ISSN 1563 – 0277
eISSN 2617 – 4871

ӘЛ-ФАРАБИ атындағы ҚАЗАҚ ҰЛТТЫҚ УНИВЕРСИТЕТІ

ХАБАРШЫ

Математика, механика, информатика сериясы

КАЗАХСКИЙ НАЦИОНАЛЬНЫЙ УНИВЕРСИТЕТ имени АЛЬ-ФАРАБИ

ВЕСТНИК

Серия математика, механика, информатика

AL-FARABI KAZAKH NATIONAL UNIVERSITY

Journal of Mathematics, Mechanics and Computer Science

№2 (110)

Алматы
«Қазақ университеті»
2021

Зарегистрирован в Министерстве информации и коммуникаций Республики Казахстан, свидетельство №16508-Ж от 04.05.2017 г. (Время и номер первичной постановки на учет №766 от 22.04.1992 г.). Язык издания: казахский, русский, английский. Выходит 4 раза в год. Тематическая направленность: теоретическая и прикладная математика, механика, информатика.

Индексируется

Научное издание
Вестник КазНУ. Серия "Математика, механика, информатика", № 2 (110) 2021.
Редактор – С.Е. Айтжанов. Компьютерная верстка – С.Е. Айтжанов

ИБ N 14672
Формат 60 × 84 1/8. Бумага офсетная. Печать цифровая. Объем 9,8 п.л.
Заказ N 6505. Издательский дом "Қазақ университетi"
Казахского национального университета им. аль-Фараби. 050040, г. Алматы, пр.аль-Фараби, 71, КазНУ.
Отпечатано в типографии издательского дома "Қазақ университетi".

1-бөлім **Раздел 1** **Section 1**

Математика **Математика** **Mathematics**

IRSTI 27.29.19; 27.29.23 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.01

S.I. Mitrokhin
Russian Academy of Natural Sciences, Lomonosov Moscow State University,
Research Computing Center, Russia, Moscow
E-mail: mitrokhin-sergey@yandex.ru

ASYMPTOTICS OF THE EIGENVALUES OF A PERIODIC BOUNDARY VALUE PROBLEM FOR A DIFFERENTIAL OPERATOR OF ODD ORDER WITH SUMMABLE OPERATOR

The paper is devoted to the study of spectral properties of differential operators of arbitrary odd order with a summable potential and periodic boundary conditions. For large values of the spectral parameter the asymptotics of the solutions of the differential equation that defines the differential operator is obtained. Differential equation that defines the differential operator is reduced to the Volterra integral equation. The integral equation is solved by Picards method of successive approximations. The method of studing of operators with a summable potential is an extension of the method of studing operators with piecewise smooth coefficients. The study of periodic boundary conditions leads to the study of the roots of the entire function represented in the form of an arbitrary odd order determinant. To obtain the roots of this function, the indicator diagram has been examined. The roots of this equation are in the sectors of an infinitesimal angle, determined by the indicator diagram. In the paper the asymptotics of eigenvalues of the differential operator under consideration is found. The obtained formulas make it impossible to study the spectral properties of the eigenfunctions and to derive the formula for the first regularized trace of the differential operator under study.

Key words: Differential operator of odd order, spectral parameter, summable potential, periodic boundary conditions, indicator diagram, asymptotics of solutions, asymptotics of eigenvalues.

С.И. Митрохин
Ресей жаратылыстану ғылымдары академиясы, М.В. Ломоносов атындағы Мәскеу мемлекеттік университетінің ғылыми-зерттеу орталығы, Ресей, Мәскеу қ.
E-mail: mitrokhin-sergey@yandex.ru

Қосылуыш операторы бар тақ ретті дифференциалдық оператор үшін периодтық шекара есебінің меншік мәндерінің асимптотикасы

Бұл жұмыста дифференциалдық операторлардың спектрлік қасиеттерін зерттеуге арналған. Спектрлік параметрдің үлкен мәндері үшін дифференциалдық операторды анықтайтын дифференциалдық теңдеу шешімдерінің асимптотикасы алынады. Дифференциалдық операторды анықтайтын дифференциалдық теңдеу Вольтерраның интегралдық теңдеуіне келтірілген. Интегралдық теңдеу Пикард әдісімен шешіледі. Жиынтық әлеуеті бар операторларды оқыту әдісі біртектес коэффициенттері бар операторларды оқыту әдістемесін кеңейту болып табылады. Периодтық шекаралық шарттарды зерттеу тақ тәрізді ерікті детерминант ретінде ұсынылған бүтін функцияның түбірлерін зерттеуге алып келеді. Бұл функцияның тамырларын білу үшін индикаторлық диаграмма зерттелді. Бұл теңдеудің түбірлері индикаторлық диаграммамен анықталған шексіз бұрыштың секторларында жатыр. Бұл мақалада дифференциалдық оператордың меншікті мәндерінің асимптотикасы келтірілген. Нәтижеде алынған формулалар меншікті функциялардың спектрлік қасиеттерін зерттеуге және зерттелетін дифференциалдық оператордың алғашқы регулирленген ізінің формуласын шығаруға мүмкіндік береді.

Түйін сөздер: Тақ ретті дифференциалдық оператор, спектрлік параметр, жиынтық потенциал, периодтық шекаралық шарттар, индикаторлық диаграмма, ерітінділердің асимптотикасы, меншікті мәндердің асимптотикасы.

С.И. Митрохин
Российская академия естественных наук, Научно-исследовательский вычислительный центр Московского государственного университета имени М.В. Ломоносова, Россия, г. Москва
E-mail: mitrokhin-sergey@yandex.ru

Асимптотика собственных значений периодической краевой задачи для дифференциального оператора нечетного порядка с суммируемым оператором

Работа посвящена изучению спектральных свойств дифференциальных операторов произвольного нечетного порядка с суммируемым потенциалом и периодическими граничными условиями. При больших значениях спектрального параметра получена асимптотика решений дифференциального уравнения, определяющего дифференциальный оператор. Дифференциальное уравнение, определяющее дифференциальный оператор, сводится к интегральному уравнению Вольтерра. Интегральное уравнение решается методом последовательных приближений Пикара. Метод обучения операторов с суммируемым потенциалом является расширением метода обучения операторов с кусочно гладкими коэффициентами. Изучение периодических граничных условий приводит к изучению корней целой функции, представленной в виде произвольного определителя нечетного порядка. Для получения корней этой функции была исследована индикаторная диаграмма. Корни этого уравнения лежат в секторах бесконечно малого угла, определяемого диаграммой индикатора. В статье найдена асимптотика собственных значений рассматриваемого дифференциального оператора. Полученные формулы не позволяют исследовать спектральные свойства собственных функций и вывести формулу для первого регуляризованного следа исследуемого дифференциального оператора.

Ключевые слова: Дифференциальный оператор нечетного порядка, спектральный параметр, суммируемый потенциал, периодические граничные условия, индикаторная диаграмма, асимптотика решений, асимптотика собственные значения.

1 Statement of the problem

Let us investigate the spectrum of a differential operator arbitrary odd order defined on an interval $[0;\pi]$ by a differential equation of the form

$$y^{(2N+1)}(x) + q(x)y(x) = \lambda a^{2N+12} y(x), \quad 0 \le x \le \pi, \quad a > 0, \quad N = 1, 2, 3, \dots, \tag{1}$$

with periodic boundary conditions

$$y(0) = y(\pi), \quad y^{(m)}(0) = y^{(m)}(\pi), \quad m = 1, 2, \dots, 2N, \tag{2}$$

where the number λ is called the spectral parameter, the function $q(x)$ is called the potential and we assume that the potential $q(x)$ is a summable function on the segment $[0;\pi]$

$$q(x) \in L_1[0;\pi] \Leftrightarrow \left(\int\limits_0^x q(t)dt \right)'_x = q(x) \tag{3}$$

for almost all values x from the segment $[0;\pi]$.

The spectral properties of differential operators were first studied in the case when the coefficients of the differential equations defining these operators were sufficiently smooth functions. The asymptotic formulas for the roots of quasipolynomials, which are obtained in the study of higher-order operators with regular boundary conditions with smooth coefficients, were obtained in paper [1].

In paper [2], the traces of higher-order ordinary differential operators with sufficiently smooth coefficients were calculated.

In work [3], the author studied the spectral properties of differential operators with piecewise smooth coefficients. In paper [4], we studied a differential operator in which not only the potential is a discontinuous function, but the weight function was also piecewise smooth.

In paper [5], a second-order operator with a summable potential was studied, the asymptotics of the eigenvalues and eigenfunctions of the Sturm—Liouville boundary value problem on a segment were calculated. A new method for studying differential operators with summable coefficients, whose order is higher than the second, was developed by the author in papers [6–8]. In all these works, the boundary conditions were separated. The periodic boundary conditions that we study in this paper are a classic example of nonseparated boundary conditions.

The spectral properties of differential operators with periodic boundary conditions with smooth potentials were studied in papers [9–11]. Interest in the study of such operators is caused by physical applications: in the case of fourth-order operators, they describe a model of a beam or plate with a hinged joint or with fixed ends. Operators with periodic boundary conditions were also studied in papers [12, 13].

In papers [14, 15] the author studied model examples of differential operators with summable potential with periodic boundary conditions. The differential operators of odd order with periodic boundary conditions have not actually been studied. A third-order operator on the real axis with periodic boundary conditions was studied in paper [16].

2 Asymptotics of solutions of differential equation (1) for large values of the spectral parameter λ

Let us introduce the following notation: $\lambda = s^{2N+1}$, $s = \sqrt[2N+1]{\lambda}$, while fixing that branch of the arithmetic root for which $\sqrt[2N+1]{1} = +1$. Let us denote by ω_k $(k = 1, 2, \ldots, 2N+1)$ the various roots of the $(2N+1)$-th degree from the unity:

$$\begin{aligned} &\omega_k^{2N+1} = 1,\ \omega_k = e^{\frac{2\pi i(k-1)}{2N+1}}\ (k = 1, 2, \ldots, 2N+1);\\ &\omega_1 = 1;\ \omega_2 = e^{\frac{2\pi i}{2N+1}} = \cos\left(\frac{2\pi}{2N+1}\right) + i\sin\left(\frac{2\pi}{2N+1}\right) = z^2; \ldots;\\ &\omega_m = z^{m-1},\ m = 1, 2, \ldots, 2N+1. \end{aligned} \tag{4}$$

The numbers ω_k $(k = 1, 2, \ldots, 2N+1)$ from (4) decide the unit circle into $(2N+1)$ an equal part, and they satisfy the following relations:

$$\sum_{k=1}^{2N+1} \omega_k^m = 0,\ m = 1, 2, \ldots, 2N; \quad \sum_{k=1}^{2N+1} \omega_k^m = 2N+1,\ m = 0,\ m = 2N+1. \tag{5}$$

The following statement is established by the method of variation of arbitrary constants under the condition (3) of the summability of the potential $q(x)$.

Theorem 1 *The solution $y(x,s)$ of the differential equation* (1) *is the solution to the following integral equation:*

$$y(x,s)=\sum_{k=1}^{2N+1}C_ke^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_ke^{a\omega_k sx}\int\limits_0^x q(t)e^{-a\omega_k st}y(t,s)dt, \tag{6}$$

where C_k $(k=1,2,\ldots,2N+1)$ are arbitrary constants.

Proof. Let us prove that the function $y(x,s)$ from (6) is indeed a solution of the differential equation (1). Since, in view of condition (3) $[q(x)\in L_1[0;\pi]$, the function $e^{-a\omega_k sx}$ is infinitely differentiable with respect to the variable x, the function $y(x,s)$ must satisfy the equation (1), which means that it must be $(2N+1)$ times differentiable with respect to variable x, then the function $G(x,s)=q(x)e^{-a\omega_k sx}y(x,s)\in L_1[0;\pi]$ and then the relation

$$\left(\int\limits_0^x G(t,s)dt\right)_x'=\left(\int\limits_0^x a(t)e^{-a\omega_k st}y(t,s)dt\right)_x'=q(x)e^{-a\omega_k sx}y(x,s) \tag{7}$$

holds for almost all x from segment $[0;\pi]$.

