - (\vec{u}, \Pi_h^* \vec{v}_h)| \leq Ch^{m+n} \|\vec{u}\|_m \|\vec{v}_h\|_n, \quad n = 0, 1. \tag{20}$$

Lemma 3 *The operator* Π_h^* *satisfies the following inequalities for all* $\vec{v}_h \in U_h$ *and* $w_h \in M_h$ *provided* $\nabla \cdot \vec{v}_h = 0$ *[14]:*
1) $\|\Pi_h^* \vec{v}_h\|_{(L^2(\Omega))^2} \leq \|\vec{v}_h\|_{(L^2(\Omega))^2}$,
2) $\|\vec{v}_h - \Pi_h^* \vec{v}_h\|_{(L^2(\Omega))^2} \leq Ch \|\vec{v}_h\|_{H(\mathrm{div};\Omega)}$,
3) $l_h(\Pi_h^* \vec{v}_h, w_h) = -(\nabla \cdot \vec{v}_h, w_h)$,
4) $\left((k\lambda(\eta_h))^{-1} \vec{v}_h, \Pi_h^* \vec{v}_h\right) \geq C \|\vec{v}_h\|^2_{H(\mathrm{div};\Omega)}$.

Let us introduce the norm $|||\vec{u}_h|||_0 = (\vec{u}_h, \Pi_h^* \vec{u}_h)^{1/2}$. It is shown in [13] that $|||\cdot|||_0$ equivalent to $\|\cdot\|_0$ on U_h.

Let us formulate the following result obtained in [14] without proof.

Theorem 1 *([14]) Let* $(\vec{u}, p)$ *and* $(\vec{u}_h, p_h)$ *be the solutions of (6)-(7) and (16)-(17), respectively. Then there exists a positive constant* C *independent of* h *such that*

$$\|\vec{u} - \vec{u}_h\|_{(L^2(\Omega))^2} + \|p - p_h\|_0 \leq Ch^2 \left(\|\vec{u}\|_{(H^1(\Omega))^2} + \|p\|_1\right), \tag{21}$$

$$\|\nabla \cdot (\vec{u} - \vec{u}_h)\|_0 \leq Ch \|\nabla \cdot \vec{u}\|_1, \tag{22}$$

provided $\vec{u}(t) \in (H^1(\Omega))^2$, $\nabla \cdot \vec{u}(t) \in H^1(\Omega)$ *and* $p(t) \in H^1(\Omega)$.

Now we prove the main results of the paper.

Theorem 2 *The sequence of solutions* $(u_h^n, p_h^n, s_h^n) \in U_h \times M_h \times W_h$, $n = 1, 2, ..., N$ *of Problem (16)-(19) satisfies the inequality*

$$\|\vec{u}_h^n\|_{H(\mathrm{div};\Omega)} + \|s_h^n\|_0 + \tau \sqrt{\frac{\gamma_0}{2T}} \|\nabla s_h^n\|_0 \leq C \|s_0\|_1. \tag{23}$$

Proof. Taking $\varphi_h = \overline{s}_h^n$ in (18), we get:

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) + b_h\left(u_h^{n-1/2}, s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) +$$

$$+ d_h\left(s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) + d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = 0. \tag{24}$$

Estimate the terms in (24):

$$\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = \frac{1}{2\tau} |||s_h^n|||_0^2 - \frac{1}{2\tau} \left|\left|\left|s_h^{n-1}\right|\right|\right|_0^2,$$

$$d_h\left(s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) = \frac{\gamma}{2} \left\|\nabla\left(s_h^n + s_h^{n-1}\right)\right\|_0^2,$$

$$d_{1h}\left(\Delta_t s_h^{n-1/2}, \rho_h^* s_h^{n-1/2}\right) \geq \frac{L\gamma_1}{2\tau}\left(\|\nabla s_h^n\|_0^2 - \left\|\nabla s_h^{n-1}\right\|_0^2\right).$$

Then it follows from (24) that

$$|||s_h^n|||_0^2 - \left|\left|\left|s_h^{n-1}\right|\right|\right|_0^2 + \frac{\tau\gamma}{2}\left\|\nabla\left(s_h^n + s_h^{n-1}\right)\right\|_0^2 + L\gamma_1\left(\|\nabla s_h^n\|_0^2 - \left\|\nabla s_h^{n-1}\right\|_0^2\right) \leq 0. \tag{25}$$

Sum (25) by n from 1 to n:

$$|||s_h^n|||_0^2 + \frac{\tau\gamma}{2}\sum_{i=1}^{n}\left\|\nabla\left(s_h^i + s_h^{i-1}\right)\right\|_0^2 + L\gamma_1\|\nabla s_h^n\|_0^2 \leq |||s_0|||_0^2 + L\gamma_1\|\nabla s_0\|_0^2.$$

By extracting the square root from the last inequality, and using Lemma 3, we arrive at the statement of the theorem.

Lemma 4 *Let $\Theta_h : W \to W_h$ be the projection operator such that there exists $\Theta_h s^n \in W_h$ for $s^{n-1}, s^n \in W$, $s_h^{n-1} \in W_h$, and $\vec{u}_h^n \in U_h$ satisfying [13]*

$$\left(\Theta_h\Delta_t s^{n-1/2} - \Delta_t s^{n-1/2}, w_h\right) + d_0\left(\Theta_h s^n - s^n, w_h\right) + d_0\left(\Theta_h s^{n-1} - s^{n-1}, w_h\right) +$$

$$+a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, w_h\right) - a\left(\vec{u}^{n-1/2}, s^{n-1/2}, w_h\right) = 0$$

for any $w_h \in W_h$, where $d_0(s, w_h) = \frac{1}{2}\left(d(s, w_h) + d_1(s, w_h)\right)$. Moreover,

$$\|\Theta_h s^n - s^n\|_0 + \tau\|\nabla\left(\Theta_h s^n - s^n\right)\|_0 \leq Ch^2\|s_0\|_2, \quad n = 0, 1, ..., N, \tag{26}$$

provided $s^n \in H^2(\Omega) \cap W$.

Theorem 3 *Let $(\vec{u}, p, s)$ and $(\vec{u}_h^n, p_h^n, s_h^n)$ be the solutions of Problem (1)-(5) and Problem (16)-(19), respectively. Then*

$$\|\vec{u}(t_n) - \vec{u}_h^n\|_{(L^2(\Omega))^2} + \|p(t_n) - p_h^n\|_0 + \|s(t_n) - s_h^n\|_0 +$$

$$+c\tau\|\nabla\left(s(t_n) - s_h^n\right)\|_0 \leq Ch^2 \tag{27}$$

provided $\tau = O(h)$.

Proof. First, consider the difference of Problems (11)-(14) and (16)-(19) to obtain

$$(s^n, \varphi_h) - (s_h^n, \rho_h^*\varphi_h) + \tau a\left(\vec{u}^{n-1/2}, s^{n-1/2}, \varphi_h\right) - \tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\varphi_h\right) +$$

$$+\tau d\left(s^{n-1/2}, \varphi_h\right) - \tau d_h\left(s_h^{n-1/2}, \rho_h^*\varphi_h\right) + d_1\left(s^n - s^{n-1}, \varphi_h\right) - d_{1h}\left(s_h^n - s_h^{n-1}, \rho_h^*\varphi_h\right) =$$

$$= \left(s^{n-1}, \varphi_h\right) - \left(s_h^{n-1}, \rho_h^*\varphi_h\right). \tag{28}$$

By applying obvious transformations and the projection defined in Lemma 4, we obtain

$$(\Theta_h s^n - s_h^n, \varphi_h) - (s_h^n, \rho_h^*\varphi_h - \varphi_h) + \tau a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \varphi_h\right) -$$

$$-\tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\varphi_h\right) + \tau d\left(\Theta_h s^{n-1/2} - s_h^{n-1/2}, \varphi_h\right) +$$

$$+d_1\left(\Theta_h\left(s^n - s^{n-1}\right) - \left(s_h^n - s_h^{n-1}\right), \varphi_h\right) = \left(\Theta_h s^{n-1} - s_h^{n-1}, \varphi_h\right) - \left(s_h^{n-1}, \rho_h^*\varphi_h - \varphi_h\right).$$

Let $\psi^n = \Theta_h s^n - s_h^n$ and choose $\varphi_h = \psi^{n-1/2}$:

$$\frac{1}{2}\left\|\psi^n\right\|_0^2 + \tau a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \psi^{n-1}\right) - \tau a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\psi^{n-1}\right) +$$

$$+\tau\gamma\left\|\nabla\psi^{n-1}\right\|_0^2 + \frac{\gamma_1}{2}\left\|\nabla\psi^n\right\|_0^2 - \frac{\gamma_1}{2}\left\|\nabla\psi^{n-1}\right\|_0^2 \le$$

$$= \frac{1}{2}\left\|\psi^{n-1}\right\|_0^2 + \left(s_h^n - s_h^{n-1}, \rho_h^*\psi^{n-1/2} - \psi^{n-1/2}\right). \tag{29}$$

Let us estimate the scalar products in (29):

$$\left|a\left(\vec{u}_h^{n-1/2}, \Theta_h s^{n-1/2}, \psi^{n-1/2}\right) - a_h\left(\vec{u}_h^{n-1/2}, s_h^{n-1/2}, \rho_h^*\psi^{n-1/2}\right)\right| \le$$

$$\le \frac{1}{4\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2 + Ch^4, \tag{30}$$

$$\left|\left(s_h^n - s_h^{n-1}, \rho_h^*\psi^{n-1/2} - \psi^{n-1/2}\right)\right| \le$$
$$\le Ch\left(\left\|\psi^n\right\|_0^2 + \left\|\psi^{n-1}\right\| + Ch^4\right) + \frac{\tau}{8\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2 + Ch^2\left\|s^n - s^{n-1}\right\|_1\left\|\nabla\psi^{n-1/2}\right\|_0 \le$$
$$\le Ch\left(\left\|\psi^n\right\|_0^2 + \left\|\psi^{n-1}\right\| + Ch^4\right) + C\tau^2h^3 + \frac{\tau}{4\gamma}\left\|\nabla\psi^{n-1/2}\right\|_0^2. \tag{31}$$

Taking into account the inequalities (30), (31) and the assumption $\tau = O(h)$, it follows from (29) that

$$\frac{1}{2}\left\|\psi^n\right\|_0^2 + c\tau\left\|\nabla\psi^{n-1/2}\right\|_0^2 + \frac{\gamma_1}{2}\left(\left\|\nabla\psi^n\right\|_0^2 - \left\|\nabla\psi^{n-1}\right\|_0^2\right) \le$$

$$\le \frac{1}{2}\left\|\psi^{n-1}\right\|_0^2 + C\tau\left(\left\|\psi^{n-1}\right\|_0^2 + h^4 + \tau^2h^2\right) + C\tau h^4. \tag{32}$$

Summing (32) with respect to n from 1 to n, we obtain

$$\frac{1}{2}\left\|\psi^n\right\|_0^2 + c\tau\sum_{i=1}^n\left\|\nabla\psi^{i-1/2}\right\|_0^2 \le \frac{1}{2}\left\|\psi^0\right\|_0^2 + \frac{\gamma_1}{2}\left\|\nabla\psi^0\right\|_0^2 + C\tau\sum_{i=1}^n\left\|\psi^{i-1}\right\|_0^2 + Ch^4.$$

Applying the discrete Gronwall's lemma yields

$$\left\|\psi^n\right\|_0^2 + c\tau\sum_{i=1}^n\left\|\nabla\psi^{i-1/2}\right\|_0^2 \le C\left(\left\|\psi^0\right\|_0^2 + \left\|\nabla\psi^0\right\|_0^2\right) + Ch^4.$$

Finally, by taking into account Theorem 1, we arrive at the statement of the theorem.