Differentiating the function $y(x,s)$ from (6) with respect to the variable x, using the properties (3) and (7), we get:

$$y'(x,s)=\sum_{k=1}^{2N+1}C_k(a\omega_k s)e^{a\omega_k sx}-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)e^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_1(x,s), \tag{8}$$

$$M_N=(2N+1)a^{2N}s^{2N},\quad \phi_1(x,s)=-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_ke^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s)\overset{(2.2)}{=}0. \tag{9}$$

From formulas (8), (9), using the properties (3) and (7), we have:

$$y''(x,s)=\sum_{k=1}^{2N+1}C_k(a\omega_k s)^2e^{a\omega_k sx}-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^2e^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_2(x,s), \tag{10}$$

$$\phi_2(x,s)=-\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s)=-\frac{asq(x)y(x,s)}{M_N}\sum_{k=1}^{2N+1}\omega_k^2\overset{(2.2)}{=}0. \tag{11}$$

Similarly, the following formulas are derived:

$$\begin{gathered}y^{(n)}(x,s)=\sum_{k=1}^{2N+1}C_k(a\omega_k s)^ne^{a\omega_k sx}-\\ -\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^ne^{a\omega_k sx}\int\limits_0^x G(t,s)dt-\frac{1}{M_N}\phi_n(x,s),n=3,4,\ldots,2N,\end{gathered} \tag{12}$$

$$\begin{gathered}\phi_n(x,s) = -\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{n-1}e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s) = \\ = -\frac{(as)^{n-1}q(x)y(x,s)}{M_N}\sum_{k=1}^{2N+1}(\omega_k)^n \overset{(2.2)}{=} -\frac{(as)^{n-1}q(x)y(x,s)\cdot 0}{M_N} = 0, \quad n = 3,4,\ldots,2N.\end{gathered} \tag{13}$$

Substituting the formulas (12), (13) at $n = 2N$ in the differential equation (1), we obtain;

$$\begin{gathered}y^{2N+1)}(x,s) + q(x)y(x,s) - \lambda^{2N+1}y(x,s) \overset{(2.3)}{=} \sum_{k=1}^{2N+1}C_k(a\omega_k s)^{2N+1}e^{a\omega_k sx} - \\ -\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{2N+1}e^{a\omega_k sx}\int_0^x G(t,s)dt - \\ -\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k(a\omega_k s)^{2N}e^{a\omega_k sx}q(x)e^{-a\omega_k sx}y(x,s) + q(x)y(x,s) - \\ -\lambda a^{2N+1}\sum_{k=1}^{2N+1}C_k e^{a\omega_k sx} + \lambda a^{2N+1}\frac{1}{M_N}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}\int_0^x G(t,s)dt = \\ = -q(x)y)x,s) + q(x)y(x,s) = 0\end{gathered} \tag{14}$$

for almost all x from the interval $[0;\pi]$, which means that the function $y(x,s)$ from (6) is indeed a solution of the differential equation (1).

(In equation (14)), the first and fifth terms cancel out, the second and sixth terms cancel out due the fact that $\lambda = s^{2N+1}$, $\omega_k^{2N+1} = 1$, $M_N = (2N+1)a^{2N}s^{2N}$.

Futher, the asymptotics of solution of the differential equation (1) will be found by the method of successive approximations of Picard: from the formula (6) we obtain the function $y(t,s)$ and substitute it into equation (6):

$$\begin{gathered}y(x,s) = \sum_{k=1}^{2N+1}C_k e^{a\omega_k sx} - \frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_k e^{a\omega_k sx}\int_0^x q(t)e^{-a\omega_k st}\times \\ \times\left[\sum_{n=1}^{2N+1}C_n e^{a\omega_n st} - \frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{k=1}^{2N+1}\omega_n e^{a\omega_n st}\int_0^t q(\xi)e^{-a\omega_n s\xi}y(\xi,s)d\xi\right]dt.\end{gathered} \tag{15}$$

Changing the order of summation in formula (15), performing the necessary transformations and estimates similar to the monograph [1, chapter 2], we come to the conclusion that the following statement is true.

Theorem 2 *The general solution of the differential equation* (1) *is represented in the following form:*

$$
\begin{aligned}
&y(x,s)=\sum_{k=1}^{2N+1} C_k y_k(x,s);\\
&y^{(m)}(x,s)=\sum_{k=1}^{2N+1} C_k y_k^{(m)}(x,s),\ \ m=1,2,\dots,2N,
\end{aligned}
\tag{16}
$$

where C_k $(k=1,2,\dots,2N+1)$ *are arbitrary constants, and the following asymptotic expansions and estimates are valid for the fundamental system of solutions* $\{y_k(x,s)\}_{k=1}^{2N+1}$*:*

$$
\begin{aligned}
&y_k(x,s)=e^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{n=1}^{2N+1}\omega_n e^{a\omega_n sx}\int_0^x q(t)e^{a(\omega_k-\omega_n)st}dt_{akn}+\\
&++\underline{O}\left(\frac{e^{|\Im s|ax}}{s^{4N}}\right),\quad k=1,2,\dots,2N+1,\quad y_k(0,s)=1;
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
&y_k^{(m)}(x,s)=(as)^m\Bigg\{\omega_k^m e^{a\omega_k sx}-\frac{1}{(2N+1)a^{2N}s^{2N}}\sum_{n=1}^{2N+1}\omega_n^{m+1}e^{a\omega_n sx}\times\\
&\times\int_0^x q(t)e^{a(\omega_k-\omega_n)st}dt_{akn}+\underline{O}\left(\frac{e^{|\Im s|ax}}{s^{4N}}\right)\Bigg\},\\
&k=1,2,\dots,2N+1,\quad m=1,2,\dots,2N;\quad y_k^{(m)}(0,s)=(as)^m\omega_k^m.
\end{aligned}
\tag{18}
$$

3 The study of boundary conditions (3)

Using the formulas (16), from the boundary conditions (2) we get:

$$
\begin{cases}
y(\pi,s)\overset{(1.2)}{=}y(0,s)\Leftrightarrow \sum\limits_{k=1}^{2N+1} C_k y_k(\pi,s)=\sum\limits_{k=1}^{2N+1} C_k y_k(0,s)\Leftrightarrow\\
\Leftrightarrow \sum\limits_{k=1}^{2N+1} C_k y_k[(\pi,s)-y_k(0,s)]=0\overset{(2.7)}{\Leftrightarrow}\sum\limits_{k=1}^{2N+1} C_k[y_k(\pi,s)-1]=0;
\end{cases}
\tag{19}
$$

$$
\begin{cases}
\dfrac{y^{(m)}(\pi,s)}{(as)^m}\overset{(1.2)}{=}\dfrac{y^{(m)}(0,s)}{(as)^m}\Leftrightarrow \sum\limits_{k=1}^{2N+1} C_k\left[\dfrac{y_k^{(m)}(\pi,s)}{(as)^m}-\dfrac{y_k^{(m)}(0,s)}{(as)^m}\right]=0,\\
m=1,2,\dots,2N.
\end{cases}
\tag{20}
$$

Theorem 3 *The eigenvalue equation of the differential operator ((1))–((3)) has the following form:*

$$\begin{vmatrix} y_1(\pi,s)-y_1(0,s) & y_2(\pi,s)-y_2(0,s) & \dots & y_{2N+1}(\pi,s)-y_{2N+1}(0,s) \\ \frac{y_1'(\pi,s)}{as}-\dfrac{y_1'(0,s)}{as} & \frac{y_2'(\pi,s)}{as}-\dfrac{y_2'(0,s)}{as} & \dots & \frac{y_{2N+1}'(\pi,s)}{as}-\dfrac{y_{2N+1}'(0,s)}{as} \\ \dots & \dots & \dots & \dots \\ \frac{y_1^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_1^{(2N)}(0,s)}{(as)^{2N}} & \frac{y_2^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_2^{(2N)}(0,s)}{(as)^{2N}} & \dots & \frac{y_{2N+1}^{(2N)}(\pi,s)}{(as)^{2N}}-\dfrac{y_{2N+1}^{(2N)}(0,s)}{(as)^{2N}} \end{vmatrix}=0. \quad (21)$$

Applying the asymptotic formulas (17), (18), we rewrite the equation (21) in the following form:

$$f(s)=\begin{vmatrix} D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{1,2N+1} \\ \omega_1 D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n^2 e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{2,2N+1} \\ \dots & \dots & \dots \\ \omega_1^{2N} D_1-\frac{1}{M_N}\sum\limits_{n=1}^{2N+1}\omega_n^{2N+1} e^{a\omega_n s\pi}\left(\int\limits_0^\pi\dots\right)_{a1n}+\underline{O}\left(\frac{1}{s^{4N}}\right) & \dots & B_{2N+1,2N+1} \end{vmatrix}=0, \quad (22)$$

$$D_m=e^{a\omega_n s\pi}-1; \quad M_n=(2N+1)a^{2N}s^{2N};$$

$$B_{m,2N+1}=\omega_{2N+1}^{m-1}D_{2N+1}-\frac{1}{M_N}\sum_{n=1}^{2N+1}\omega_n^m e^{a\omega_n s\pi}\left(\int_0^\pi\dots\right)_{a,2N+1,n}+\underline{O}\left(\frac{1}{s^{4N}}\right);$$

$$m=1,2,\dots,2N+1.$$

Expanding the determinant $f(s)$ from (22) into columns into the sum of determinants, we obtain:

$$f(s)=f_0(s)-\frac{f_{2N}(s)}{(2N+1)a^{2N}s^{2N}}+\underline{O}\left(\frac{1}{s^{4N}}\right)=0, \quad (23)$$

$$f_0(s)=\Delta_{00}[e^{a\omega_1 s\pi}-1][e^{a\omega_2 s\pi}-1][e^{a\omega_3 s\pi}-1](\dots)[e^{a\omega_{2N+1}s\pi}-1], \quad (24)$$

$$f_0(s)=\begin{vmatrix} 1\cdot[e^{a\omega_1 s\pi}-1] & 1\cdot[e^{a\omega_2 s\pi}-1] & \dots & 1\cdot[e^{a\omega_{2N+1}s\pi}-1] \\ \omega_1[e^{a\omega_1 s\pi}-1] & \omega_2[e^{a\omega_2 s\pi}-1] & \dots & \omega_{2N+1}[e^{a\omega_{2N+1}s\pi}-1] \\ \dots & \dots & \dots & \dots \\ \omega_1^{2N}[e^{a\omega_1 s\pi}-1] & \omega_2^{2N}[e^{a\omega_2 s\pi}-1] & \dots & \omega_{2N+1}^{2N}[e^{a\omega_{2N+1}s\pi}-1] \end{vmatrix}; \quad (25)$$

Δ_{00} is the Wandermonde's determinant of the numbers $\omega_1, \omega_2, \ldots, \omega_{2N+1}$:

$$\Delta_{00} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ \omega_1 & \omega_2 & \omega_3 & \ldots & \omega_{2N} & \omega_{2N+1} \\ \omega_1^2 & \omega_2^2 & \omega_3^2 & \ldots & \omega_{2N}^2 & \omega_{2N+1}^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_1^{2N} & \omega_2^{2N} & \omega_3^{2N} & \ldots & \omega_{2N}^{2N} & \omega_{2N+1}^{2N} \end{vmatrix} = \tag{26}$$

$$= \prod_{\substack{k>n \\ k,n=1,2,\ldots,2N+1}} (\omega_k - \omega_n) \neq 0,$$

$$\begin{aligned} f_{2N}(s) = \Delta_{00} \Bigg\{ & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \ldots \right)_{a1k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 1}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) + \\ + & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \ldots \right)_{a2k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 2}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) + \cdots + \\ + & \sum_{k=1}^{2N+1} \omega_k \left(\int_0^\pi \ldots \right)_{a,2N+1,k} e^{a\omega_k s\pi} \left(\prod_{\substack{n=1 \\ n\neq 2N+1}}^{2N+1} (e^{a\omega_n s\pi} - 1) \right) \Bigg\}. \end{aligned} \tag{27}$$

For the determinant Δ_{00} from ((26)) the following property holds: if (δ_{mk}) $(m, k = 1, 2, \ldots, 2N+1)$ is the matrix of algebraic minors to the elements $b_{m,k}$ $(m, k = 1, 2, \ldots, 2N+1)$ of the determinant Δ_{00} of ((26)), then

$$(\delta_{mn}) = \begin{pmatrix} \delta_{11} & \delta_{12} & \ldots & \delta_{1,2N+1} \\ \delta_{21} & \delta_{22} & \ldots & \delta_{2,2N+1} \\ \cdots & \cdots & \cdots & \cdots \\ \delta_{2N+1,1} & \delta_{2N+1,2} & \ldots & \delta_{2N+1,2N+1} \end{pmatrix} =$$

$$= \frac{\Delta_{00}}{2N+1} \begin{pmatrix} 1 & -1 & 1 & -1 & \ldots & -1 & 1 \\ -\omega_1^{-1} & \omega_2^{-1} & -\omega_3^{-1} & \omega_4^{-1} & \ldots & \omega_{2N}^{-1} & -\omega_{2N+1}^{-1} \\ \omega_1^{-2} & -\omega_2^{-2} & \omega_3^{-2} & -\omega_4^{-2} & \ldots & -\omega_{2N}^{-2} & \omega_{2N+1}^{-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -\omega_1^{-2N-1} & \omega_2^{-2N-1} & -\omega_3^{-2N-1} & \omega_4^{-2N-1} & \ldots & \omega_{2N}^{-2N-1} & -\omega_{2N+1}^{-2N-1} \\ \omega_1^{-2N} & -\omega_2^{-2N} & \omega_3^{-2N} & -\omega_4^{-2N} & \ldots & -\omega_{2N}^{-2N} & \omega_{2N+1}^{-2N} \end{pmatrix}. \tag{28}$$

The proof of property (28) can be found in the work of the author [2]. The formula (27) is derived using the property (28). To find the roots of the equation $f_0(s)$ from (24), it is necessary to study the indicator diagram of this equation (see [3, chapter 12]).