3 Results

To validate the finite volume element scheme (16)-(19), the following five-spot test problem was solved. The problem (1)-(5) in the square $\Omega = [-1, 1] \times [-1, 1]$ was considered in which an injection and a production well were placed in the lower left and upper top corner of Ω, respectively. The following dimensionless values were taken as initial data: $k = 1$, $\mu_w = \mu_o = 1$, $\tau = 10^{-3}$, $\gamma = \gamma_1 = 1$, $L = 1$, $s_0(x) \equiv 0$, and the functions $k_\alpha(s)$ were defined as $k_w(s) = s^2$, $k_o(s) = (1 - s)^2$.

Since the exact solution to the problem cannot be found analytically, more attention is paid to the qualitative characteristics of the solutions obtained using the scheme (16)-(19). To programmatically construct the dual grids used in the finite volume element method, we first introduced a quasiuniform triangulation in Ω. In the numerical test, the triangular decomposition of Ω containing 357 nodes, 648 triangles, and 1004 edges was used. A coarser grid was chosen intentionally to assess the stability of the scheme to the appearance of non-physical oscillations. Then, in construction of the dual grid used for the velocity and pressure equations, the algorithm given in [14] was utilized. The dual grid for the saturation equation was built based on an algorithm presented in [21].

Firstly, the total velocity and water saturation obtained by the scheme (16)-(19) at $t = 5600\tau$ are shown in Figure 1.

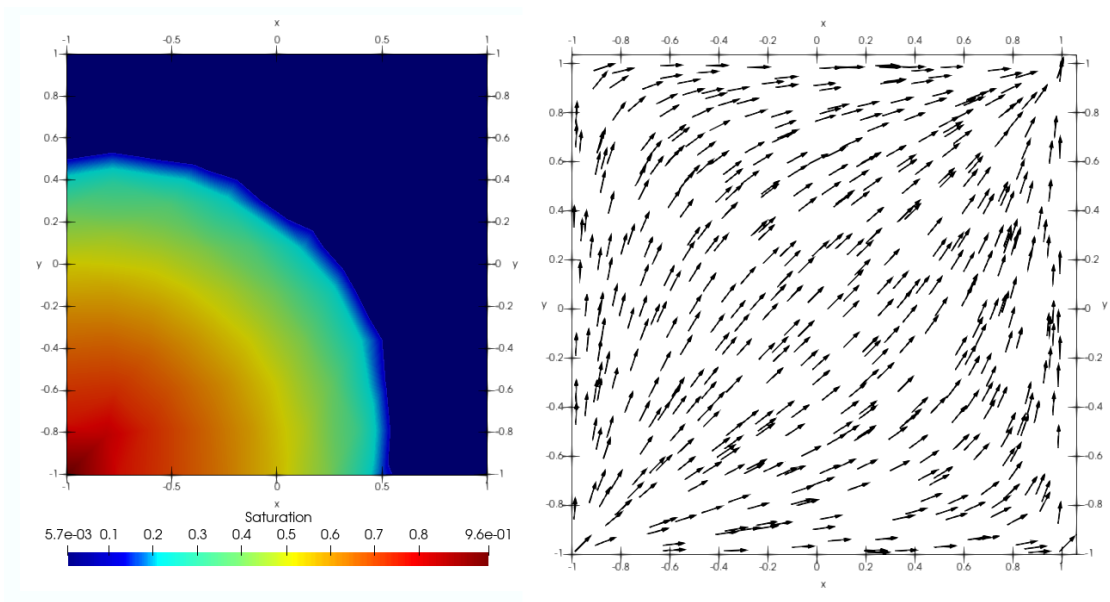

Figure 1: Numerical solution of the model problem at $t = 5600\tau$, saturation (left), total velocity (right)

In our previous work [22], a stabilized finite element method was applied to solving the problem of two-phase nonequilibrium flow. Using the model convection-diffusion equation as an example, it was shown that the use of the standard Galerkin method can lead to the appearance of nonphysical oscillations near the phase separation line. It is shown that their suppression without stabilization can be partially implemented by thickening the grid near the

discontinuity line. Stabilization of the saturation equation significantly reduces nonphysical oscillations; however, the choice of the stabilization parameter itself is a separate problem. Using the stabilization approach, in [22] the model problem of two-phase nonequilibrium flow with the parameters indicated above was solved. It was noted that the stabilization of the equation imposes an additional computational complexity associated with the need to recalculate the stabilization parameter for each finite element at each time layer.

Due to its local conservativeness, the finite volume element scheme (16)-(19) did not lead to the appearance of non-physical oscillations which in turn did not require the addition of stabilizing terms in the scheme. As can be seen from Figure 1, the scheme allows obtaining non-oscillating solutions even on a coarse mesh. In addition, the scheme turned out to be simpler in software implementation.

4 Conclusion

Thus, in this paper, a fully discrete mixed finite volume element method was studied for the problem of two-phase non-equilibrium flow in porous media. It was shown on a synthetic example that the constructed method can be considered as an alternative method that allows obtaining non-oscillating solutions to the problem without the stabilization of the equation, and also requires less computational complexity in comparison, for example, with discontinuous Galerkin methods. The constructed method can be generalized to solve filtration problems with more real input data. A separate work will be devoted to this problem.

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References

[1] Xiong Q., Baychev T.G., and Jivkov A.P., "Review of pore network modelling of porous media: Experimental characterisations, network constructions and applications to reactive transport", *Journal of Contaminant Hydrology,* 192 (2016): 101–117.

[2] Al-Bayati D., Saeedi A., Ktao I., Myers M., White C., and Xie Q., "Insight into Influence of Crossflow in layered Sandstone porous media during Miscible and Immiscible CO2 WAG Flooding", InterPore 2019 Valencia: Book of Abstracts (2019): 66.

[3] Allen G.H. and Sahimi M., "Flow, Transport, and Reaction in Porous Media: Percolation Scaling, Critical-Path Analysis, and Effective Medium Approximation" (AGU Publications, 2017).

[4] Hassanizadeh S.M. and Graw W.G., "Toward an improved description of the physics of twophase flow", *Advances in Water Resources,* 16 (1993): 53–67.

[5] O'Carroll D.M., Phelan T.J., and Abriola L.M., "Exploring dynamic effects in capillary pressure in multistep outflow experiments", *Water Resour Res.,* 41 (2005): W11419.

[6] Cuesta C., van Duijn C.J., and Hulshof J., "Infiltration in porous media with dynamic capillary pressure: travelling waves", *Eur. J. Appl. Math.,* 11 (2000): 381–397.

[7] DiCarlo D.A., Juanes R., LaForce T., and Witelski T.P., "Nonmonotonic travelling wave solutions of infiltration in porous media", *Water Resour. Res.,* 44 (2008): W02406.

[8] Barenblatt G.I., "Filtratsiya dvuh nesmeshivayuschihsya zhidkostei v odnorodnoi poristoi srede [Filtration of two nonmixing fluids in a homogeneous porous medium]", *Mehanika gazov i zhidkostei,* 5 (1971): 57–64 (in Russian).

[9] Cao X. and Mitra K., "Error estimates for a mixed finite element discretization of a two-phase porous media flow model with dynamic capillarity", *Journal of Computational and Applied Mathematics,* 353 (2019): 164–178.

[10] Nicaise S. and Bbekkouche F., "A posteriori error estimates for a fully discrete approximation of sobolev equations", *Confluentes Math.,* 11 (2019): 3–28.

[11] Fan Y., and Pop I.S., "A class of pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization", *Math. Meth. Appl. Sci.,* 34 (2011): 2329–2339.

[12] Bouadjila K., Mokrane A., Saad A.S., and Saad M., "Numerical analysis of a finite volume scheme for two incompressible phase flow with dynamic capillary pressure", *Computers and Mathematics with Applications,* 75:10 (2018): 3614–3631.

[13] Luo Z.D., "A Stabilized Crank-Nicolson Mixed Finite Volume Element Formulation for the Non-stationary Incompressible Boussinesq Equations", *Journal of Scientific Computing,* 66 (2016): 555–576.

[14] Kumar S., "Finite volume element methods for incompressible miscible displacement problems in porous media (Ph.D. thesis)" (Department of mathematics indian institute of technology, Bombay, 2008).

[15] Vasilevsky Yu.V. and Kapyrin I.V., "Dve shemy rasshheplenija dlja nestacionarnoj zadachi konvekcii-diffuzii na tetrajedralnyh setkah [Two splitting schemes for the nonstationary convection-diffusion problem on tetrahedral meshes]", *Zhurnal vychislitel'noj matematiki i matematicheskoj fiziki,* 48 (2008): 1429–1447 (in Russian).

[16] Luo Z., Li H., and Sun P., "A fully discrete stabilized mixed finite volume element formulation for the non-stationary conduction–convection problem", *Journal of Mathematical Analysis and Applications,* 404 (2013): 71–85.

[17] Ferraris S., Bevilacqua I., Canone D., Pognant D., and Previati M., "The Finite Volume Formulation for 2D Second-Order Elliptic Problems with Discontinuous Diffusion/Dispersion Coefficients", *Mathematical Problems in Engineering,* 187634 (2012): 1–24.

[18] Nikitin K.D., "Nelinejnyj metod konechnyh obyomov dlja zadach dvuhfaznoj filtracii [Nonlinear finite volume method for two-phase filtration problems]", *Matematicheskoe modelirovanie,* 22 (2010): 131–147 (in Russian).

[19] Lin Y., Liu J., and Yang M., "Finite volume element methods: an overview on recent developments", *International Journal of Numerical analysis and modeling, Series B.,* 4 (2013): 14–34.

[20] Sayevand K. and Arjang F., "Finite volume element method and its stability analysis for analyzing the behavior of sub-diffusion problems", *Applied Mathematics and Computation,* 290 (2016): 224–239.

[21] Voller V. R., "Basic control volume finite element methods for fluids and solids" (World Scientific Publishing, 2009).

[22] Omariyeva D., Temirbekov N., Madiyarov M., "Stabilized finite element method for solving the saturation equation in the two-phase non-equilibrium flow problem", *Joint Issue of Bulletin of the National Engineering Academy of the Republic of Kazakhstan and Computational Technologies,* 3:1 (2020): 209–216.

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ONE CLASS OF SMOOTH BOUNDED SOLUTIONS TO THE CAUCHY PROBLEM FOR A THREE-DIMENSIONAL FILTRATION MODEL WITH DARCY'S LAW

The first results on the use of the apparatus of four-dimensional mathematics for solving the three-dimensional model of the Navier-Stokes equations by the analytical method were obtained by the Kazakh mathematician Professor M.M. Abenov. After the author of this article with other researchers proved the theorem on the existence of a solution to the Cauchy problem for a three-dimensional model of filtration theory.