For the equation $f_0(s) = 0$ from (24) – (25) the following relation is valid:

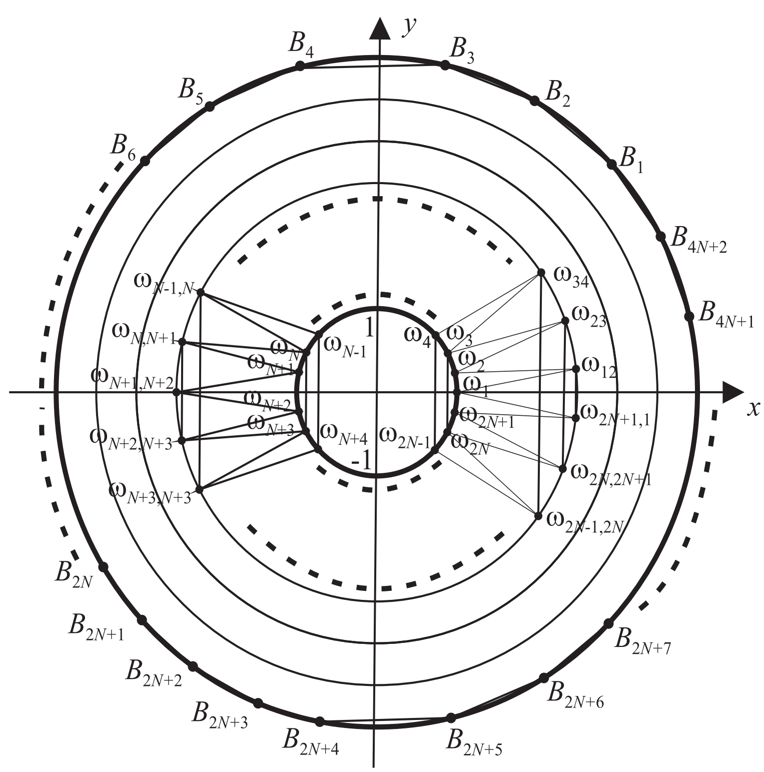
$$f_0(s) = \Delta_{00}\Big\{ -1 + \sum_{n_1=1}^{2N+1} e^{a\omega_{n_1}s\pi} - \sum_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{2N+1} e^{a(\omega_{n_1}+\omega_{n_2})s\pi} + \\ + \sum_{\substack{n_1,n_2,n_3=1\\ n_1\neq n_2, n_1\neq n_3, n_2\neq n_3}}^{2N+1} e^{a(\omega_{n_1}+\omega_{n_2}+\omega_{n_3})s\pi} - \sum_{\substack{n_1,n_2,n_3,n_4=1\\ n_k\neq n_m,(k\neq m)}}^{2N+1} e^{a(\omega_{n_1}+\omega_{n_2}+\omega_{n_3}+\omega_{n_4})s\pi} + \ldots \Big\} = 0. \tag{29}$$

To construct the indicator diagram of the equation (29), it is necessary to study the convex hulls of the sets

$$\{\omega_{n_1}\}_{n_1=1}^{2N+1},\ \{\omega_{n_1}+\omega_{n_2}\}_{\substack{n_1,n_2=1\\ n_1\neq n_2}}^{2N+1},\ \{\omega_{n_1}+\omega_{n_2}+\omega_{n_3}\}_{n_1,n_2,n_3=1}^{2N+1},\ \Big\{\sum_{m=1}^{4}\omega_{m_n}\Big\}_{\substack{n_m=1\\ m=1,2,3,4}}^{2N+1},\ldots$$

The indicator diagram has the following form:

$$\omega_{k,m} = \omega_k + \omega_n, \quad \omega_{n_1,n_2,\ldots,n_k} = \omega_{n_1} + \omega_{n_2} + \cdots + \omega_{n_k} \tag{30}$$



In figure (30) the following designations are introduce: the points $B_1, B_2, B_3, B_4, B_5, \ldots,$ $B_{2N}, B_{2N+1}, \ldots, B_{2N+5}, B_{2N+6}, \ldots, B_{4N+1}, B_{4N+2}$ correspond to exponents with exponents $\omega_{1,2,\ldots,N}$; $\omega_{1,2,\ldots,N,N+1}$; $\omega_{2,3,\ldots,N,N+1}$; $\omega_{2,3,\ldots,N+1,N+2}$; $\omega_{3,4,\ldots,N+1,N+2}; \ldots;$ $\omega_{N,N+1,\ldots,2N}$; $\omega_{N+1,N+2,\ldots,2N}$; $\omega_{N+1,N+2,\ldots,2N,2N+1}$; $\omega_{N+2,N+3,\ldots,2N,2N+1}$; $\omega_{N+2,N+3,\ldots,2N+1,1}$; $\omega_{N+3,N+4,\ldots,2N+1,1}$; $\omega_{N+3,N+4,\ldots,2N+1,1,2}; \ldots;$ $\omega_{2N+1,1,2,3,\ldots N-1}$; $\omega_{N+1,1,2,3,\ldots,2N-1,N}$, where

$\omega_{n_1,n_2,\dots,n_k}$ corresponds to the sum $\omega_{n_1}+\omega_{n_2}+\cdots+\omega_{n_k}$ indices n_k, n_m do not coincide in pairs at $k \neq m$.

In figure (30), the circle of the smallest radius $R_1 = 1$ is the set of points $\{\omega_k\}_{k=1}^{2N+1}$ from (4) that divide the unit circle $(2N+1)$ equal parts. The circle of the second largest radius $R_2 = |\omega_1+\omega_2| > 1$ is a set of points $\{\omega_k+\omega_m\}_{\substack{k,m\neq 1\\ k\neq m}}^{2N+1}$ that are constructed according to the parallelogram rule, while only points $\omega_1+\omega_2$, $\omega_2+\omega_3$, $\omega_3+\omega_4,\dots$, $\omega_{2N}+\omega_{2N+1}$, $\omega_{2N+1}+\omega_1$ appear on the circle, the point $\omega_{n_1}+\omega_{n_2}$ under the condition $|n_1-n_2| \geqslant 2$ fall inside the circle of radius R_2 and do not affect the asymptotics of the roots of equation (23) – (27) (see [3, chapter 12]).

The third largest circle of radius $R_3 = |\omega_1+\omega_2+\omega_3| > R_2$ is a set of the points $\{\omega_k+\omega_m+\omega_n\}_{k,m,n=1}^{2N+1}$ only points $\omega_1+\omega_2+\omega_3$,$\omega_2+\omega_3+\omega_4$, $\omega_3+\omega_4+\omega_5,\dots$, $\omega_{2N-2}+\omega_{2N-1}+\omega_{2N}$, $\omega_{2N-1}+\omega_{2N}+\omega_{2N+1}$, $\omega_{2N}+\omega_{2N+1}+\omega_1$, $\omega_{2N+1}+\omega_1+\omega_2$ are located on the boundary of the circle, the remaining points are inside this circle, and the asymptotics of the roots of equation (23) – (27) are not affected. Next come the circles of the radius $R_4 = |\omega_1+\omega_2+\omega_3+\omega_4| > R_3$ (this is a set of points $\{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}\}_{\substack{k_m=1\\ m=1,2,3,4}}^{2N+1}$), the circle of radius $R_5 = |\sum\limits_{k=1}^{5}\omega_k| > R_4$ (this is the set of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_5}\}_{\substack{k_m=1\\ m=1,2,\dots,5}}^{2N+1}$),..., the circle of radius $R_{N-1} = |\sum\limits_{k=1}^{N-1}\omega_k| > R_{N-2}$ (this is the set of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_{N-1}}\}_{\substack{k_m=1\\ m=1,2,\dots,N-1}}^{2N+1}$), and finally, the circle of the largest radius $R_N = |\sum\limits_{k=1}^{N}\omega_k| > R_{N+1} = |\sum\limits_{k=1}^{N}\omega_k|$ due to equality $\omega_1+\omega_2+\omega_3+\cdots+\omega_{2N}+\omega_{2N+1} \overset{(2.2)}{=} 0$ (these are the sets of points $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_N}\}_{\substack{k_m=1\\ m=1,2,\dots,N}}^{2N+1}$ and $\{\omega_{k_1}+\omega_{k_2}+\cdots+\omega_{k_N}+\omega_{k_{N-1}}\}_{\substack{k_m=1\\ m=1,2,\dots,N+1}}^{2N+1}$).

The circle of he radius $R_{N+2} = |\sum\limits_{k=1}^{N+2}\omega_k|$ coincides with the circle of the radius R_{N-1}, the circle of the radius $R_{N+m} = |\sum\limits_{k=1}^{N+m}\omega_k|$ coincide with the circles of the radius $R_{N-m+1} = |\sum\limits_{k=1}^{N-m+1}\omega_k|$ $(m = 2,3,\dots,N)$ due to equality $\sum\limits_{k=1}^{2N+1}\omega_k \overset{(2.2)}{=} 0$ they are located inside the indicator diagram (30) and such exponentials do not affect the asymptotics of the roots of equation (23) – (27).

The roots of equation (23) – (27) are located in the sectors $1), 2),\dots,(4N+1)), (4N+2))$ of an infinitesimal opening, the bisectors of which are perpendicular to the segment $[B_m; B_{m+1}]$ $(m = 1,2,\dots,4N+2)$ and pass through the midpoints of the segments.

4 The asymptotics of the eigenvalues of the differential operator (1) – (2) in the sector 1) of the indicator diagram (30)

To find the roots of the roots of the equation $f(s) = 0$ from (23) – (27) in the sector 1) of the indicator diagram (1), only the exponents with the exponents $\omega_{1,2,3,\dots,N} = \sum\limits_{k=1}^{N}\omega_k$ and

$\omega_{1,2,3,\dots,N,N+1} = \sum\limits_{k=1}^{N+1} \omega_k$ should be left in this equation.