This paper is devoted to the study of a three-dimensional model of the filtration theory in one of the spaces of four-dimensional numbers. The purpose of this article is to obtain an analytical solution of the three-dimensional Cauchy problem for the mathematical model of linear filtration model by the method of four-dimensional regular functions.

In this study, a class of infinitely differentiable and bounded functions of the initial conditions of the Cauchy problem, satisfying the Cauchy-Riemann condition, with five degrees of freedom for a specific four-dimensional function is found, and also a class of infinitely differentiable and bounded solutions of this problem is found that satisfy the linear Darcy law.

Key words: continuity equation, Darcy's law, four-dimensional function, Cauchy-Riemann condition.

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ДАРСИ ЗАҢЫ БАР ҮШ ӨЛШЕМДІ ФИЛЬТРАЦИЯ МОДЕЛІНІҢ КОШИ ЕСЕБІНІҢ ТЕГІС ШЕКТЕЛГЕН ШЕШІМДЕРІНІҢ БІР КЛАСЫ

Төртөлшемді математика аппаратын Навье-Стокс теңдеулерінің үш өлшемді моделін аналитикалық әдіспен шешу үшін қолданудың алғашқы нәтижелерін қазақстандық математик профессор М.М. Әбенов қол жеткізді. Кейиниректе осы мақаланың авторы басқа ізденушілермен қатар фильтрация теориясының үш өлшемді моделі үшін Коши мәселесінің шешімі бар туралы теореманы дәлелдеген.

Бұл жұмыс төрт өлшемді сандар кеңістіктерінің бірінде фильтрация теориясының үш өлшемді моделін зерттеуге арналған. Аталып отырған мақаланың мақсаты төрт өлшемді тұрақты функциялар әдісімен сызықтық фильтрацияның математикалық моделі үшін үш өлшемді Коши есебінің аналитикалық шешімін алу болып табылады.

Бұл жұмыста нақты төрт өлшемді функция үшін бес еркіндік дәрежесі бар Коши-Риман шартын қанағаттандыратын Коши есебінің бастапқы шарттарының шексіз дифференциалданатын және шектелген функцияларының класы табылды, сонымен қатар

шексіз дифференциалданатын және шектелген шешімдерінің класы табылды. Табылған шешімдер класы Дарси заңын қанағаттандырады.

Түйін сөздер: үздіксіздік теңдеуі, Дарси заңы, төрт өлшемді функция, Коши-Риман шарты.

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ОДИН КЛАСС ГЛАДКИХ ОГРАНИЧЕННЫХ РЕШЕНИЙ ЗАДАЧИ КОШИ ДЛЯ ТРЕХМЕРНОЙ МОДЕЛИ ФИЛЬТРАЦИИ С ЗАКОНОМ ДАРСИ

Первые результаты по применению аппарата четырехмерной математики для решения трехмерной модели уравнений Навье-Стокса аналитическим методом были получены казахстанским математиком профессором М.М. Абеновым. После авторам настоящей статьи и другими исследователями была доказана теорема о существовании решения задачи Коши для трехмерной модели теории фильтрации.

Настоящая работа посвящена исследованию трехмерной модели теории фильтрации в одном из пространств четырехмерных чисел. Целью настоящей статьи является получение аналитического решения трехмерной задачи Коши для математической модели линейной фильтрации методом четырехмерных регулярных функций.

В данной работе найден класс бесконечно дифференцируемых и ограниченных функций начальных условий задачи Коши, удовлетворяющие условию Коши-Римана, с пятью степенями свободы для конкретной четырехмерной функции, а также найден класс бесконечно дифференцируемых и ограниченных решений этой задачи, которые удовлетворяют линейному закону Дарси.

Ключевые слова: уравнение неразрывности, закон Дарси, четырехмерная функция, условие Коши-Римана.

1 Introduction

The theory of filtration is one of the main areas of scientific research due to its economic importance in connection with the extraction of oil and gas products, the development of subsoil, where the main extraction technologies are managed by the laws of filtration theory. The main law of the filtration theory, which describes the movement in a porous medium, is the linear Darcy law [1-14].

A large number of scientific works are devoted to the study of three-dimensional models of fluid motion. Basically, such problems of hydrodynamics and filtration are solved by numerical methods, since the solution by the analytical method is difficult. To avoid this difficulty, professor M.M. Abenov [15-16] proposed a new method for solving the continuity equation in four-dimensional space. In the work [15] M.M. Abenov provides an analytical solution of the Navier-Stokes equations using four-dimensional regular functions.

To obtain an analytical solution of the three-dimensional model of the filtration theory, in the article [19] a new approach to solving the problem in the space of four-dimensional numbers M5 is studied. The norm and properties of the space M5 were studied in [17-18]. In this paper, we study the Cauchy problem for a three-dimensional non-stationary mathematical model of filtration with the linear Darcy law.

2 Problem statement

Consider the motion of a compressible fluid in a porous medium. The continuity equation has the following form

$$m\frac{\partial \rho}{\partial t} + div\,(\rho \overrightarrow{v}) = 0 \tag{1}$$

where $m = m(x, y, z)$ – known medium porosity, $\rho(x, y, z, t)$ – liquid density, $\overrightarrow{v}\,(x, t) = (v_1\,(x, y, z, t)\,, v_2\,(x, y, z, t)\,, v_3(x, u, z, t))$ – filtration velocity, $x = (x, y, z) \in R^3$ – spatial coordinates, $t \in R_+$ - time.

Let's write Darcy's law in the form

$$-\nabla P = \frac{\mu}{k}\overrightarrow{v}, \tag{2}$$

where μ – liquid viscosity, $k = k(x, y, z)$ – permeability coefficient depending only on the properties of the porous medium, $P(x, y, z, t)$ – fluid pressure. The system of equations (1) - (2) describes the non-stationary motion of a fluid in a porous medium in a three-dimensional space R^3. From physical considerations, it is obvious that the solutions of this system should be bounded functions in the entire space.

The initial conditions for the system (1) - (2) are set in the form:

$$\rho\,(x, y, z, 0) = \rho_0(x, y, z), 0 < \rho_{min} \leq \rho_0\,(x, y, z) \leq \rho_{max} < \infty, \tag{3}$$

$$v_i\,(x, y, z, 0) = \varphi_i(x, y, z), (x, y, z) \in R^3, \quad i = 1, 2, 3., \tag{4}$$

where the functions on the right-hand side are smooth and bounded functions in the entire space. In the paper [19], the existence theorem for a solution to the Cauchy problem (1) – (4) was proved under certain conditions on the initial data, and was obtained explicit analytical formulas for the solution.

In this paper, it is interested to find a specific type of initial conditions under which the solution of the considered problem is smooth and bounded in the entire space R^3 for any values of $t > 0$.

3 Methodology

Let us rewrite (1) as follows [15]

$$m\frac{\partial \rho}{\partial t} + \overrightarrow{v} \cdot grad\ \rho + \rho\ div\overrightarrow{v} = 0$$

Let us introduce the function $\delta\,(x, y, z, t)$ and define it as follows

$$m\frac{\partial \rho}{\partial t} + \overrightarrow{v} \cdot grad\ \rho = \frac{\rho}{c}\frac{\partial \delta}{\partial t} \tag{5}$$

where c – some characteristic filtration velocity. It is easy to see that $\delta\,(x, y, z, t)$ has the dimension of velocity. Dividing both sides of equation (5) by $\rho\,(x, y, z, t)$ we obtain the equation

$$m\frac{\partial \theta}{\partial t} + \overrightarrow{v} \cdot grad\theta = \frac{1}{c}\frac{\partial \delta}{\partial t}, \tag{6}$$

where $\delta\left(x,y,z,t\right)=ln\rho(x,y,z,t)$.

Rewrite (1) taking into account (5)

$$\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}+\frac{1}{c}\frac{\partial\delta}{\partial t}=0 \tag{7}$$

For equation (7), we set the initial Cauchy data in the form

$$v_1|_{t=0}=\varphi_1\left(x,y,z\right);v_2|_{t=0}=\varphi_2\left(x,y,z\right);$$
$$v_3|_{t=0}=\varphi_3\left(x,y,z\right);\ \ \delta|_{t=0}=\delta_0\left(x,y,z\right). \tag{8}$$

where $\delta_0(x,y,z)$ can be determined through $\rho_0(x,y,z)$ solving the equation (6).

Similarly [19], the solution of problem (7) – (8) is sought in the form

$$\begin{aligned}
v_1&=q_1w_1(p_1x,p_2y,p_3z,p_4ct)\\
v_2&=q_2w_2(p_1x,p_2y,p_3z,p_4ct)\\
v_3&=q_3w_3(p_1x,p_2y,p_3z,p_4ct)\\
\delta&=q_4w_4(p_1x,p_2y,p_3z,p_4ct)
\end{aligned} \tag{9}$$

Using the Cauchy-Riemann conditions from (9) can be obtained formulas relating the components of the 4-vectors of velocity:

$$\begin{aligned}
&\tfrac{1}{p_1q_1}\tfrac{\partial v_1}{\partial x}=\tfrac{1}{p_2q_2}\tfrac{\partial v_2}{\partial y}=\tfrac{1}{p_3q_3}\tfrac{\partial v_3}{\partial z}=\tfrac{1}{p_4q_4}\tfrac{\partial\delta}{\partial t}\\
&\tfrac{1}{p_1q_2}\tfrac{\partial v_2}{\partial x}=-\tfrac{1}{p_2q_1}\tfrac{\partial v_1}{\partial y}=\tfrac{1}{p_3q_4}\tfrac{\partial\delta}{\partial z}=-\tfrac{1}{p_4q_3}\tfrac{\partial v_3}{\partial t}\\
&\tfrac{1}{p_1q_3}\tfrac{\partial v_3}{\partial x}=\tfrac{1}{p_2q_4}\tfrac{\partial\delta}{\partial y}=-\tfrac{1}{p_3q_1}\tfrac{\partial v_1}{\partial z}=-\tfrac{1}{p_4q_2}\tfrac{\partial v_2}{\partial t}\\
&\tfrac{1}{p_1q_4}\tfrac{\partial\delta}{\partial x}=-\tfrac{1}{p_2q_3}\tfrac{\partial v_3}{\partial y}=-\tfrac{1}{p_3q_2}\tfrac{\partial v_2}{\partial z}=\tfrac{1}{p_4q_1}\tfrac{\partial v_1}{\partial t}
\end{aligned}$$