Theorem 4 *The equation for the eigenvalues of the differential operator* (1) – (3) *in the sector 1) of the indicator diagram* (30) *has the following form:*

$$v_1(s) = v_{1,0}(s) - \frac{v_{1,2N}(s)}{(2N+1)a^{2N}s^{2N}} + \underline{O}\left(\frac{1}{s^{4N}}\right) = 0, \tag{31}$$

$$v_{1,0}(s) \overset{(3.6)}{=} \Delta_{00}[e^{a(\omega_1+\omega_2+\dots+\omega_N)s\pi} - e^{a(\omega_1+\omega_2+\dots+\omega_N+\omega_{N+1})s\pi}], \tag{32}$$

while the main approximation has the form $v_{1,0}(s)$

$$\begin{aligned} \frac{v_{1,2N}(s)}{\Delta_{00}} &= \omega_1 \left(\int\limits_0^\pi \dots\right)_{a11} \frac{v_{1,0}(s)}{\Delta_{00}} + \omega_2 \left(\int\limits_0^\pi \dots\right)_{a22} \frac{v_{1,0}(s)}{\Delta_{00}} + \dots + \\ &+\omega_N \left(\int\limits_0^\pi \dots\right)_{aNN} \frac{v_{1,0}(s)}{\Delta_{00}} + \omega_{N+1} \left(\int\limits_0^\pi \dots\right)_{a,N+1,N+1} (-1)h_{N+1}(s) + \\ &+ \sum_{k=1}^N \omega_k h_N(s) \left(\int\limits_0^\pi \dots\right)_{a,N+1,k} + \sum_{k=1}^N \omega_k \left(\int\limits_0^\pi \dots\right)_{a,N+2,k} \frac{v_{1,0}(s)}{\Delta_{00}} + \\ &+\omega_{N+1} \left(\int\limits_0^\pi \dots\right)_{a,N+2,N+1} (-1)h_{N+1}(s) + \dots + \sum_{k=1}^N \omega_k \left(\int\limits_0^\pi \dots\right)_{a,2N+1,k} \frac{v_{1,0}(s)}{\Delta_{00}} + \\ &+\omega_{N+1} \left(\int\limits_0^\pi \dots\right)_{a,2N+1,N+1} (-1)h_{N+1}(s), \end{aligned} \tag{33}$$

Where the notation $h_N(s) = e^{a(\omega_1+\omega_2+\dots+\omega_N)s\pi}$ and $h_{N+1}(s) = e^{a(\omega_1+\omega_2+\dots+\omega_N+\omega_{N+1})s\pi}$ are introduced.

Dividing in the equation (31) – (33) by $(-1)\Delta_{00}h_N(s) \neq 0$ we obtain:

$$\begin{aligned} v_1(s) = [e^{a\omega_{N+1}s\pi} - 1] - \frac{1}{(2N+1)a^{2N}s^{2N}} \Bigg\{ \int\limits_0^\pi q(t)dt_{a11} \sum_{k=1}^N \omega_k [e^{a\omega_{N+1}s\pi} - 1] + \\ +\omega_{N+1} \left(\int\limits_0^\pi \dots\right)_{a,N+1,N+1} e^{a\omega_{N+1}s\pi} - \sum_{k=1}^N \omega_k \left(\int\limits_0^\pi \dots\right)_{a,N+1,k} + \\ + \sum_{k=1}^N \omega_k \left(\int\limits_0^\pi \dots\right)_{a,N+2,k} [e^{a\omega_{N+1}s\pi} - 1] + \omega_{N+1} \left(\int\limits_0^\pi \dots\right)_{a,N+2,N+1} e^{a\omega_{N+1}s\pi} + \end{aligned}$$

$$+\sum_{k=1}^{N}\omega_k\left(\int_0^\pi\ldots\right)_{a,N+3,k}[e^{a\omega_{N+1}s\pi}-1]+\omega_{N+1}\left(\int_0^\pi\ldots\right)_{a,N+3,N+1}e^{a\omega_{N+1}s\pi}+\cdots+$$
$$+\sum_{k=1}^{N}\omega_k\left(\int_0^\pi\ldots\right)_{a,2N+1,k}[e^{a\omega_{N+1}s\pi}-1]+\omega_{N+1}\left(\int_0^\pi\ldots\right)_{a,2N+1,N+1}e^{a\omega_{N+1}s\pi}\Bigg\}+ \tag{34}$$
$$+\underline{O}\left(\frac{1}{s^{4N}}\right)=0,$$

at the same time

$$\left(\int_0^\pi\ldots\right)_{a11}=\left(\int_0^\pi\ldots\right)_{a22}=\cdots=\left(\int_0^\pi\ldots\right)_{akk}\overset{(2.14)}{=}\int_0^\pi q(t)dt_{a11},$$
$$k=1,2,\ldots,2N+1.$$

The basic approximation of equation ((34)) has the form:

$$e^{a\omega_{N+1}s\pi}-1=0\Leftrightarrow e^{a\omega_{N+1}s\pi}=1=e^{2\pi ik}\Leftrightarrow s_{k,1,\text{bas}}=\frac{2ik}{a\omega_{N+1}},\quad k\in\mathbb{N}. \tag{35}$$

The following statement follows from the formula (35) and the general theory of finding the roots of quasipolynomes of the form (34) (see [4], [5]).

Theorem 5 *The asymptotics of the eigenvalues of the differential operator* (1) – (3) *in the sector 1) of the indicator diagram* (30) *has the following form:*

$$s_{k,1}=\frac{2i}{a\omega_{N+1}}\left[k+\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{2N}}\right)\right],\quad k\in\mathbb{N}. \tag{36}$$

Proof. Applying the Maclaurin's formulas, we have:

$$e^{a\omega_{N+1}s\pi}\big|_{s_{k,1}}=\exp\left[a\omega_{N+1}\pi\frac{2i}{a\omega_{N+1}}\left(k+\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\right)\right]=$$
$$=e^{2\pi ik}\exp\Big[2\pi i\Big(\frac{d_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\Big)\Big]=1+\frac{2\pi id_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)\Big); \tag{37}$$

$$\frac{1}{s^{2N}}\bigg|_{s_{k,1}}=\frac{a^{2N}\omega_{N+1}^{2N}}{2^{2N}i^{2N}}\frac{1}{k^{2N}}\left(1-\frac{2Nd_{2N,k,1}}{k^{2N+1}}\underline{O}\left(\frac{1}{k^{4N+1}}\right)\right). \tag{38}$$

Substituting formulas (36) – (38) into equation (34), we obtain:

$$\left[1+\frac{2\pi id_{2N,k,1}}{k^{2N}}+\underline{O}\left(\frac{1}{k^{4N}}\right)-1\right]-\frac{a^{2N}\omega_{N+1}^{2N}}{(2N+1)a^{2N}2^{2N}i^{2N}}\frac{1}{k^{2N}}\left(1+\underline{O}\left(\frac{1}{k^{2N+1}}\right)\right)\times$$
$$\times\Big\{\int_0^\pi q(t)dt_{a11}\sum_{k=1}^{N}\omega_k\underline{O}\left(\frac{1}{k^{2N}}\right)+\omega_{N+1}\int_0^\pi q(t)dt_{a11}\left(1+\underline{O}\left(\frac{1}{k^{2N}}\right)\right)-$$
$$-\sum_{k=1}^{N}\omega_k\left(\int_0^\pi\ldots\right)_{a,N+1,k}+\sum_{k=1}^{N}\omega_k\left(\sum_{m=2}^{N+1}\left(\int_0^\pi\ldots\right)_{a,N+m,k}\right)+\underline{O}\left(\frac{1}{k^{2N}}\right)+ \tag{39}$$
$$+\omega_{N+1}\sum_{m=2}^{N+1}\left(\int_0^\pi\ldots\right)_{a,N+m,N+1}\left(1+\underline{O}\left(\frac{1}{k^{2N}}\right)\right)\Big\}+\underline{O}\left(\frac{1}{k^{4N}}\right)=0.$$

For k^0 in (39) we obtain the correct equality $1-1=0$, which means that the form of asymptotic formula (36) is chosen correctly. For k^{-2N} in (39) we have:

$$d_{2N,k,1}=\frac{1}{2\pi i}\frac{\omega_{N+1}^{2N}}{(2N+1)2^{2N}(-1)^N}\Big[\omega_{N+1}\int\limits_0^\pi q(t)dt_{a11}-\sum_{k=1}^{N}\omega_k\left(\int\limits_0^\pi\ldots\right)_{a,N+1,k}+$$
$$+\omega_{N+1}\sum_{k=N+2}^{2N+1}\left(\int\limits_0^\pi\ldots\right)_{a,k,N+1}\Big],\quad k\in\mathbb{N}. \tag{40}$$

In the formula (40) the first term is transformed to the following form:

$$\frac{1}{2\pi i}\frac{\omega_{N+1}^{2N}}{(2N+1)2^{2N}(-1)^N}\omega_{N+1}\int\limits_0^\pi q(t)dt\overset{(2.1)}{=}\frac{(-1)^{N+1}}{(2N+1)\pi 2^{2N+1}}\int\limits_0^\pi q(t)dt. \tag{41}$$

The second and the third terms in formula (40) will be calculated as follows:

$$H_N=-\sum_{m=1}^{N}\omega_m\left(\int\limits_0^\pi\ldots\right)_{a,N+1,m}+\omega_{N+1}\sum_{m=N+2}^{2N+1}\left(\int\limits_0^\pi\ldots\right)_{a,m,N+1}=$$
$$=\sum_{k=1}^{N}\left[\omega_{N+1}\left(\int\limits_0^\pi\ldots\right)_{a,2N+2-m,N+1}-\omega_m\left(\int\limits_0^\pi\ldots\right)_{a,N+1,m}\right]\overset{(2.14)}{=}$$
$$=\sum_{m=1}^{N}\omega_{N+1}\int\limits_0^\pi q(t)e^{a(\omega_{2N+2-m}-\omega_{N+1})st}dt_{a,2N+2-m,N+1}-$$
$$-\omega_m\int\limits_0^\pi q(t)e^{a(\omega_{N+1}-\omega_m)st}dt_{a,N+1,m}\overset{(2.1),(4.5)}{=}$$
$$=\sum_{m=1}^{N}\Big[e^{\dfrac{2\pi iN}{2N+1}}\int\limits_0^\pi q(t)\exp\left[at\frac{2ik}{a\omega_{N+1}}\left(e^{\dfrac{2\pi i(2N+1-m)}{2N+1}}-e^{\dfrac{2\pi iN}{2N+1}}\right)\right]dt_{a,2N+2-m,N+1}-$$
$$-e^{\dfrac{2\pi i(m-1)}{2N+1}}\int\limits_0^\pi q(t)\exp\left[at\frac{2ik}{a\omega_{N+1}}(\omega_{N+1}-\omega_m)\right]dt_{a,N+1,m}\Big]dt_{a,N+1,m},$$

this expression will be transformed and simplified as follows:

$$H_N=\sum_{m=1}^{N}e^{\dfrac{\pi i(N+m-1)}{2N+1}}\Big[e^{\dfrac{\pi i(N-m+1)}{2N+1}}\int\limits_0^\pi q(t)e^{-2kti}\exp\left[2kti\exp\left(\frac{2\pi i(1-m+N)}{2N+1}\right)\right]dt-$$

$$-e^{-\dfrac{\pi i(N-m+1)}{2N+1}}\int\limits_0^\pi q(t)e^{2kti}\exp\left[(-2kti)\exp\left(\frac{2\pi i(m-1-N)}{2N+1}\right)\right]dt\Bigg]=$$

$$=\sum_{m=1}^{N}e^{\dfrac{\pi i(N+m-1)}{2N+1}}\int\limits_0^\pi q(t)e^{-2kti}\exp\Big[2kti\Big(\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)+$$

$$+i\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\Big)\Big]e^{\dfrac{\pi i(N-m+1)}{2N+1}}dt-$$

$$-\int\limits_0^\pi q(t)e^{2kti}e^{-\dfrac{\pi i(N-m+1)}{2N+1}}\exp\Big[(-2kti)\Big[\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-$$

$$-i\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\Big]dt\Big],$$

as a result of which we will receive:

$$H_N=(-2i)\sum_{m=1}^{N}\exp\left(\frac{\pi i(N+m-1)}{2N+1}\right)\int\limits_0^\pi q(t)\sin\Big[2kt-2kt\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-$$
$$-\frac{\pi(N-m+1)}{2N+1}\Big]dt\exp\left(-2kti\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\right)dt_{bNm}. \tag{42}$$

Substituting the formulas (41), (42) into formula (40), we find:

$$d_{2N,k,1}=\frac{\omega_{N+1}^{2N}}{(2\pi i)(2N+1)2^{2N}(-1)^N}\left[\omega_{N+1}\int\limits_0^\pi q(t)dt_{a11}+H_N\right]=$$
$$=\frac{(-1)^{N+1}i}{(2N+1)\pi 2^{2N+1}}\int\limits_0^\pi q(t)dt-2ie^{-\dfrac{2\pi iN}{2N+1}}\sum_{m=1}^{N}\exp\left(\frac{\pi i(N+m-1)}{2N+1}\right)\int\limits_0^\pi q(t)\times \tag{43}$$
$$\times\sin\left[2kt-2kt\cos\left(\frac{2\pi(N-m+1)}{2N+1}\right)-\frac{\pi(N-m+1)}{2N+1}\right]\times$$
$$\times\exp\left(-2kt\sin\left(\frac{2\pi(N-m+1)}{2N+1}\right)\right)dt_{bNm},\quad k=1,2,3,\ldots;\quad N=1,2,3,\ldots.$$

We have proved that all the coefficients $d_{2N,k,1}$ $(k=1,2,3,\ldots)$ of formula (36) are found in a unique way, in formula (43) we have given explicit formulas for calculating them, so theorem 5 is completely proved. Studying in a similar way sectors $2),3),\ldots,(4N+2))$ of the indicator diagram (30), we come to the following statement.