From these equalities can obtain the following problems and their solutions for the functions $v_1\left(x,y,z,t\right),\ v_2\left(x,y,z,t\right),\ v_3\left(x,y,z,t\right),\ \delta\left(x,y,z,t\right)$ [19]

$$\begin{cases}
\frac{\partial^2v_1}{\partial t^2}=\frac{{p_4}^2c^2}{{p_1}^2}\frac{\partial^2v_1}{\partial x^2}\\
v_1|_{t=0}=\varphi_1\left(x,y,z\right)\\
\frac{\partial v_1}{\partial t}|_{t=0}=\frac{p_4q_1c}{p_1q_4}\frac{\partial\delta_0}{\partial x}
\end{cases}$$

$$v_1\left(x,y,z,t\right)=\frac{1}{2}\left(\varphi_1\left(x+\frac{p_4c}{p_1}t,y,z\right)+\varphi_1\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)$$

$$+\frac{1}{2}\frac{q_1}{q_4}\left(\delta_0\left(x+\frac{p_4c}{p_1}t,y,z\right)-\delta_0\left(x-\frac{p_4c}{p_1}t,y,z\right)\right) \tag{10}$$

$$\begin{cases}
\frac{\partial^2v_2}{\partial t^2}=\frac{{p_4}^2c^2}{{p_1}^2}\frac{\partial^2v_2}{\partial x^2}\\
v_2|_{t=0}=\varphi_2\left(x,y,z\right)\\
\frac{\partial v_2}{\partial t}|_{t=0}=-\frac{p_4q_2c}{p_1q_3}\frac{\partial\varphi_3}{\partial x}
\end{cases}$$

$$v_2(x,y,z,t) = \frac{1}{2}\left(\varphi_2\left(x+\frac{p_4c}{p_1}t,y,z\right)+\varphi_2\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)$$

$$-\frac{1}{2}\frac{q_2}{q_3}\left(\varphi_3\left(x+\frac{p_4c}{p_1}t,y,z\right)-\varphi_3\left(x-\frac{p_4c}{p_1}t,y,z\right)\right) \tag{11}$$

$$\begin{cases} \frac{\partial^2 v_3}{\partial t^2} = \frac{{p_4}^2c^2}{{p_1}^2}\frac{\partial^2 v_3}{\partial x^2} \\ v_3|_{t=0} = \varphi_3(x,y,z) \\ \frac{\partial v_3}{\partial t}|_{t=0} = -\frac{p_4q_3c}{p_1q_2}\frac{\partial\varphi_2}{\partial x} \end{cases}$$

$$v_3(x,y,z,t) = \frac{1}{2}\left(\varphi_3\left(x+\frac{p_4c}{p_1}t,y,z\right)+\varphi_3\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)$$

$$-\frac{1}{2}\frac{q_3}{q_2}\left(\varphi_2\left(x+\frac{p_4c}{p_1}t,y,z\right)-\varphi_2\left(x-\frac{p_4c}{p_1}t,y,z\right)\right) \tag{12}$$

$$\begin{cases} \frac{\partial^2 \delta}{\partial t^2} = \frac{{p_4}^2c^2}{{p_1}^2}\frac{\partial^2 \delta}{\partial x^2} \\ \delta|_{t=0} = \delta_0(x,y,z) \\ \frac{\partial \delta}{\partial t}|_{t=0} = \frac{p_4q_4c}{p_1q_1}\frac{\partial\varphi_1}{\partial x} \end{cases}$$

$$\delta(x,y,z,t) = \frac{1}{2}\left(\left(\delta_0\left(x+\frac{p_4c}{p_1}t,y,z\right)+\delta_0\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)\right.$$

$$\left.+\frac{q_4}{q_1}\left(\varphi_1\left(x+\frac{p_4c}{p_1}t,y,z\right)-\varphi_1\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)\right)$$

Further find $\rho(x,y,z,t)$ from [15]

$$\rho(x,y,z,t) = \rho_0[C_1,C_2,C_3]\times\exp\left[\frac{1}{2cm}\left(\left(\delta_0\left(x+\frac{p_4c}{p_1}t,y,z\right)+\delta_0\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)\right.\right.$$

$$\left.\left.+\frac{q_4}{q_1}\left(\varphi_1\left(x+\frac{p_4c}{p_1}t,y,z\right)-\varphi_1\left(x-\frac{p_4c}{p_1}t,y,z\right)\right)\right)\right] \tag{13}$$

Thus, using the methodology for solving the four-dimensional equation of the solenoidal field, can find the solution of the continuity equation (1) in the form (10)-(12) of the velocity component, (13) the density.

Let us turn to the solution of the three-dimensional Darcy equation (2). Write as follows [19]

$$P(x,y,z,t) = -\frac{\mu}{k}\int_0^x v_1(\xi,y,z,t)\,d\xi-\frac{\mu}{k}\int_0^y v_2(0,\eta,z,t)d\eta-\frac{\mu}{k}\int_0^z v_3(0,0,\zeta,t)d\zeta+C(t) \tag{14}$$

where $C(t) = p_\infty + \frac{\mu}{k}\int_0^\infty v_3(0,0,\zeta,t)\,d\zeta$.

4 Results

Consider a three-dimensional filtration model (1)-(2). Let us assume that the viscosity of the liquid and the coefficient of porosity and permeability are constant values. As initial conditions, take the components of the four-dimensional function $exp\left(-X^2\right)$, which are infinitely differentiable and bounded functions in the entire half-space:

$$\varphi_1(x,y,z)=\frac{1}{4}q_1\left[exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)+exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$\left.+exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)+exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right],$$

$$\varphi_2(x,y,z)=-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)-exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right. \tag{15}$$

$$\left.+exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)-exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right],$$

$$\varphi_3(x,y,z)=-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\left[-exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)+exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$\left.+exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)-exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right],$$

$$\delta_0(x,y,z)=-\frac{1}{4}\frac{\sqrt{s_{13}}q_1}{\sqrt{s_{12}}s_{14}}\left[exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)+exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$\left.-exp\left(-\left(p_1\left(x-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)-exp\left(-\left(p_1\left(x+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]$$

and satisfy the conditions

$$\begin{array}{l}
\frac{\partial\varphi_1(x,y,z)}{\partial x}=s_{12}\frac{\partial\varphi_2(x,y,z)}{\partial y}=s_{13}\frac{\partial\varphi_3(x,y,z)}{\partial z}\\
\frac{\partial\varphi_2(x,y,z)}{\partial x}=\frac{\partial\varphi_1(x,y,z)}{\partial y}=s_{14}\frac{\partial\delta_0(x,y,z)}{\partial z}\\
\frac{\partial\varphi_3(x,y,z)}{\partial x}=\frac{s_{12}s_{14}}{s_{13}}\frac{\partial\delta_0(x,y,z)}{\partial y}=\frac{\partial\varphi_1(x,y,z)}{\partial z}\\
\frac{\partial\delta_0(x,y,z)}{\partial x}=\frac{s_{13}}{s_{14}}\frac{\partial\varphi_3(x,y,z)}{\partial y}=\frac{s_{13}}{s_{14}}\frac{\partial\varphi_2(x,y,z)}{\partial z}
\end{array}$$

at $(x,y,z)\in R^3$, where s_{12}, s_{13}, s_{14} are given positive real constants. Then the Cauchy problem (1), (3)-(4) has the following solution:

$$v_1(x,y,z,t)=\frac{1}{4}q_1\left[exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right) \tag{16}$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\Bigg]$$

$$v_2(x,y,z,t)=-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\Bigg[exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right) \tag{17}$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\Bigg]$$

$$v_3(x,y,z,t)=-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\Bigg[-exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right) \tag{18}$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\Bigg]$$

$$\delta(x,y,z,t)=-\frac{1}{4}\frac{\sqrt{s_{13}}q_1}{\sqrt{s_{12}}s_{14}}\Bigg[exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\Bigg]$$

Then find $\rho\left(x,y,z,t\right)$

$$\rho\left(x,y,z,t\right)=\rho_0\left[C_1,C_2,C_3\right]\times\exp\left[-\frac{1}{4cm}\frac{\sqrt{s_{13}}q_1}{\sqrt{s_{12}}s_{14}}\left(exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right) \tag{19}$$

$$-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.\left.-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right)\right]$$

Let's move on to solving the three-dimensional Darcy equation. Find the required pressure by substituting (16)-(18) into the equation for pressure (14):

$$P\left(x,y,z,t\right)=-\frac{\mu}{k}\int_0^x\frac{1}{4}q_1\left[exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$+exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.+exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]d\xi$$

$$-\frac{\mu}{k}\int_0^y\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}\eta+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$-exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}\eta-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}\eta-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.\left.-exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}\eta+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)d\eta$$

$$-\frac{\mu}{k}\int_0^z\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\left[-exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{13}}}\zeta\right)\right)^2\right)\right.\right. \tag{20}$$

$$+exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{13}}}\zeta\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{13}}}\zeta\right)\right)^2\right)$$

$$\left.\left.-exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{13}}}\zeta\right)\right)^2\right)\right]\right)d\zeta+C\left(ct\right)$$

Verify the fulfillment of the pressure equation in the following way. In the direction x, the integral is solved in an obvious way

$$\frac{\partial P}{\partial x} = -\frac{1}{4}\frac{\mu}{k}q_1\left[exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right] = -\frac{\mu}{k}v_1$$

In the direction y, take the differential $\frac{\partial}{\partial y}$ under the integral, and calculate as follows

$$\frac{\partial P}{\partial y} = -\frac{\mu}{k}\int_0^x \frac{\partial}{\partial y}\left(\frac{1}{4}q_1\left[exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$+exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.\left.+exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)d\xi$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$\left.\left.-exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$\left.\left.-exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)$$

$$=-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\right)\left(\int_0^x exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$\left.d\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right)$$

$$-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\right)\left(\int_0^x exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$\left.d\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$\left.\left.-exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)$$

$$+\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$\left.\left.-exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)$$

$$=-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{12}}}\left[exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.\right.$$

$$-exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.\left.-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]\right)=-\frac{\mu}{k}v_2.$$

In the direction z, take the differential $\frac{\partial}{\partial z}$ under the integral, and calculate as follows

$$\frac{\partial P}{\partial z}=-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^x exp\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\xi-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^y exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)^2\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^y exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^y exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{\mu}{k}\left(\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\right)\int_0^y exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$d\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{1}{4}\frac{\mu}{k}\frac{q_1}{\sqrt{s_{13}}}exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+\frac{1}{4}\frac{\mu}{k}\frac{q_1}{\sqrt{s_{13}}}exp\left(-\left(p_1\left(\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+\frac{1}{4}\frac{\mu}{k}\frac{q_1}{\sqrt{s_{13}}}exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$-\frac{1}{4}\frac{\mu}{k}\frac{q_1}{\sqrt{s_{13}}}exp\left(-\left(p_1\left(-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$=-\frac{1}{4}\frac{q_1}{\sqrt{s_{13}}}\left[-exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right.$$

$$+exp\left(-\left(p_1\left(x+\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$+exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct-\frac{1}{\sqrt{s_{12}}}y-\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)$$

$$\left.-exp\left(-\left(p_1\left(x-\frac{\sqrt{s_{12}}}{\sqrt{s_{13}}}s_{14}\left(1+\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)ct+\frac{1}{\sqrt{s_{12}}}y+\frac{1}{\sqrt{s_{13}}}z\right)\right)^2\right)\right]=-\frac{\mu}{k}v_3.$$

Indeed, the equality is satisfied, from (20) have explicitly obtained equalities in directions

$$\frac{\partial P}{\partial x} = -\frac{\mu}{k}v_1, \ \frac{\partial P}{\partial y} = -\frac{\mu}{k}v_2, \frac{\partial P}{\partial z} = -\frac{\mu}{k}v_3.$$

Thus, obtained a class of infinitely differentiable and bounded in the entire half-space solutions of the three-dimensional model of filtration theory with five degrees of freedom $q_1,\ p_1,\ s_{12},\ s_{13},\ s_{14}$ of system (1)-(2) with initial conditions (3), (4) in the form (16)-(20).