Theorem 6 *1) The asymptotics of the eigenvalues of the differential operator* (1) – (3) *in the sectors* $2),3),\ldots,(4N+2))$ *of the indicator diagram* (30) *satisfies the following law:*

$$s_{k,2}=s_{k,1}e^{-\dfrac{2\pi i}{4N+2}};\quad s_{k,3}=s_{k,2}e^{-\dfrac{2\pi i}{4N+2}}=s_{k,1}e^{-\dfrac{4\pi i}{4N+2}};\ldots;$$

$$s_{k,m} = s_{k,m-1} e^{-\dfrac{2\pi i}{4N+2}} = s_{k,1} e^{-\dfrac{2\pi i(m-1)}{4N+2}},$$
$$m = 1, 2, 3, \ldots, 4N+2,$$

where $s_{k,1}$ *satisfies formulas* (36), (43).

2) Wherein $\lambda_{k,m} = s_{k,m}^{2N+1}$, $m = 1, 2, 3, \ldots, 4N+2$; $k = 1, 2, 3, \ldots$.

References

[1] Lidskii V. B., Sadovnichii V. A., "Asymptotic formulas for the zeros of a class of entire functions", *Math. USSR-izv* 2 (3) (1968): 585–600.

[2] Sadovnichii V. A., "The trace of ordinary differential operators of higher orders", *Math. USSR-Sb* 1 (2) (1967): 263–288.

[3] Mitrokhin S. I., "On spectral properties of differential operators with discontinuous coefficients", *Differential Equations* 28 (3) (1992): 530–532.

[4] Mitrokhin S. I., "On some spectral properties of second-order differential operators with a discontinuous weight function", *Reports of the Russian Academy of Sciences* 356 (1) (1997): 13–15.

[5] Vinokurov V. A., Sadovnichy V. A., "Asymptotics of any order for the eigenvalues and eigenfunctions of the Sturm-Liouville boundary-value problem on a segment with a summable potential", *Izv. Math.* 64 (4) (2000): 695–754.

[6] Mitrokhin S. I., "Spectral properties of a fourth-order differential operator with integrable coefficients", *Proc. Steklov Inst. Math.* 270 (2010): 184–193.

[7] Mitrokhin S. I., "On spectral properties of odd order differential operators with summable potential", *Differential Equations* 47 (2) (2011): 1808–1811.

[8] Mitrokhin S. I., "Spectral properties of the family of even order differential operators with a summable potential", *Moscow University Mathematics Bulletin* 72 (4) (2017): 137–148.

[9] Badanin A. V., Korotyaev E. L., "Spectral estimates for a periodic fourth-order operator", *St. Petersburg Math. J.* 22 (5) (2011): 703–736.

[10] Dzhakov P., Mityagin B. S., "Instability zones of one-dimensional periodic Schrodinger and Dirac operators", *Russian Math. Surveys* 61 (4) (2006): 663–766.

[11] Polyakov D. M., "Spectral analysis of a fourth-order differential operator with periodic and antiperiodic boundary conditions", *St. Petersburg Math. J.* 27 (5) (2016): 789–811.

[12] Veliev O. A., "On the nonself-adjoint ordinary differential operator with periodic boundary conditions", *Israel J. Math.* 27 (2010): 195–207.

[13] Mikhailets V., Moliboga V., "Uniform estimates for the semi-periodic eigenvalues of the singular differential operators", *Methods Funct. Anal. Topology* 10 (4) (2004): 30–57.

[14] Mitrokhin S. I., "The asymptotics of the spectrum of a periodic boundary value problem for a differential operator with a summable potential", *Proceedings of IMM Uro RAS* 25 (1) (2019): 136–149.

[15] Mitrokhin S. I., "Periodic boundary value problem for a fourth-order differential operator with a summable potential", *Vladikavkaz mathematical journal* 19 (4) (2017): 35–49.

[16] Badanin A. V., Korotyaev E. L., "Third-order operator with periodic coefficients on the real axis", *St. Petersburg Math. J.* 25 (5) (2014): 713–734.

[17] Naimark M. A., *Linear differential operators* (Moscow: Nauka, 1969): 528.

[18] Bellman R., Cook K. L., *Differential-difference equations* (Moscow: Mir, 1967): 548.

IRSTI 27.29.15 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.02

K.N. Ospanov, A.N. Yesbayev*
L.N. Gumilyov Eurasian National University, Kazakhstan, Nur-Sultan
*e-mail: adilet.e@gmail.com

THE TWO-SIDED ESTIMATES OF THE FREDHOLM RADIUS AND COMPACTNESS CONDITIONS FOR THE OPERATOR ASSOCIATED WITH A SECOND-ORDER DIFFERENTIAL EQUATION

In this paper we consider the properties of the resolvent of a linear operator corresponding to a degenerate singular second-order differential equation with variable coefficients, considered in the Lebesgue space. The singularity of the specified differential equation means that it is defined in a noncompact domain — on the whole set of real numbers, and its coefficients are unbounded functions. The conditions for the compactness of the resolvent were obtained, as well as a double-sided estimate of its fredgolm radius. The previously known compactness conditions of the resolvent were obtained under the assumption that the intermediate term of the differential operator either is missing or, in the operator sense, is subordinate to the sum of the extreme terms. In the current paper these conditions are not met due to the rapid growth at infinity of the intermediate coefficient of the differential equation, and the minor coefficient can change sign. The property of compactness of the resolvent allows, in particular, to justify the process of finding an approximate solution of the associated equation. The Fredholm radius of a bounded operator characterizes its closeness to the Fredholm operator. The operator coefficients are assumed to be smooth functions, but we do not impose any constraints on their derivatives. The result on the invertibility of the operator and the estimation of its maximum regularity obtained by the authors earlier is essentially used in this paper.

Key words: second-order differential operator, Fredholm radius, resolvent, compactness, differential equation in an unbounded domain, differential operator with unbounded coefficients.

К.Н. Оспанов, А.Н. Есбаев*
Л.Н. Гумилёв атындағы Еуразия ұлттық университеті, Қазақстан, Нұр-Сұлтан қ.
*e-mail: adilet.e@gmail.com

Фредгольм радиусының екі жақты бағалаулары және екінші ретті дифференциалдық теңдеумен байланысты оператордың компактылығының шарттары

Бұл жұмыста Лебег кеңістігінде берілген айнымалы коэффициентті нұқсанды екінші ретті сингулярлы дифференциалдық теңдеуге сәйкес келетін сызықты оператордың резольвентасының қасиеттері зерттелген. Аталған дифференциалдық теңдеудің сингулярлы болуын оның шексіз облыста — бүкіл сан осінде — берілуі мен оның коэффициенттерінің шенелмегендігі білдіреді. Резолютентаның компактылығының шарттары, сондай-ақ оның Фредгольм радиусының екі жақты бағасы алынды. Резольвентаның компактылығының бізге бұрыннан белгілі шарттары дифференциалдық оператордың аралық мүшесі жоқ немесе ол оператор мағынасында шеткі мүшелердің қосындысына бағынады деген болжамда алынған. Ал бұл жұмыста дифференциалдық теңдеудің аралық коэффициентінің шексіз алыс нүкте аймағында жылдам өсуі мен төменгі коэффициенттің таңбасы өзгермелі болуына байланысты аталған шарттар орындалмайды. Резольвентаның компактылық қасиетінің болуы, мысалы, онымен байланысты теңдеудің жуықталған шешімін табу процесін негіздеуге мүмкіндік береді. Шенелген оператордың Фредгольм радиусы оның фредгольмдік операторларға жақындығын сипаттайды. Оператордың коэффициенттері тегіс функциялар деп есептеледі, бірақ біз олардың туындыларына ешқандай шектеу қоймаймыз. Жұмыста авторлар өздері осыған дейін алған оператордың қайтарымдылығы жайлы нәтиже мен оның максималды регулярлығының бағасына сүйенеді.

Түйін сөздер: екінші ретті дифференциалдық оператор, Фредгольм радиусы, резольвента, компактылық, шенелмеген облыстағы дифференциалдық теңдеу, коэффициенттері шенелмеген дифференциалдық теңдеу.

К.Н. Оспанов, А.Н. Есбаев*
Евразийский национальный университет имени Л.Н. Гумилёва, Казахстан, г. Нур-Султан
*e-mail: adilet.e@gmail.com

Двухсторонние оценки радиуса фредгольмовости и условия компактности оператора, связанного с дифференциальным уравнением второго порядка

В настоящей работе изучаются свойства резольвенты линейного оператора, соответствующего вырожденному сингулярному дифференциальному уравнению второго порядка с переменными коэффициентами, рассматриваемому в пространстве Лебега. Сингулярность указанного дифференциального уравнения означает, что он задан в некомпактной области — на всей числовой оси, а его коэффициенты являются неограниченными функциями. Получены условия компактности резольвенты, а также двусторонняя оценка его радиуса фредгольмовости. Ранее известные условия компактности резольвенты были получены в предположении, что промежуточный член дифференциального оператора либо отсутствует, либо в операторном смысле подчиняется сумме крайних членов. В настоящей работе эти условия не выполняются вследствие быстрого роста на бесконечности промежуточного коэффициента дифференциального уравнения, а младший коэффициент может менять знак. Наличие свойства компактности резольвенты позволяет, в частности, обосновать процесс нахождения приближенного решения связанного с нею уравнения. Радиус фредгольмовости ограниченного оператора характеризует его близость к фредгольмовому оператору. Коэффициенты оператора предполагаются гладкими функциями, однако мы не накладываем какие-либо ограничения на их производные. В работе существенно использован результат об обратимости этого оператора и оценка его максимальной регулярности, полученные авторами ранее.

Ключевые слова: дифференциальный оператор второго порядка, радиус фредгольмовости, резольвента, компактность, дифференциальное уравнение в неограниченной области, дифференциальный оператор с неограниченными коэффициентами.

1 Introduction

For the Sturm-Liouville operator

$$Ly = -y'' + q(x)y,\ q \geqslant 1,\ x \in \mathbb{R} = (-\infty, +\infty)$$

the following fact proven by Molchanov [1] is known: for the inverse operator L^{-1} to be compact in the space $L_2(\mathbb{R})$ it is necessary and sufficient that for each $d > 0$ the coefficient $q(x)$ satisfies the condition

$$\lim_{|x|\to+\infty} \int\limits_{x-d}^{x+d} q(t)dt = +\infty.$$

This result is generalized in [2, 3] for rather general classes of elliptic operators with non-smooth and oscillating coefficients. In the case of non self-adjoint operators, such results were obtained using the properties that are inherent to semi-bounded self-adjoint operators. However, these results have been established in cases when the intermediate term of a second-order differential operator is either equal to zero or in the operator sense is subordinate to the sum of the extreme terms. The mentioned requirements are not satisfied when the intermediate coefficient grows strongly (see [4,5]), this case is considered in the current paper.

Fredholm operators are extensively studied and have a rich theory similar to the theory of second kind integral operators. One of the characteristics of the adjacency of a given bounded operator to a Fredholm operator is its so-called Fredholm radius [6]. In this paper we give an estimate of the Fredholm radius as well as a criterion for compactness of the resolvent of one class of second-order differential equations with a fast-growing intermediate coefficient. We will essentially use the maximum regularity estimate of the solution of this equation, which was established by us in [7].

Consider the equation

$$l_0 y = -y'' + ry' + sy = f(x), \tag{1}$$

where $x \in \mathbb{R}$, r is a continuously differentiable function, and s is a continuous function, $f \in L_p = L_p(\mathbb{R})$. Let $\widetilde{Q} \subseteq \mathbb{R}$. We denote $C_0^{(k)}(\widetilde{Q})$ $(k = 1, 2, \dots)$ as a set of k-times continuously differentiable functions with compact support in $\widetilde{Q}$. Let $D(l_0) = C_0^{(2)}(\mathbb{R})$ and l is a closure of the operator l_0 in the norm of L_p. We call a function $y \in D(l)$ such that $ly = f$ as a solution of the equation (1). It follows that the unique solvability of the equation (1) is equivalent to the bounded invertibility of the operator l.

2 Material and methods

We base on the maximum regularity estimates (2) below obtained in the Lemma 2. The embedding and compactness theorems of weighed functional classes of the Sobolev type are used. The properties of the average M. Otelbaev function are also used in obtaining the necessary and sufficient condition for the discreteness of the spectrum of the operator l. In addition, Hardy-type integral inequalities on the real axis and semi-axis are used to estimate the norm of the element from the domain of the operator l with singular weight.

3 Auxiliary statements

Let $g(x)$ and $h(x) \neq 0$ be some real continuous functions, $q = \frac{p}{p-1}$ and $\|\cdot\|_p$ is the norm of L_p. Let us introduce the following notations

$$\alpha_{g,h}(t) = \|g\|_{L_p(0,t)} \left\|h^{-1}\right\|_{L_q(t,+\infty)} \ (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_p(\tau,0)} \left\|h^{-1}\right\|_{L_q(-\infty,\tau)} \ (\tau < 0),$$

$$\alpha_{g,h} = \sup_{t>0} \alpha_{g,h}(t), \ \beta_{g,h} = \sup_{\tau<0} \beta_{g,h}(\tau), \ \gamma_{g,h} = \max\left(\alpha_{g,h}, \beta_{g,h}\right).$$

In [7] the following statements were proved.

Lemma 1 *If $g(x)$ and $h(x) \neq 0$ are continuous functions with $\gamma_{g,h} < +\infty$, then*

$$\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leqslant C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx, \quad \forall y \in C_0^{(1)}(\mathbb{R}).$$

Moreover, if C is the smallest constant for which this inequality holds, then

$$(\min\left(\alpha_{g,h}, \beta_{g,h}\right))^p \leqslant C \leqslant \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p.$$

Lemma 2 *Let $1 < p < +\infty$, $r(x) \geqslant 1$ be continuously differentiable function, $s(x)$ be continuous function such that $\gamma_{1,r^{1/p}} < +\infty$ and $\gamma_{s,r} < +\infty$. If there exist $C_1 > 1$ such that*

$$C_1^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C_1, \; x, \nu \in \mathbb{R} : |x - \nu| \leqslant 1,$$

then for any $f \in L_p$, the equation (1) has a unique solution $y(x)$, which satisfies the following inequality:

$$\|y''\|_p + \|ry'\|_p + \|sy\|_p \leqslant C_2 \|f\|_p, \tag{2}$$

where C_2 depends only on $\gamma_{s,r}$ and p.

4 Main results

4.1 Estimation of the Fredholm radius of a degenerate operator in L_p

Definition 1 *We call the following value as the Fredholm radius of the bounded operator $A : L_p \to L_p$*

$$\rho_A = \left[\inf_{T \in \sigma_\infty(L_p)} \|A - T\|_{L_p \to L_p} \right]^{-1},$$

where σ_∞ is the set of all compact operators in L_p.

Let $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$. If $\gamma_{1,r} < +\infty$ then according to the Theorem 2 [10] $\|ry'\|_{L_p(\mathbb{R}^+)}$ and $\|ry'\|_{L_p(\mathbb{R}^-)}$ are the norms.

Let X and Y are normed spaces. The transformation $E : X \to Y$ that matches each element $a \in X$ with the same element from space Y is called an embedding operator. By $H_p(r, \mathbb{R}^+)$ we denote the completion of the set $C_0^{(2)}(\mathbb{R}^+)$ by a norm $\|u\|_{H_p(r,\mathbb{R}^+)} = \|u''\|_{L_p(\mathbb{R}^+)} + \|ru'\|_{L_p(\mathbb{R}^+)}$.

The following lemma is a special case of the Theorem 3 [8].

Lemma 3 *Let the function $Q(x) \geqslant \delta > 0$ be continuous on $\mathbb{R}^+$ and there exists a positive constant C such that $C^{-1} \leqslant \frac{Q(x)}{Q(\nu)} \leqslant C, x, \nu \in \mathbb{R}^+ : |x - \nu| \leqslant 1$. Suppose that the embedding operator $E_+ : H_p(Q, \mathbb{R}^+) \to L_p(\mathbb{R}^+)$ is bounded. Then for the Fredholm radius ρ_{E_+} of the operator E_+ the following estimates hold*

$$C_1^{-1} \leqslant \rho_{E_+} \gamma_+ \leqslant C_1,$$

where $\gamma_+ = \lim\limits_{t \to +\infty} \alpha_{1,Q}(t)$, and $C_1 > 1$ does not depend on $Q(x)$.

The following statement holds.

Lemma 4 *Let the function $Q(x) \geqslant \delta > 0$ be continuous on $\mathbb{R}^-$ and there exists a positive constant C such that $C^{-1} \leqslant \frac{Q(x)}{Q(\nu)} \leqslant C, x, \nu \in \mathbb{R}^- : |x - \nu| \leqslant 1$. Suppose that the embedding operator $E_- : H_p(Q, \mathbb{R}^-) \to L_p(\mathbb{R}^-)$ is bounded. Then for the Fredholm radius ρ_{E_-} of the operator E_- the following estimates hold*

$$C_2^{-1} \leqslant \rho_{E_-} \gamma_- \leqslant C_2,$$

where $\gamma_- = \lim\limits_{\tau \to -\infty} \beta_{1,Q}(\tau)$, and $C_2 > 1$ does not depend on $Q(x)$.

Proof. Let $u \in H_p(Q, \mathbb{R}^-)$ thus $E_- u \in L_p(\mathbb{R}^-)$. Suppose $u_1(x) = u(-x) \in H_p(Q, \mathbb{R}^+)$ then $E_- u = E_+ u_1$. Let $T_1 \in \sigma_\infty(L_p(\mathbb{R}^-))$, $T \in \sigma_\infty(L_p(\mathbb{R}^+))$ and $T_1 u = T u_1$. Then $(E_- - T_1)u = (E_+ - T)u_1$. Further

$$\|E_- - T_1\|_{L_p(\mathbb{R}^-)} = \sup_{u \neq 0} \frac{\|(E_- - T_1)\, u\|_{L_p(\mathbb{R}^-)}}{\|u\|_{L_p(\mathbb{R}^-)}} = \sup_{u_1 \neq 0} \frac{\left[\int\limits_{-\infty}^{0} |(E_- - T_1)\, u_1(-x)|^p \, dx\right]^{\frac{1}{p}}}{\left[\int\limits_{-\infty}^{0} |u_1(-x)|^p \, dx\right]^{\frac{1}{p}}} =$$

$$= \sup_{u_1 \neq 0} \frac{\left[\int\limits_{0}^{+\infty} |(E_+ - T)\, u_1(x)|^p \, dx\right]^{\frac{1}{p}}}{\left[\int\limits_{0}^{+\infty} |u_1(x)|^p \, dx\right]^{\frac{1}{p}}} = \sup_{u_1 \neq 0} \frac{\|(E_+ - T)\, u_1\|_{L_p(\mathbb{R}^+)}}{\|u_1\|_{L_p(\mathbb{R}^+)}} = \|E_+ - T\|_{L_p(\mathbb{R}^+)}.$$

That means $\rho_{E_+} = \rho_{E_-}$, it follows by the Lemma 3 that $C_2^{-1} \leqslant \rho_{E_-} \lim\limits_{t \to +\infty} \alpha_{1,Q(-x)}(t) \leqslant C_2$. Moreover

$$\lim_{t \to +\infty} \alpha_{1,Q(-x)}(t) = \lim_{t \to +\infty} t^{\frac{1}{p}} \left(\int\limits_{t}^{+\infty} \frac{dx}{|Q(x)|^q}\right)^{\frac{1}{q}} = \lim_{\tau \to -\infty} (-\tau)^{\frac{1}{p}} \left(\int\limits_{-\infty}^{\tau} \frac{dx}{|Q(x)|^q}\right)^{\frac{1}{q}} = \gamma_-.$$

The lemma is proved.

Here is another lemma.

Lemma 5 *[9] Let $U = \bigcup\limits_{k=1}^{n} U_k$, $V = \bigcup\limits_{k=1}^{n} V_k$ are unions of mutually disjoint intervals and an operator $T = \sum\limits_{k=1}^{n} T_k$ is such that $T_k : L_p(U_k) \to L_q(V_k)$, $k = \overline{1,n}$ and $T : L_p(U) \to L_q(V)$, $0 < p \leqslant q < +\infty$. Then*

$$\|T\|_{L_p(U) \to L_q(V)} = \sup_{1 \leqslant k \leqslant n} \|T_k\|_{L_p(U_k) \to L_q(V_k)}.$$

The main result of this section is the following statement.

Theorem 1 *Let the functions r, s satisfy the conditions of Lemma 2. Then there exists a constant C_3 that for the Fredholm radius $\rho_{l^{-1}}$ of the inverse to l operator l^{-1} the following estimates hold*

$$C_3^{-1} \leqslant \rho_{l^{-1}} \gamma_{1,r} \leqslant C_3. \tag{3}$$

Proof. By $H_p(r, \mathbb{R})$ we denote the completion of the set $C_0^{(2)}(\mathbb{R})$ by a norm $\|u\|_{H_p(r,\mathbb{R})} = \|u''\|_{L_p(\mathbb{R})} + \|ru'\|_{L_p(\mathbb{R})}$. Let $f \in L_p(\mathbb{R})$, we denote

$$f_+(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ f(x), & x \in [0, +\infty), \end{cases} \qquad f_-(x) = \begin{cases} f(x), & x \in (-\infty, 0), \\ 0, & x \in [0, +\infty). \end{cases}$$

Then obviously $f = f_- + f_+$. Let E be an embedding operator of the space $H_p(r, \mathbb{R})$ in $L_p(\mathbb{R})$, $T \in \sigma_\infty(L_p(\mathbb{R}))$, $(E-T)_-$ and $(E-T)_+$ be restrictions of the operator $E-T$ in the spaces $L_p(\mathbb{R}^+)$ and $L_p(\mathbb{R}^-)$ respectively. According to Lemma 5

$$\|E-T\|_{L_p(\mathbb{R})} = \max\left(\|(E-T)_-\|_{L_p(\mathbb{R}^-)}, \|(E-T)_+\|_{L_p(\mathbb{R}^+)}\right). \tag{4}$$

Hence

$$\|E-T\|_{L_p(\mathbb{R})} \leqslant \|(E-T)_-\|_{L_p(\mathbb{R}^-)} + \|(E-T)_+\|_{L_p(\mathbb{R}^+)},$$

which implies

$$\rho_E^{-1} = \inf_{T\in\sigma_\infty(L_p(\mathbb{R}))} \|E-T\|_{L_p(\mathbb{R})} \leqslant \inf_{T_-\in\sigma_\infty(L_p(\mathbb{R}^-))} \|E_- - T_-\|_{L_p(\mathbb{R}^-)} +$$

$$+ \inf_{T_+\in\sigma_\infty(L_p(\mathbb{R}^+))} \|E_+ - T_+\|_{L_p(\mathbb{R}^+)} = \rho_{E_-}^{-1} + \rho_{E_+}^{-1} \leqslant C_2\gamma_- + C_1\gamma_+. \tag{5}$$

It also follows from (4)

$$\|(E-T)_-\|_{L_p(\mathbb{R}^-)} + \|(E-T)_+\|_{L_p(\mathbb{R}^+)} \leqslant 2\|E-T\|_{L_p(\mathbb{R})},$$

which means

$$2\rho_E^{-1} \geqslant \rho_{E_-}^{-1} + \rho_{E_+}^{-1} \geqslant C_2^{-1}\gamma_- + C_1^{-1}\gamma_+.$$

An estimate follows from the last inequality and (5)

$$\widetilde{C_3}^{-1} \leqslant \rho_E\gamma_{1,r} \leqslant \widetilde{C_3}$$

According to Lemma 2, the operator l^{-1} is bounded from L_p into the space W with the norm $\|u\|_W = \|u''\|_p + \|ru'\|_p + \|su\|_p$, which coincides with $H_p(r, \mathbb{R})$ by virtue of the condition $\gamma_{s,r} < +\infty$. It is clear that l is bounded from $H_p(r, \mathbb{R})$ to L_p operator. So, l is a one-to-one relationship between $H_p(r, \mathbb{R})$ and $L_p(\mathbb{R})$. Therefore the last inequalities lead to estimates (3). The theorem is proved.