5 Conclusion

In this paper, was studied a three-dimensional model of the filtration theory with the linear Darcy law. A class of infinitely differentiable and bounded in the entire half-space initial conditions and solutions of a three-dimensional model of filtration theory with five degrees of freedom $q_1,\ p_1,\ s_{12},\ s_{13},\ s_{14}$ is found. The method for solving the problem of three-dimensional filtration model is the application of the theory of functions of four-dimensional variables M5.

References

[1] Donald, A. Nield, Adrian Bejan. "Convection in Porous Media." USA: Springer; Third edition (2006).

[2] Frank, A. Coutelieris, J.M.P.Q Dlgado. "Transport Processes in Porous Media." USA: Springer (2012).

[3] Oliver, Coussy. "Mechanics and Physics of Porous Solids." UK: Wiley (2010).

[4] IHAR (International Association for Hydraulic Research). "Fundamentals of Transport Phenomena in Porous Media." Israel: Elsevier (1972).

[5] Kambiz, Vafai. "Handbook of Porous Media." USA: Taylor & Francis Group; Second Edition (2005).

[6] Entov, V.N. "Filtration theory." Sorosov educational journal, no. 2. (1998): 121-128.

[7] Polubarinova-Kochina, P.Ya. "Теория движения грунтовых вод." M.: Nauka (1977).

8. Dullien, F.A.L. "Porous Media: Fluid Transport and Pore Structure." USA: Academic Press; Second Edition (1991).

[8] Whintaker, S. "The Forchheimer equation: A theoretical development." Transport in Porous Media , no. 25 (1996): 27–61.

[9] Derek, B. Ingham, Ioan Pop. "Transport Phenomena in Porous Media." USA: Great Britain (2005).

[10] Tamer, O.S., Toropov, E.S., Shevnina, T.E., Vorobieva, T.I. "Research of Reservoir Rock Properties in Violation of Darcy's Linear Law. Transport and Storage of Hydrocarbons." IOP Conf. Series: Materials Science and Engineering 154 (2016). https://doi:10.1088/1757-899X/154/1/012006

[11] Hassanizadeh, S.M., Gray, W.G. "High Velocity Flow in Porous Media." Transport in Porous Media 2 (1987): 521-531.

[12] Bear, J. "Dynamics of fluids in porous media." Elsevier, New York (1972).

[13] Alekseev, B.V. "Analytical solution of the Leibenson equation and filtration theory." Fine chemical technologies 11, no. 1 (2016): 34-39.15.

[14] Abenov M.M. Chetirehmernaya matematika. Metody i prilozheniya. Nauchnaya monographia [Four-dimensional mathematics: Methods and applications. Scientific monograph]. Almaty.: Publishing House Kazakh University, 2019. - 176.

[15] Abenov M.M., Gabbassov M.B. Anyzotropnie chetirehmernie prostranstva ili novie kvaternioni [Anisotropic four-dimensional spaces or new quaternions]. Preprint, Nur-Sultan. 2020.

[16] Rakhymova A.T., Gabbassov M.B., Shapen K.M., "On one space of four-dimensional numbers," *Journal of Mathematics, Mechanics and Computer Science (Vol 4)* (2020): 199-225.

[17] Rakhymova A.T., Gabbassov M.B., Shapen K.M., "Functions in one space of four-dimensional numbers," *Journal of Mathematics, Mechanics and Computer Science (Vol 2)* (2021): 139-154.

[18] Rakhymova A.T., Gabbassov M.B., Ahmedov A.A. Analytical Solution of the Cauchy Problem for a Nonstationary Three-dimensional Model of the Filtration Theory. Journal of Advanced Research in Fluid Mechanics and Thermal Sciences. Vol. 87 No. 1: November (2021): 118 – 133.

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SOLUTION OF A NONLINEAR HEAT TRANSFER PROBLEM BASED ON EXPERIMENTAL DATA

The paper develops a method for solving nonlinear equations of heat conduction. Two-layer container complexes have been created, the side faces of which are thermally insulated so that the 1D heat equation can be used. In order not to solve the boundary value problem with a contact discontinuity and lose the accuracy of the solution method, a temperature sensor was placed at the junction of two media, and a mixed boundary value problem is solved in each area (container). To provide initial data for the initial boundary value problem, three temperature sensors were used: two sensors measure air temperatures at the left and right boundaries of the container complex; the third sensor measures the temperature of the soil at the junction of two media. The paper numerically investigates the initial-boundary problem of heat conduction with nonlinear coefficients of heat conduction, heat capacity, heat transfer and density of the material. To solve a nonlinear initial-boundary value problem, the grid method is used. Two types of difference schemes are constructed: linearized and nonlinear. The linearized difference scheme is implemented numerically by the scalar sweep method, and the nonlinear difference problem is solved by the Newton method. On the basis of an a priori estimate of the solution of a nonlinear difference problem, we prove the convergence of the second degree of Newton's method. The numerical calculations carried out show that, for small time intervals, the solutions of the linearized difference problem differ little from the solution of the nonlinear difference problem (1 - 3%). And for long periods of time, tens of days or months, the solutions of the two methods differ significantly, sometimes exceeding 20%.
Key words: thermal conductivity, nonlinearity, difference problem, convergence, inverse problem, differentiation with respect to a parameter.

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Эксперименттік мәліметтер негізінде сызықты емес жылу алмасу есебін шешу

Жұмыста жылуөткізгіштіктің сызыққ емес теңдеулерін шешу әдісі әзірленген. 1D жылу теңдеуін қолдану үшін бүйір беттері жылу оқшауланған екі қабатты контейнерлік кешендер жасалды. Контакті үзілісімен шекаралық есептерді шешпеу және шешу әдісінің дәлдігін жоғалтпау үшін екі ортаның түйіскен жеріне температура датчигі қойылып, әр аймаққа (контейнер) аралас шекаралық есеп шығарылды. Бастапқы шекаралық есептің бастапқы деректерін қамтамасыз ету үшін үш температура датчигі пайдаланылды: екі датчиктер контейнер кешенінің сол және оң жақ шекараларында ауа температурасын өлшейді; үшінші сенсор екі ортаның түйіскен жеріндегі топырақ температурасын өлшейді. Жұмыста жылу өткізгіштіктің сызықты емес коэффициенттері, жылу сыйымдылығы, жылу беру және материалдың тығыздығы бар жылу өткізгіштіктің бастапқы-шекаралық есептері сандық түрде зерттеледі. Сызықты емес бастапқы-шекаралық есептерді шешу үшін тор әдісі қолданылады. Айырмашылық схемалардың екі түрі құрастырылады: сызықтық және сызықтық емес. Сызықтық айырым схемасы скалярлық сыпыру әдісімен сандық түрде жүзеге асырылады, ал сызықтық емес айырмашылық мәселесі Ньютон әдісімен шешіледі. Сызықты емес айырмашылық есебінің шешімін априорлық бағалау негізінде Ньютон әдісінің екінші дәрежелі жинақтылығын дәлелдейміз. Жүргізілген сандық есептеулер шағын уақыт аралықтары үшін сызықтық айырым есебінің шешімдерінің сызықтық емес айырмашылық есебінің шешімінен (1 - 3%) айырмашылығы аз екенін көрсетеді. Ал ұзақ уақыт бойы, ондаған күндер немесе айлар үшін екі әдістің шешімдері айтарлықтай ерекшеленеді, кейде 20% -дан асады.

Түйін сөздер: жылу өткізгіштік, сызықтық емес, айырықша есеп, жинақтылық, кері есеп, параметрге қатысты дифференциалдау.

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Решение нелинейной задачи теплопередачи, основанной на экспериментальных данных

В работе разрабатывается метод решения нелинейной уравнений теплопроводности. Созданы двухслойные комплексы контейнеров, боковые грани которых теплоизолированные, чтобы можно было воспользоваться 1D уравнением теплопроводности. Чтобы не решать краевую задачу с контактным разрывом и терять точность метода решения, на стыке двух сред поставили датчик температуры, и в каждой области (контейнера) решается смешанная краевая задача. Чтобы обеспечить исходными данными начально граничную задачу, использовали три датчика температуры: два датчика измеряет температуры воздуха на левой и правой границе комплекса контейнеров; третьи датчик измеряет температуру грунта на стыке двух сред. В работе численно исследуется начально-краевая задача теплопроводности с нелинейными коэффициентами теплопроводности, теплоемкости, теплоотдачи и плотности материала. Чтобы решить нелинейную начально-краевую задачу используется метод сеток. Строятся два вида разностных схем: линеаризованная и нелинейная. Линеаризованная разностная схема реализуется численно методом скалярной прогонки, а нелинейная разностная задача решается методом Ньютона. На основе априорной оценки решения нелинейной разностной задачи доказывается сходимость второй степени метода Ньютона. Проведенные численные расчеты показывают, что при небольших промежутках времени решения линеаризованной разностной задачи мало отличаются от решения нелинейной разностной задачи (1 - 3%). А при больших промежутках времени, десятки дней или месяцы, решения двух методов значительно отличаются, порой переваливая за 20%.

Ключевые слова: теплопроводность, нелинейность, разностная задача, сходимость, обратная задача, дифференцирование по параметру.

1 Introduction

It is well known that the main source of information about thermophysical properties is the performance of a physical experiment [1, 4]. The law of conservation of energy is used for the theoretical basis of the method for finding the thermophysical characteristics of the medium, the consequence of which is the nonlinear differential equation of heat conduction [1, 5–7]. Where κ is the coefficient of thermal conductivity, ρ is the density, c is the specific heat, h is the heat transfer coefficient, all coefficients depend on the temperature of the material and determine the process of heat transfer in the medium (solid or liquid). Temperature is one of the main factors affecting the thermal conductivity of the soil. It has been experimentally established that the nature of the influence of temperature on the thermophysical parameters of the soil is non-linear [8–10]. In this regard, there is an urgent need to solve the inverse problem of the nonlinear heat equation.

Therefore, the main purpose of our study is the conduction of a thermophysical experiment and on the basis of the obtained data, numerical calculation of the nonlinear inverse problem of heat conduction to find the thermophysical coefficients [2, 3, 5, 11–13]. At the beginning, the linearized direct problem of heat transfer is solved with respect to the input parameters (temperature values at the boundaries). Then, using the solution of the linearized scheme for the initial data in the Newton method, the nonlinear heat equation is solved. Differentiation of the nonlinear difference problem with respect to the parameter and experimental temperature

values at the accessible soil-ground boundary make it possible to find thermophysical characteristics in inverse coefficient problems of heat transfer and heat fluxes in inverse boundary problems [14–18].

The article is structured as follows: the 2 section shows a demonstration of a mathematical model for describing the physical phenomenon of heat conduction. Additionally one can find there the discretization of the computational domain and the mathematical model. Section 3 presents a numerical method for solving the direct problem of a nonlinear differential heat equation, which is the Newton's method. And the main goal of this section is proof of the Newton's method convergence. The 4 section provides us scheme and an overview of the experimental setup. The 5 section illustrates the solution of a numerical method for solving the problem of a nonlinear differential heat equation by differentiating a nonlinear difference problem with respect to the required parameter.