4.2 Degenerate operator resolvent compactness criterion in L_p

For a continuous function $r(x) \geqslant 1$ we denote the following notation (see [8]):

$$r^*(x) = \inf_{d>0}\left\{ d^{-1} : d^{1-p} \geqslant \int_{x-d}^{x+d} r^p(t)dt \right\}, \qquad x \in \mathbb{R}.$$

Lemma 6 *Let $r(x) \geqslant 1$ be a continuous function, and there exists a $C > 1$ such that*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \text{ for } x, \nu \in \mathbb{R} : |x-\nu| < 1. \tag{6}$$

Then there exists $C_1 > 1$ such that

$$C_1^{-1} r(x) \leqslant r^*(x) \leqslant C_1 r(x), \quad x \in \mathbb{R}.$$

Proof. Let $d_x = (r^*(x))^{-1}$. The continuity of the function $r(x)$ and the condition $r(x) \geqslant 1$ imply that

$$d_x^{-1} = r^*(x) = \left(\int\limits_{x-d_x/2}^{x+d_x/2} r^p(t)dt \right)^{\frac{1}{p-1}} \geqslant \left(\int\limits_{x-d_x/2}^{x+d_x/2} dt \right)^{\frac{1}{p-1}} = d_x^{\frac{1}{p-1}}.$$

Since $\frac{1}{p-1} > 0$, then $d_x \leqslant 1$, so the condition (6) is satisfied when $|x-\nu| < d_x$. Hence, taking into account the previous equality, we have

$$r^*(x) \geqslant \left(C^{-1} r^p(x) \int\limits_{x-d_x/2}^{x+d_x/2} dx \right)^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) d_x^{\frac{1}{p-1}} = C^{-\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) (r^*(x))^{-\frac{1}{p-1}},$$

$$r^*(x) \leqslant \left(C r^p(x) \int\limits_{x-d_x/2}^{x+d_x/2} dx \right)^{\frac{1}{p-1}} = C^{\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) d_x^{\frac{1}{p-1}} = C^{\frac{1}{p-1}} r^{\frac{p}{p-1}}(x) (r^*(x))^{-\frac{1}{p-1}}.$$

By putting $C_1 = C^{\frac{1}{p}}$, we obtain the required estimates. The lemma is proved.

Lemma 7 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \textit{ for } x, \nu \in \mathbb{R}^+ : |x-\nu| \leqslant 1,$$

and $E_+ : H_p(r, \mathbb{R}^+) \to L_p(r, \mathbb{R}^+)$ is an embedding operator. Then the operator E_+ is compact if and only if

$$\lim_{t \to +\infty} \alpha_{1,r}(t) = 0.$$

Proof. Due to the Lemma 6, the equality $\lim\limits_{t\to+\infty} \alpha_{1,r}(t) = 0$ holds if and only if $\lim\limits_{t\to+\infty} \alpha_{1,r^*}(t) = 0$. Therefore, taking into account Theorem 2 [8], we obtain the statement of the lemma. The lemma is proved.

Lemma 8 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition*

$$C^{-1} \leqslant \frac{r(x)}{r(\nu)} \leqslant C \textit{ for } x, \nu \in \mathbb{R}^- : |x-\nu| \leqslant 1,$$

and $E_- : H_p(r, \mathbb{R}^-) \to L_p(r, \mathbb{R}^-)$ is an embedding operator. Then the operator E_- is compact if and only if

$$\lim_{\tau \to -\infty} \beta_{1,r}(\tau) = 0.$$

Proof. Let $u \in H_p(r, \mathbb{R}^-)$, and the functions $u_1(t) = u(-t)$ and $r_1(t) = r(-t)$ are defined on $\mathbb{R}^+$. Then the compactness of the embedding operator E_- is equivalent to the compactness of the operator $E_1 : H_p(r_1, \mathbb{R}^+) \to L_p(r_1, \mathbb{R}^+)$, where $E_1 u_1 = E_- u$. Due to the previous lemma, the operator E_1 is compact if and only if the following equality holds $\lim\limits_{t\to+\infty} \alpha_{1,r_1}(t) = 0$. On the other hand

$$\lim_{t\to+\infty} \alpha_{1,r_1(x)}(t) = \lim_{t\to+\infty} \beta_{1,r_1(-x)}(-t) = \lim_{\tau\to-\infty} \beta_{1,r(x)}(\tau),$$

where $\tau = -t$. The lemma is proved.

Lemma 9 *Let the function $r(x) \geqslant 1$ be continuous and satisfy the condition* (6) *on $\mathbb{R}$, and $E : H_p(r,\mathbb{R}) \to L_p(r,\mathbb{R})$ is an embedding operator. Then E is compact if and only if*

$$\lim_{t\to+\infty} \alpha_{1,r}(t) = 0, \quad \lim_{\tau\to-\infty} \beta_{1,r}(\tau) = 0. \tag{7}$$

Proof. We denote

$$r_+(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ r(x), & x \in [0,+\infty), \end{cases} \qquad r_-(x) = \begin{cases} r(x), & x \in (-\infty, 0), \\ 0, & x \in [0,+\infty). \end{cases}$$

Then $r(x) = r_-(x) + r_+(x)$ and $\|r\|_{L_p(\mathbb{R})} = \|r_-\|_{L_p(\mathbb{R}^-)} + \|r_+\|_{L_p(\mathbb{R}^+)}$. Let us prove that embedding E is compact if and only if the following embedding operators are compact: $E_- : H_p(r,\mathbb{R}^-) \to L_p(\mathbb{R}^-)$ and $E_+ : H_p(r,\mathbb{R}^+) \to L_p(\mathbb{R}^+)$.

Let E_- and E_+ be compact. Consider the function $u_1 \in H_p(r,\mathbb{R}^+)$. According to the Lemma 1

$$\|u_1\|_{L_p(N,+\infty)} \leqslant pq^{\frac{p}{q}} \left(\sup_{t\geqslant N} \alpha_{1,r}(t)\right)^p \|ru_1'\|_{L_p(N,+\infty)} \leqslant C \left(\sup_{t\geqslant N} \alpha_{1,r}(t)\right)^p \|u_1\|_{H_p(r,\mathbb{R}^+)}.$$

Applying the first condition in (7), we obtain for any $u_1 \in H_p(r,\mathbb{R}^+)$

$$\lim_{N\to+\infty} \|u_1\|_{L_p(N,+\infty)} = 0.$$

Similarly, considering the function $u_2 \in H_p(r,\mathbb{R}^-)$ and using the second condition in (7), we obtain

$$\lim_{N\to-\infty} \|u_2\|_{L_p(-\infty,N)} = 0.$$

Therefore, due to the Frechet-Kolmogorov theorem the operator E is compact.

On the other hand, if E is a compact operator, let us prove that the embedding operator E_+ is compact. Consider a Cauchy sequence $\{u_n\}_{n=1}^{+\infty}$ from the unit ball of $H_p(r,\mathbb{R}^+)$: $\{u_n\}_{n=1}^{+\infty} \subset H_p(r,\mathbb{R}^+), \|u_n\|_{H_p(r,\mathbb{R}^+)} \leqslant 1$. Since the set $C_0^{(2)}(\mathbb{R}^+)$ is dense in $H_p(r,\mathbb{R}^+)$, then without loss of generality we can assume that $u_n(x) = 0$ for $x \in (0,a)$ for some $a > 0$. Let
$$v_n(x) = \begin{cases} 0, & x \in (-\infty, 0], \\ u_n(x), & x \in \mathbb{R}^+. \end{cases}$$

The sequence $\{v_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in the $H_p(r,\mathbb{R})$ therefore it converges in the $L_p(\mathbb{R})$ since the operator E is compact. Then by construction, the sequence $\{u_n\}_{n=1}^{+\infty}$ converges in the $L_p(\mathbb{R}^+)$, hence E_+ is also compact operator.

The compactness of the operator E_- is proved similarly. The lemma is proved.

Theorem 2 *Let the conditions of Lemma 2 hold. Then the resolvent l^{-1} is compact in $L_p(\mathbb{R})$ if and only if*

$$\lim_{t\to+\infty} \alpha_{1,r}(t) = 0, \quad \lim_{\tau\to-\infty} \beta_{1,r}(\tau) = 0. \tag{7}$$

Proof. According to the Lemma 2, the resolvent l^{-1} is bounded from the space $L_p(\mathbb{R})$ into the space $H_p(r, \mathbb{R})$. Due to the Lemma 9 condition (7) and compactness of the embedding operator $E : H_p(r, \mathbb{R}) \to L_p(\mathbb{R})$ are equivalent. The theorem is proved.

5 Conclusion

In the current paper we have investigated an operator l corresponding to a second-order differential equation (1) with unbounded coefficients, with an intermediate term that does not subordinate to the sum of the extreme terms. The main results of the paper are Theorem 1 on the estimation of the Fredholm radius of the resolvent l^{-1}, as well as Theorem 2, which gives the compactness conditions of the resolvent l^{-1} in L_p.

6 Acknowledgements

The first author was financially supported by the Committee of Science of the Ministry of Education and Science of the Republic of Kazakhstan (grant № AR08856281 "Nonlinear elliptic equations with unbounded coefficients").

References

[1] Molchanov, A. M., "Ob usloviyah diskretnosti spektra samosopryazhennyh differencial'nyh uravnenij vtorogo poryadka [On conditions for discreteness of the spectrum of self-adjoint differential equations of the second order.]", *Trudy Moskovskogo matematicheskogo obshchestva* V. 2 (1953): 169-200.

[2] Otelbaev M., "Teoremy vlozheniya prostranstv s vesom i ih primeneniya k izucheniyu spektra operatora Shredingera [Imbedding theorems for spaces with weight and their use in investigating the spectrum of the Schrödinger operator]", *Trudy Mat. Inst. Steklov.* V. 150 (1979): 265-305.

[3] Birman M.S., Pavlov V.S., "O polnoj nepreryvnosti nekotoryh operatorov vlozheniya [On the complete continuity of some embedding operators]", *Vestnik Leningrad. Gos. Univ., Seriya "Matematika, mekhanika i astronomiya"* No 1 (1961): 61-74.

[4] Ospanov K.N., "Maximal L_p-regularity for a second-order differential equation with unbounded intermediate coefficient", *Elec. J. Qual. Th. Dif. Eq.,* 65 (2019): 1-13.

[5] Yesbayev A.N., Ospanov K.N., "Maximal regularity estimate for a differential equation with oscillating coefficients", *Vestnik KazNU,* 1(109) (2021): 25-34.

[6] Apyshev, O. D.; Otelbaev, M., "O spektre odnogo klassa differencial'nyh operatorov i nekotorye teoremy vlozheniya [The spectrum of a class of differential operators and some imbedding theorems.]", *Izv. Akad. Nauk SSSR Ser. Mat.* V. 43, No 4 (1979): 739-764.

[7] Ospanov K., Yesbayev A., "Solvability and maximal regularity results for a differential equation with diffusion coefficient", *Turk. J. Math.*, 44:4 (2020): 1304-1316.

[8] Otelbaev M., "Kriterij diskretnosti spektra odnogo vyrozhdennogo operatora i nekotorye teoremy vlozheniya [A criterion for discreteness of the spectrum for a certain degenerate operator, and certain imbedding theorems]", *Differencial?nye Uravneniya* V. 13, No 1 (1977): 111-120. .