2 Mathematical model

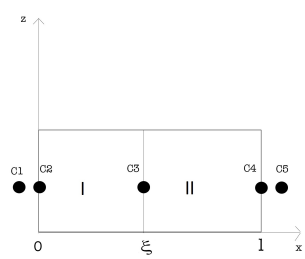

Figure 1: Two-chamber container

Problem formulation. Fig.1 schematically shows a two-chamber container, which has the thermally insulated side faces, and the end faces bound with the environment (air).

Taking into account these limitations, instead of using the three-dimensional heat equation, we consider the one-dimensional non-stationary heat equation (1).

$$c\left(u\right)\rho\left(u\right)\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\left(u\right)\frac{\partial u}{\partial x}\right),\ x\in\left(0,\xi\right)\times\left(\xi,l\right),\ t\in\left(0,4t_{max}\right). \tag{1}$$

where $u\left(x,t\right)$ is the temperature distribution inside of the container chambers; x is the coordinate of the complex along the Ox axis; t is the current time. At the initial time of observation, the temperature distribution of the both chambers is: $t=0,\ u\left(x,0\right)=u_0\left(x\right), x\in\left(0,l\right)$. The ambient temperature at the left boundary of the region at $x=0$ will be denoted by $u_{ins}\left(t\right)$, and at the right boundary at $x=l$ we will denote by $u_{out}\left(t\right)$. In engineering calculations, the parameters c,ρ and κ are usually considered like constants [1]. However, many scientists [10,12,13,16,17] come to the conclusion that the study of nonlinear processes is of great practical interest. Since the vast majority of processes occurring in nature are non-linear. Taking into account the nonlinearity - greatly complicates the solution of the mathematical model.

The boundary conditions that determine the features of the process on the wall surface are given as follows: the left and right boundaries of the region $\Omega = (0, \xi) \times (\xi, l)$ are in contact with the gaseous medium (air), therefore, at these boundaries, it is advisable to formulate the Robin boundary condition - the relationship between the heat flux due to thermal conductivity from the solid wall and the heat flux from the gaseous medium. Thus, the boundary conditions on the left and right boundaries are written as follows:

$$x = 0: \ k_1(u) \frac{\partial u}{\partial x} = h_{ins}(u)(u - u_{ins}(t)), \tag{2}$$

$$x = l: \ k_2(u) \frac{\partial u}{\partial x} = -h_{out}(u)(u - u_{out}(t)), \tag{3}$$

here $u_{ins}(t)$, $u_{out}(t)$ is the ambient temperature; $h_{ins}(u)$, $h_{out}(u)$ are heat transfer coefficients; $k_1(u), k_2(u)$ are thermal conductivity coefficients of the «I» and «II» mediums (Fig.1).

Usually, boundary conditions are set on the contact surface of the layers $x = \xi$, which determine the equality of temperatures and heat fluxes at the junction of materials:

$$u_1(\xi, t) = u_2(\xi, \ t), \tag{4}$$

$$k_1(u) \frac{\partial u_1}{\partial x}(\xi, t) = k_2(u) \frac{\partial u_2}{\partial x}(\xi, t). \tag{5}$$

Here $u_1(x, t)$ and $u_2(x, t)$ are the temperatures of the material layers in contact. When solving problems with contact conditions of the form (4) - (5), the rate of convergence of the homogeneous difference scheme becomes very low [2]. Therefore, in order to avoid this problem, as well as to solve the inverse problem, we placed a separate sensor at the point $x = \xi$, which measures the change in soil temperature at the contact point of two media. Due to this, the original task is split into two tasks, i.e. using the measured data in each container, solves its own inverse problem of nonlinear thermal conductivity. Further, for brevity, we will describe the method for solving the inverse problem only on the left container (I), shown in Fig.1.

In addition to $u_{ins}(t)$, $u_{out}(t)$, the initial temperature values are measured - $T_{ins}(t), T_\xi(t), T_{out}(t)$, $t \in [0, 4t_{max}]$ where T_{ins}, T_ξ, T_{out} - respectively measured temperatures of materials at points $x = 0$, $x = \xi$ and $x = l$. For convenience of notation, we introduce the notation $h_{ins}(u) = h_1(u)$

Problem. Using the measured values $u_{ins}(t), T_{ins}(t), T_\xi(t), t \in [0, 4t_{max}]$, it is required to develop a method for finding the environment parameters $\rho_1(u)$, $c_1(u)$, $k_1(u)$, $h_1(u)$.

Since we consider only the left container, for simplicity we omit the indexes of the coefficients and use $k(u), c(u), \rho(u), h(u)$. We introduce the notation

Based on (1) - (5), the inverse problem is formulated as follows: in the region $Q_1 = (0, \xi) \times (0, 4t_{max})$ the following system is being studied

$$c(u)\rho(u) \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(u) \frac{\partial u}{\partial x}\right) \tag{6}$$

$$u(x, 0) = u_0(x) \tag{7}$$

$$k_1(u) \frac{\partial u}{\partial x} = h_1(u)(u - u_{ins}(t)), \quad x = 0 \tag{8}$$

$$u(\xi, t) = T_\xi(t) \tag{9}$$

where the following dependences on temperature are used for thermophysical characteristics: $c(u) = c_0 + c_1 u,\ \rho(u) = \rho_0 + \rho_1 u,\ k(u) = k_0 + k_1 u + k_2 u^2 + k_3 u^3,\ h(u) = h_0 + h_1 u + h_2 u^2$.

And the experimentally measured value on the left boundary of the region

$$T_{ins}(t), t \in [0, 4t_{max}]. \tag{10}$$

It is required to develop a method for finding thermophysical parameters $\rho(u),\ c(u),\ k(u),\ h(u)$.

Grid method. The segment $(0, \xi)$ is divided into I equals with a step $\Delta x = \xi / I$. Then $\xi = I\Delta x$, where I is the node number of the contact point $x = \xi$. And the segment $(0, t_{max})$ is divided into m equal parts with the step $\Delta t = \frac{t_{max}}{m}$.

As a result of this action, we get a grid

$\omega_1 = \{x_i = i\Delta x,\ t_j = j\Delta t,\ i = 0, 1, ..., I;\ j = 0, 1, ..., 4m\}$

Further, the problem (6) - (10) is solved by splitting in the grid domain

$\omega_{1s} = \{x_i = i\Delta x,\ t_j = j\Delta t,\ i = 0, 1, ..., I,\ j = (s-1)m, ..., s \cdot m;\ s = 1,\ 2,\ 3,\ 4\}$.

The difference scheme L_{11} is studied in the grid domain:

$$\begin{aligned}
&\rho\left(u_i^{j+1}\right) c\left(u_i^{j+1}\right) \frac{u_i^{j+1} - u_i^j}{\Delta t} = \\
&= \frac{1}{\Delta x}\left(k\left(u_{i+1/2}^{j+1}\right) \frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x} - k\left(u_{i-1/2}^{j+1}\right) \frac{u_i^{j+1} - u_{i-1}^{j+1}}{\Delta x}\right), \\
&i = 1, 2, ..., I-1;\ j = 0, 1, ..., m-1, \\
&u_i^0 = u_0(x_i),\ i = 0, 1, ..., I, \\
&u_I^{j+1} = T_\xi(t_{j+1}),\ j = 0, 1, ..., m-1, \\
&k\left(u_{1/2}^{j+1}\right) \frac{u_1^{j+1} - u_0^{j+1}}{\Delta x} = h\left(u_0^{j+1}\right)\left(u_0^{j+1} - u_{ins}^{j+1}\right),\ j = 0, 1, ..., m-1,
\end{aligned} \tag{11}$$

where $u_{i+\frac{1}{2}} = \frac{u_{i+1}^{j+1} + u_i^{j+1}}{2},\ i = 0, 1, ..., I-1$.

3 An iterative method for solving a nonlinear difference problem

For simplicity of presentation, consider the nonlinear difference problem (11), when $c = const$, $h = const$. Let's rewrite (11) as

$$\begin{aligned}
&C \cdot \frac{v_i^{j+1} - v_i^j}{\Delta x} = \frac{1}{\Delta x}\left[k\left(\frac{v_{i+1}^{j+1} + v_i^{j+1}}{2} + T_\xi^{j+1}\right) \frac{v_{i+1}^{j+1} - v_i^{j+1}}{\Delta x} - \right. \\
&\left. - k\left(\frac{v_i^{j+1} + v_{i-1}^{j+1}}{2} + T_\xi^{j+1}\right) \frac{v_i^{j+1} - v_{i-1}^{j+1}}{\Delta x}\right] - C \cdot \frac{T_\xi^{j+1} - T_\xi^j}{\Delta t}, \\
&\quad i = 1, 2, \ldots, N-1,\ j = 0, 1, \ldots,\ m-1,
\end{aligned} \tag{12}$$

$$v_i^0 = N_i^0, \quad i = 0, 1, \ldots,\ I, \quad N_i^0 = u_0(x_i) - T_\xi^0,$$

$$v_I^{j+1} = 0, \quad j = 0, 1, \ldots,\ m-1,$$

$$k\left(\frac{v_1^{j+1}+v_0^{j+1}}{2}+T_\xi^{j+1}\right)\cdot\frac{v_1^{j+1}-v_0^{j+1}}{\Delta x}=h\cdot v_0^{j+1}+h\left(T_\xi^{j+1}-u_{ins}^{j+1}\right),$$
$$j=0,1,\ldots,\ m-1.$$

Замечание 1 *The solution to the problem* (11) *and* (12) *are related by the equality*

$$U_i^{j+1}=V_i^{j+1}+T_\xi^{j+1}, i=0,1,\ldots,I;\quad j=0,1,\ldots,m-1.$$

The following statements are proved in the work:

Lemma 1 *If* $u_0(x)\in l_2(\Omega=(0,\xi)), T_\xi(t), u_{ins}(t)\in l_2[0,t_{max}]$, *then the solution of the problem* (12) *satisfies the estimate*

$$C\|V^{j+1}\|^2+2k\sum_{j=0}^{J}\|V_{\bar x}^{j+1}\|^2\Delta t+h\sum_{j=0}^{J}(V_0^{j+1})^2\Delta t\le C_3,$$

where C_3 is the limited value.

Lemma 2 *If the original functions have the property* $u_0(x)\in l_\infty(0,\xi), T_\xi(t)-u_{ins}(t)\in l_\infty(0,t_{max}), T_{\xi,\bar t}^{j+1}\in l_\infty(0,t_{max})$, *then the solution to the difference scheme* (12) *satisfies the estimate*

$$\max_{0\le i\le i-1}|V_i^{j+1}|\le\max_j|N_i^0|+\max_j|T_\xi^{j+1}-u_{ins}^{j+1}|+\max_j T_{\xi,\bar t}^{j+1}|=C_4,$$

where C_4 *is the limited value.*

Lemma 3 *Let* $|\sum_{i=1}^{I-1}V_{i,\bar t}^{j+1}\Delta t|$ *be limited value, then*

$$\max_i|V_{i,\bar x}^{j+1}|\le C_5<\infty,$$

where C_5 *is limited value, for every* j *from 0 to* $m-1$.

Lemma 4 *If there is a condition*

$$u_0(x)\in l_\infty(0,\xi), T'_\xi(t)-u'_{ins}(t)\in l_2(0,t_{max}), T_{\xi,\bar t t}^{j+1}\in l_2(0,t_{max})$$

and inequality

$$C\left|\sum_{i=1}^{I}V_{i,t}^j\right|\Delta t<\infty,\quad i=1,2,\ldots,I; j=0,1,\ldots,m-1,$$

then the solution to the difference scheme (12) *satisfies the estimate*

$$\|V_t^j\|^2+\sum_{j=0}^{J}\|V_{\bar x,t}^j\|^2\Delta t+\sum_{j=0}^{J}\left(V_{(0,t)}^j\right)^2\Delta t\le C_6<\infty.$$

where C_6 *is the limited value.*

The system of nonlinear algebraic equations (12) is solved by an iterative method for each

$$j = 0, 1, \ldots, m-1.$$

We fix the variable j and look for the desired value v_i^{j+1} in the form v_i^s, where s is the number of iterations. The initial approximation of the problem (12) when $s = 0$ is determined from the solution of the linearized problem for a fixed j :

$$\begin{aligned}
&C \cdot \frac{v_i^0 - v_i^j}{\Delta t} = \frac{1}{\Delta x}\left[k\left(\frac{v_{i+1}^j + v_i^{j+1}}{2} + T_\xi^{j+1}\right)\frac{v_{i+1}^0 - v_i^0}{\Delta x} - \right.\\
&\left. - k\left(\frac{v_i^j + v_{i-1}^j}{2} + T_\xi^{j+1}\right)\frac{v_i^0 - v_{i-1}^0}{\Delta x}\right] - C \cdot \frac{T_x i^{j+1} - T_\xi^j}{\Delta t},\\
&k\left(\frac{v_1^j + v_0^j}{2} + T_\xi^{j+1}\right) \cdot \frac{v_1^1 - v_0^0}{\Delta x} = h \cdot v_0^0 + h\left(T_\xi^{j+1} - u_{ins}^{j+1}\right),\\
&i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, m-1,\\
&v_i^0 = N_i^0, \quad i = 0, 1, \ldots, N,\\
&v_N^0 = 0.
\end{aligned} \tag{13}$$

Before proceeding to solve the (12) problem by the iterative method, let's estimate the error of the initial approximation described by the (13) system. To do this, subtract the system (13) from (12). We introduce the notation

$$\Delta v_i^0 = v_i^{j+1} - v_i^0, \qquad i = 0, 1, \ldots, I.$$

Then

$$\begin{aligned}
&C \cdot \frac{\Delta v_i^0}{\Delta t} = \left[k'\left(\theta v_{i+\frac{1}{2}}^{j+1} + (1-\theta)\, v_{i+\frac{1}{2}}^j + T_\xi^{j+1}\right)\frac{1}{2}\left(v_{i+1}^{j+1} + v_i^{j+1}\right)_{\bar{t}} \cdot\right.\\
&\left.\cdot \Delta t \cdot v_{i,x}^{j+1} - k\left(\frac{v_{i+1}^j + v_i^j}{2} + T_\xi^{j+1}\right) \cdot \Delta v_{i,x}^0\right]_{\bar{x}},\\
&i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, m-1,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\Delta v_i^0 = 0, \quad i = 0, 1, \ldots, N,\\
&\Delta v_N^0 = 0, \quad j = 0, 1, \ldots, m-1,
\end{aligned}$$

$$\begin{aligned}
&k'\left(\theta v_{\frac{1}{2}}^{j+1} + (1-\theta)\, v_{\frac{1}{2}}^j + T_\xi^{j+1}\right) \cdot \frac{1}{2}\left(v_1^{j+1} + v_0^{j+1}\right)_{\bar{t}} \cdot v_{1,x}^{j+1} +\\
&+ k\left(\frac{v_1^j + v_0^j}{2} + T_\xi^{j+1}\right) \cdot \Delta v_{i,x}^0 = h_1 \Delta v_0^0,\\
&j = 0, 1, \ldots, \ m-1.
\end{aligned} \tag{15}$$

Замечание 2 *It can be proved that the previous lemmas hold for* (14).

To obtain an estimate for the solution of the (14) problem, we multiply the first equation of the (14) system by $\Delta v_i^0 \Delta t\ \Delta x$ and sum over i from 1 to $I-1$. We apply the summation by parts formula over the variable i, and taking into account the boundary conditions of the problem (14) we have the equality

$$C\|\Delta v^0\| + h|\Delta v_0^0|\Delta t + \sum_{i=1}^{I} k\left(\frac{v_{i+1}^j + v_i^j}{2} + T_\xi^{j+1}\right)\cdot\left(\Delta v_{i,\bar{x}}^0\right)^2 \Delta t \Delta x =$$

$$= -\Delta t \sum_{i=1}^{I} k'\left(\theta v_{i+\frac{1}{2}}^{j+1} + (1-\theta)\, v_{i+\frac{1}{2}}^{j} + T_\xi^{j+1}\right)\cdot v_{i+\frac{1}{2},\bar{t}}\cdot v_{i,\bar{x}}^{j+1}\cdot \Delta v_{i,\bar{x}}^0 \Delta x \Delta t.$$

The third lemma says that

$$\max_i |v_{i,\bar{x}}^{j+1}| < C_7, \quad j = 0, 1, \ldots,\ m-1.$$

Taking into account this information, applying the Cauchy inequality, we have the inequality

$$C\|\Delta v^0\|^2 + h|\Delta v_0^0| + k\|\Delta v_{\bar{x}}^0\|^2 \Delta t \le C_7 \Delta t \cdot \|v_{i+\frac{1}{2},\bar{t}}^{j+1}\| \cdot \|\Delta v_{\bar{x}}^0\|.$$

But $v_{i-\frac{1}{2},\bar{t}} = \frac{1}{2}\left(v_{i-1,\bar{t}}^{j+1} + v_{i,\bar{t}}^{j+1}\right)$, therefore

$$\sum_{i=1}^{N}\left(v_{i-\frac{1}{2},\bar{t}}\right)^2 \Delta x = \sum_{i=1}^{N}\left(v_{i,\bar{t}}\right)^2 \Delta x = \|v_{\bar{t}}^{j+1}\|^2$$

.

Applying ε - Cauchy's inequality, we deduce the estimate

$$C_1\|\Delta v^0\|^2 + h_1|\Delta v_0^0|\Delta t + \left(k - \frac{\varepsilon}{2}\right)\|\Delta v_{\bar{x}}^0\|^2 \Delta t \le \frac{C_8}{2\varepsilon}\|v_{\bar{t}}^{j+1}\|^2 \Delta t.$$

Let $\varepsilon = \frac{k}{C_2}$, then the inequality

$$C\|\Delta v^0\|^2 + h|\Delta v_0^0| + \frac{k}{2}\|\Delta v_{\bar{x}}^0\|^2 \Delta t \le \frac{C_8^2}{2k}\|v_{\bar{t}}^{j+1}\|^2 (\Delta t)^2.$$

In the fourth lemma, we proved that $\|v_{\bar{t}}^{j+1}\|^2 < \infty$, so we finally get the estimate

$$q_0 = C\|v_i^{j+1} - v_i^0\|^2 + h_1|v_0^{j+1} - v_0^0|\Delta t + \frac{k}{2}\|v_{\bar{x}}^{j+1} - v_{\bar{x}}^0\|^2 \Delta t \le C_9 (\Delta t)^2. \tag{16}$$

Theorem 1 *If the initial approximation of Newton's method is taken from the solution of the linearized problem* (13)*, then the solution of the system of nonlinear algebraic equations in the iterative Newton scheme* (12) *converges, and the convergence rate estimate takes place*

$$q^{s+1} \le q_0^{2^{s+1}}, \quad s = 0, 1, \ldots,$$

$$q^s = C\frac{\|\Delta v^s\|^2}{\sqrt{\Delta t}} + k\|\Delta v_{\bar{x}}^S\|^2\sqrt{\Delta t} + h\left(\Delta v_0^S\right)^2\sqrt{\Delta t}, \quad \Delta v_i^s = v_i^{j+1} - v_i^s.$$

Figure 2: Soil containers

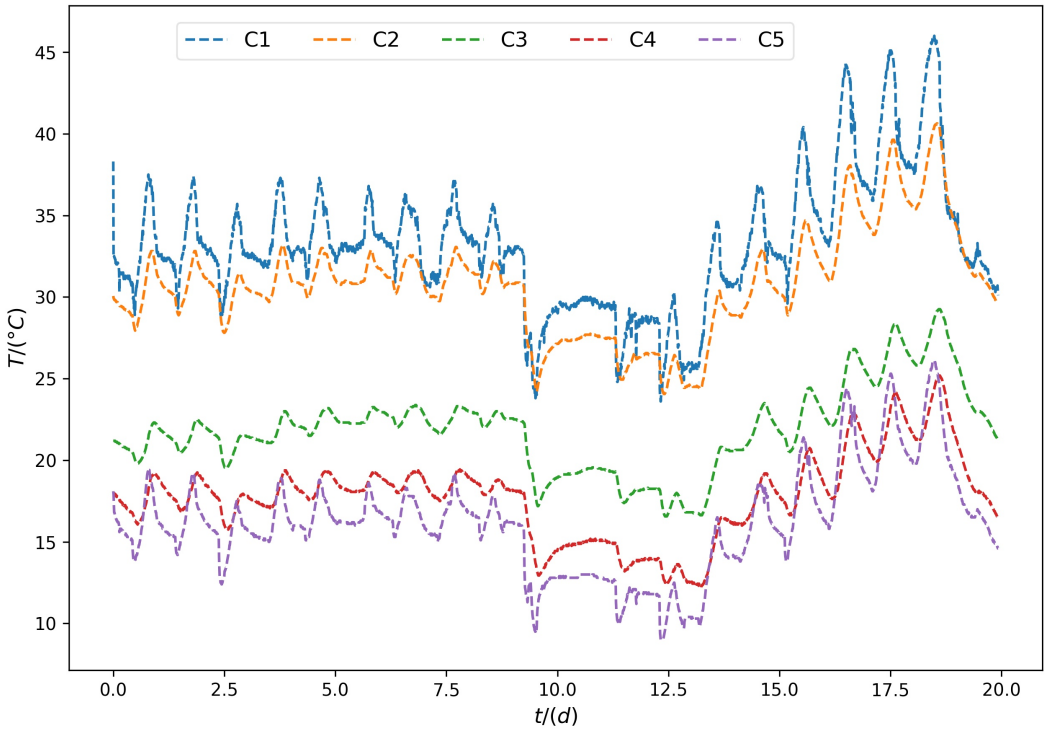

Figure 3: Measured temperature at five points of the container.

4 Experimental setup

During the experiment, the data of the problem of one-dimensional heat and mass transfer for various soils were obtained.

Photos of containers are shown in Fig.4. The side edges of the chambers consist of 2 cm thermally insulated material, and the end edges are in bound with the environment (air). In each compartment of the container, 15 cm long, there are various soils. One end side is heated with lamps. The second outer side is affected by the ambient temperature.

3 sensors (C2, C3, C4) are evenly distributed inside the material as shown in Fig.1. They produce temperature measurements with an error of 0.3 degrees Celsius according to the technical passport of the sensor. In addition to these sensors, there are 2 more sensors (C1, C5) close to the ends to measure the ambient temperature. The errors of these sensors are the same as those of the previous sensors. The temperature data measurement is taken at intervals of 10 minutes.

For calculations, a two-chamber container was considered and, accordingly, with two materials: sand and black soil. The data were measured over the course of three months, and the physical length of the entire container was determined through the interval $x \in (0, l)$, where $l = 30$ cm. The boundary of the two media is at a distance of $x = 15$ cm and there is also a temperature measurement sensor. Since there is an exchange with the environment at the end boundaries, Robin boundary conditions were considered for the numerical solution. Measurements at the points $x = 0$ cm and $x = 30$ cm determine the temperature at the end boundaries. The temperature values of the measured data can be seen in Fig.3.

5 Results

The measured temperature data were used to solve a numerical problem to find all thermophysical coefficients (thermal conductivity coefficient, specific heat capacity, specific density and heat transfer coefficient). As can be seen from Fig.4 and 5, thanks to the steepest descent method, the functionals converge fairly quickly and reach a minimum in 6 and 7 iterations. The minimization of the functional continued until the relative error between the nonlinear solution and the experimental data reached 4.3% for chernozem and 3.12% for sand, which in turn shows a fairly good accuracy of the solution. If we look at the absolute errors in two environments - 6.3% and 5.3%, we see that they also meet our expectations. Due to the fact that it is difficult for the reader to evaluate each parameter separately, in Fig.5, Fig.6 and Fig.7 functions of thermophysical coefficients are given depending on temperature for each medium. For clarity, the interval from $10°C$ to $50°C$ was used. The graphs show that with increasing temperature, the values of thermophysical parameters increase. In this case, it is possible to evaluate the behavior of the thermal conductivity and heat capacity coefficients. It can be seen that they show a stronger dependence on temperature values than the density coefficient, which in turn is confirmed by the theoretical basis. It is also worth noting that the data obtained are in good agreement with the tabular, experimental values [6]. And in the case of the heat transfer coefficient, it can be seen that it practically does not change with increasing temperature. This is explained by the fact that the heat transfer is affected by the difference between the ambient temperature and the temperature at the end boundaries of the soil-soil. In our case, according to the experimental data, it can be seen that the difference is not large.

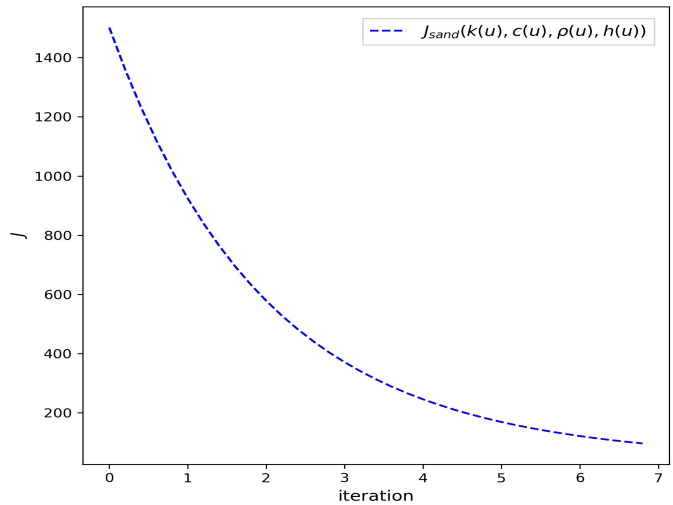

Figure 4: Convergence graph of functionals for two media: sand

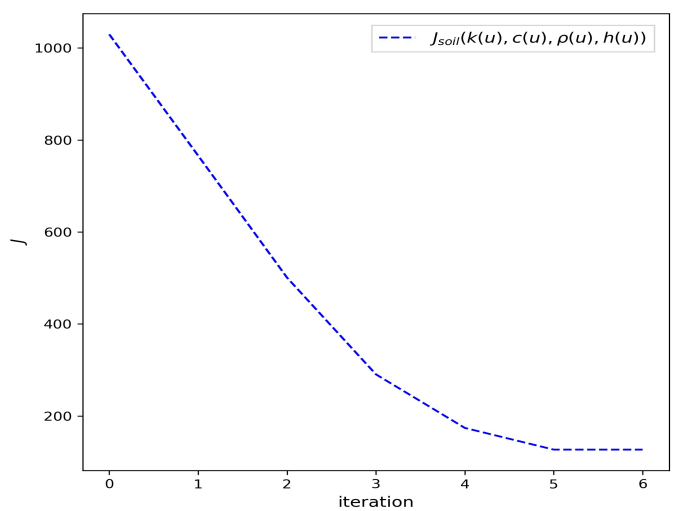

Figure 5: Convergence graph of functionals for two media: soil

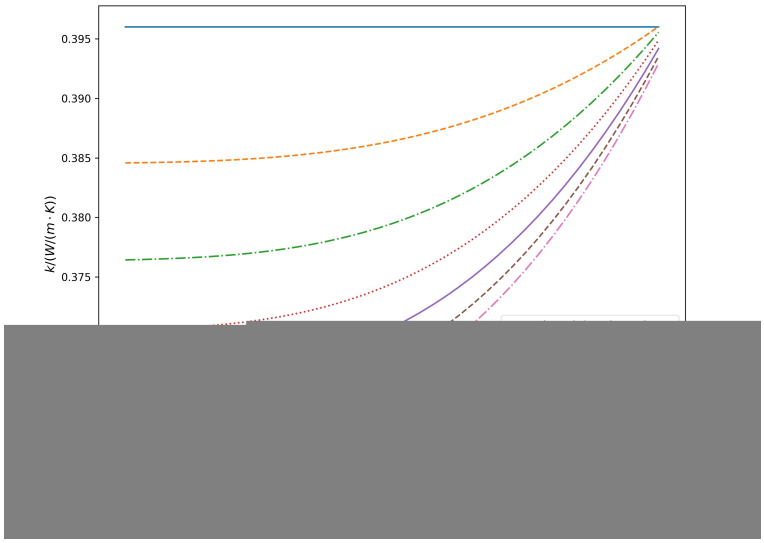

Figure 6: Changing the heat conduction function at each iteration of the calculation from the initial data. Sand

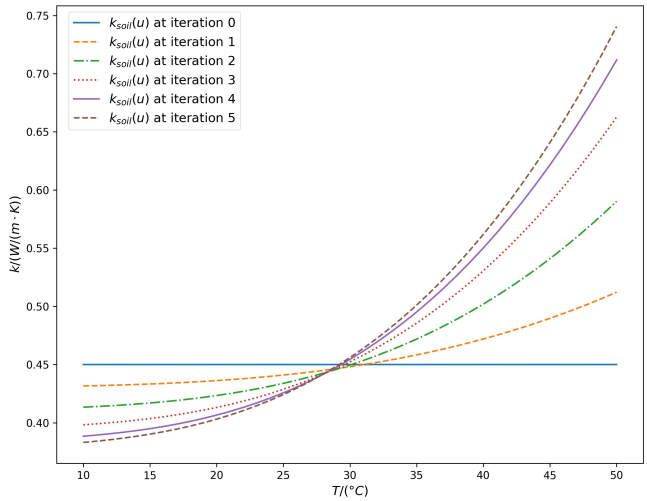

Figure 7: Changing the heat conduction function at each iteration of the calculation from the initial data. The soil

6 Conclusion

Thus, the constructed numerical method allows obtaining an approximate solution to the problem of fluid flow in a fractured porous medium with the second order in both time and

spatial variable. The results of computational experiments carried out for various orders of fractional derivatives and grid configurations fully confirm the results of theoretical analysis. The methods used and the conclusions drawn, described in the work, can be used to solve other classes of fractional differential equations.

7 Acknowledgement

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References

[1] Desta T. Z., Langmans J., Roels S., "Experimental data set for validation of heat, air and moisture transport models of building envelopes " , *Building and Environment,* (2011).

[2] Rysbaiuly B., Adamov A.A., "Matematicheskoe modelirovanie tepla i massoobmennogo processa v mnogosloynom grunte " [Monographiya] *Izdatelskyi dom "Kazak Universitety* (2020): 211.

[3] Luikov A.V., "Heat and Mass Transfer in Capillary-Porous Bodies" , (1964).

[4] Berger J., Dutykh D., Mendes N., Rysbaiuly B., "SA new model for simulating heat, air and moisture transport in porous building material" , *International Journal of Heat and Mass Transfer,* 134 (2019): 1041-1060.

[5] Tien-Mo Shih, Chao-Ho Sung, Bao Yang, "A Numerical Method for Solving Nonlinear Heat Transfer Equations" , *Numerical Heat Transfer, Part B: Fundamentals: An International Journal of Computation and Methodology,* (2008).

[6] Mazumder S., "Numerical Methods for Partial Differential Equations: Finite Difference and Finite Volume Methods" , *Academic Press* (2016).

[7] Lopushansky A., Lopushansky O., Sharyn S., "Nonlinear inverse problem of control diffusivity parameter determination for a space-time fractional diffusion equation" , *Applied Mathematics and Computation* (2021).

[8] Moore T., Jones M., "Solving nonlinear heat transfer problems using variation of parameters" , *International Journal of Thermal Sciences* (2015).

[9] Battaglia J-L., Maachou A., Malti R., Melchior P., "Nonlinear heat diffusion simulation using Volterra series expansion" , *International Journal of Thermal Sciences* (2013).

[10] Nguyen Huy Tuana, Pham Hoang Quanc, "Some extended results on a nonlinear ill-posed heat equation and remarks on a general case of nonlinear terms" , *Nonlinear Analysis: Real World Applications* (2011).

[11] Huntul M., Lesnic D., "Determination of the time-dependent convection coefficient in two-dimensional free boundary problems" , *Engineering Computations* (2020).

[12] Jumabekova A., Berger J., Dutykh D., Le Meur H., "An efficient numerical model for liquid water uptake in porous material and its parameter estimation" , *Numerical Heat Transfer, Part A: Applications* (2019).

[13] Hasanov A., "Lipschitz continuity of the Fr?chet gradient in an inverse coefficient problem for a parabolic equation with Dirichlet measured output" , *Journal of Inverse and Ill-Posed Problems* (2018).

[14] Cao K., Lesnic D., Ismailov M., "Determination of the time-dependent thermal grooving coefficient" , *Journal of Inverse and Ill-Posed Problems* (2021).

[15] Lesnic D., Hussein M.S., Kamynin V., Kostina B., "Direct and inverse source problems for degenerate parabolic equations" , *Journal of Inverse and Ill-Posed Problems* (2020).

[16] Kabanikhin S. I., Shishlenin M.A., "Theory and numerical methods for solving inverse and ill-posed problems" , *Journal of Inverse and Ill-Posed Problems* (2019).

[17] Lesnic D., "Inverse Problems with Applications in Science and Engineering" , *JRC Press* (Abingdon, UK, 2021): 349.

[18] Rysbaiuly B., Rysbaeva N., "The method of solving nonlinear heat transfer model in freezing soil", *Eurasian Journal of Mathematical and Computer Applications (EJMCA).* V.8, No 4 (2020): 83-96.

[19] Rysbaiuly B., "Obratnye zadachi nelineynoy teploperedachi", *Kazakh Universitety* (2020): 369.

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