[9] Stepanov V. D., Ushakova E. P., "Hardy operator with variable limits on monotone functions", *J. Funct. Spaces Appl.,* 1 (2003): 1-15.

[10] Muckenhoupt B., "Hardy's inequality with weights", *Studia Mathematica,* 44:1 (1972): 31-38.

IRSTI 27.29.19 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.03

Ghulam Hazrat Aimal Rasa[1,2], **Gaukhar Auzerkhan**[1*]
[1]AL-Farabi Kazakh national university, Kazakhstan, Almaty
[2]Shaheed Prof. Rabbani Education university, Afghanistan, Kabul
*E-mail: auzerkhanova@gmail.com

INCEPTION OF GREEN FUNCTION FOR THE THIRD-ORDER LINEAR DIFFERENTIAL EQUATION THAT IS INCONSISTENT WITH THE BOUNDARY PROBLEM CONDITIONS

Regarding the importance of teaching linear differential equations, it should be noted that every physical and technical phenomenon, when expressed in mathematical sciences, is a differential equation. Differential equations are an essential part of contemporary comparative mathematics that covers all disciplines of physics (heat, mechanics, atoms, electricity, magnetism, light and wave), many economic topics, engineering fields, natural issues, population growth and today's technical issues. Used cases. In this paper, the theory of third-order heterogeneous linear differential equations with boundary problems and transforming coefficients into multiple functions $p(x)$ we will consider. In mathematics, in the field of differential equations, a boundary problem is called a differential equation with a set of additional constraints called boundary problem conditions. A solution to a boundary problem is a solution to the differential equation that also satisfies the boundary conditions. Boundary problem problems are similar to initial value problems. A boundary problem with conditions defined at the boundaries is an independent variable in the equation, while a prime value problem has all the conditions specified in the same value of the independent variable (and that value is below the range, hence the term "initial value"). A limit value is a data value that corresponds to the minimum or maximum input, internal, or output value specified for a system or component. When the boundaries of boundary values in the solution of the equation to obtain constants D_1, D_2, D_3 to lay down Failure to receive constants is called a boundary problem. We solve this problem by considering the conditions given for that true Green expression function. Every real function of the solution of a set of linear differential equations holds, and its boundary values depend on the distances.

Key words: Green Function, Boundary Problem, Private Solution, Public Solution, Wronskian Determinant.

Гулам Хазрат Аймал Раса[1,2], Аузерхан Гаухар[1,*]
[1] Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
[2] Шахид проф. Раббани атындағы педагогикалық университеті, Ауғаныстан, Қабыл қ.
*E-mail: auzerkhanova@gmail.com

Үшінші ретті сызықтық дифференциалдық теңдеу үшін Грин функциясына сәйкес шекаралық есебі

Математика ғылымында сызықтық дифференциалдық теңдеулерді оқытудың маңыздылығы туралы айта кету керек, көрсетілген әрбір физикалық-техникалық құбылыс дифференциалдық теңдеу болып табылады. Дифференциалдық теңдеулер – бұл барлық физикалық пәндерді (жылу, механика, атомдар, электр, магнетизм, жарық және толқындар), көптеген экономикалық тақырыптарды, инженерлік өрістерді, табиғи мәселелерді, халық санының өсуін және заманауи техникалық мәселелерді қамтитын қазіргі салыстырмалы математиканың ажырамас бөлігі. Бұл мақалада шекаралық есептері бар үшінші ретті біртекті емес сызықтық дифференциалдық теңдеулер және коэффициенттерді бірнеше функцияға айналдыру теориясын қарастырамыз. Шектік есептер бастапқы мәнмен есептерге ұқсас. Шектерінде анықталған шарттары бар шекаралық есеп теңдеудегі тәуелсіз айнымалы болып табылады, ал қарапайым мәні бар есепте барлық шарттар болады. Тәуелсіз айнымалының бірдей мәнінде көрсетілген (және бұл мән ауқымнан төмен, демек, "бастапқы мән" термині).

Шектік мән дегеніміз – жүйеге немесе құрамдас бөлікке көрсетілген минималды немесе максималды кіріс, ішкі немесе шығыс мәндеріне сәйкес келетін деректер мәні. Теңдеу шешіміндегі шекара мәндерінің шекаралары тұрақтыларды алу үшін пайдаланады, содан кейін тұрақтыларды қосамыз. Бұл Грин шекаралық есебі деп аталады. Сызықтық дифференциалдық теңдеулер жүйесін шешудің әрбір нақты функциясы орындалады және оның шекаралық мәндері арақашықтықтарға тәуелді болады.
Түйін сөздер: Грин функциясы, шекаралық есеп, нақты шешім, вронскиян анықтамасы.

Гулам Хазрат Аймал Раса[1,2], Аузерхан Гаухар[1,*]
[1]Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
[2]Педагогический университет имени Шахид проф. Раббани, Афганистан, г. Кабул
[*]E-mail: auzerkhanova@gmail.com

Возникновение функции Грина для линейного дифференциального уравнения третьего порядка, несовместимого с условиями краевой задачи

Что касается важности обучения линейным дифференциальным уравнениям, следует отметить, что каждое физическое и техническое явление, выраженное в математических науках, является дифференциальным уравнением. Дифференциальные уравнения являются неотъемлемой частью современной сравнительной математики, которая охватывает все дисциплины физики (тепло, механику, атомы, электричество, магнетизм, свет и волны), многие экономические темы, области техники, естественные проблемы, рост населения и современные технические проблемы. В данной статье мы рассмотрим теорию неоднородных линейных дифференциальных уравнений третьего порядка с краевыми задачами и преобразованием коэффициентов в кратные функции. В области дифференциальных уравнений краевая задача называется дифференциальным уравнением с набором дополнительных ограничений, называемых условиями краевой задачи. Решение краевой задачи – это решение дифференциального уравнения, которое также удовлетворяет граничным условиям. Краевые задачи аналогичны задачам с начальным значением. Граничная задача с условиями, определенными на границах, является независимой переменной в уравнении, тогда как задача с простым значением имеет все условия, указанные в одном и том же значении независимой переменной (и это значение находится ниже диапазона, отсюда выйдет термин "начальное значение"). Предельное значение – это значение данных, которое соответствует минимальному или максимальному входному, внутреннему или выходному значению, заданному для системы или компонента. Когда границы граничных значений в решении уравнения для получения констант сложить потом получаем констант. Это называется краевой задачей Грина. Каждая действительная функция решения системы линейных дифференциальных уравнений имеет место, и ее граничные значения зависят от расстояний.

Ключевые слова: функция Грина, граничная задача, частное решение, публичное решение, определитель Вронскиана.

1 Introduction

Differential equations are one of the most interesting and widely used mathematical topics that have attracted the attention of many researchers. Differential equations in various disciplines including physics; It is especially useful in the movement of weights attached to springs, electrical circuits and free vibrations. In mathematics, in the field of differential equations, a boundary value problem is a differential equation with a set of additional constraints called boundary conditions. A solution to a boundary value problem is a solution to the differential equation that also satisfies the boundary conditions [3].

Boundary value problems arise in several branches of physics because each equation has a body differential. Wave equation problems, such as determining normal states, are often referred to as boundary value problems. A large category of important problems in the boundary value are the Storm-Liouville problems. The analysis of these problems involves special functions, the Green functions of a differential operator [4].

2 Objectives of this research

Topics for introducing heterogeneous linear differential equations of the third order with boundary conditions, and obtaining the Green function are discussed.

3 Methodology

Information has been collected in the form of libraries, websites, domestic and foreign scientific articles, undergraduate and doctoral research dissertations.

3.1 Literature review

Differential equations have been developed for nearly 300 years, and the relationship between evolutions is functions and derivatives of functions, so its history naturally dates back to the discovery of the derivative by the English scientist Isaac Newton (1772-1642) and Gottfried Gottfried Wilhelm Leibniz (Germany (1716-1646)) began. Newton worked on differential equations, including first-order differential equations, into forms. Jacob proposed the Bernoulli differential equation in 1674, but failed to prove it until Euler proved it in 1705.

In the linear differential equations of the boundary problem, Sturm-Lowville first worked, the Sturm-Lowville theory in mathematics and its applications, the classical Sturm-Lowville theory, named after Jacques François Sturm (1803-1855) and Joseph Lowville (1809-1882), the theory of linear differential equations is the second real order of form. In 1969, the Russian scientist Nymark wrote in his book Linear Differential Operators about the Green function to solve differential equations with boundary problem conditions.

4 Green function of an unperturbation boundary value problem

Problem statement. Consider the general solution of a third-order heterogeneous linear differential equation with boundary problem conditions

It should be noted that in the space $L_2(0,1)$ the operator generated by a linear differential expression of the third order with constant coefficients is considered.

$$y^{(3)}(x) + P_1(x)y^{(1)}(x) + P_0(x)y(x) = f(x) \tag{1}$$

Here $P_0(x)$ multiple functions are limited in $[0,1]$ interval $P_1(x)$ multiple functions can be changed in the interval $[0,1]$ are on. Number 3 is the order of differential expression and three times different.

In this section, we recall the known features of these operators, which weconsider with the following boundary conditions

$$U_j(y) = \alpha_j y^{(\gamma_j)}(0) + \beta_j y^{(\gamma_j)}(1) = 0, \quad j = 1, 2, 3, \tag{2}$$

where $\gamma_1 = 0,\ \ \gamma_2 = 1,\ \ \gamma_3 = 2$ chosen according to the Mikhailov-Keselman theorem are often called the strongly regular boundary conditions [4]. Therefore, the eigenvalues of the operator S_0 are asymptotic simple and separated [3], that is, there is a positive number δ for which any two eigenvalues of the operator S_0 are separated from each other by a distance greater than δ. It also follows from works [1, 2] that the system of eigenfunctions and associated functions of the operator S_0 forms a Rises basis in the space $L_2(0,1)$.

We assume

$$3 > \gamma_3 \geq \gamma_2 \geq \gamma_1 \geq 0$$

The general form of the heterogeneous linear differential equation using differential operators can also be written as follows

$$L(y) = \lambda y(x) + f(x) \tag{3}$$

We consider the general solution of equation (1), (2) to be a third-order differential

$$y(x) = y_0(x) + D_1\varphi_1(x) + D_2\varphi_2(x) + D_3\varphi_3(x) \tag{4}$$

where

$$y_0(x) = \int\limits_0^x g(x,t)f(t)dt$$

$y_0(x)$ The specific solution is a heterogeneous equation and here $\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x)$ the basic system of solutions of the equation is homogeneous when the conditions $L(\varphi_1) = 0$, $L(\varphi_2) = 0$, $L(\varphi_3) = 0$ and satisfaction with heterogeneous border conditions $\varphi_j^{(k-1)}(0) = \delta_{kj}$ function $g(x,t)$ Determined by the following formula, which I call the Green function

$$g(x,t) = \frac{P(x,t)}{W(t)}$$

Where $\delta_{kj} = \begin{cases} 1\ , k = j \\ 0\ ,\ \ k \neq j \end{cases}$ and $W(t)$ determinative Wronskian

$$W(t_1,t_2,t_3) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1^{(2)}(t) & y_2^{(2)}(t) & y_3^{(2)}(t) \end{vmatrix}$$

and it should be known that $P(x,t)$ is equal to becomes

$$P(x,t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}.$$

So you should know $g(x,t) = P(x,t)$ because $g(x,t)$ the following formula can be defined.

$$g(x,t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1(x) & y_2(x) & y_3(x) \end{vmatrix}.$$

From here we propose a specific inhomogeneous solution, below

$$y_0(x)=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix}f(t)dt$$

function $y_0(x)$ the heterogeneous solution is equation (1), (2) and to investigate it we take the first-order derivative from the specific (inhomogeneous) solution.

$$y_0^{(1)}(x)=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\end{vmatrix}f(t)dt+\begin{vmatrix} y_1(x) & y_2(x) & y_3(x)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix}f(x)$$

now we take the second-order derivative

$$y_0^{(2)}(x)=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(2)}(x) & y_2^{(2)}(x) & y_3^{(2)}(x)\end{vmatrix}f(t)dt$$

now we take the third order derivative

$$y_0^{(3)}(x)=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(3)}(x) & y_2^{(3)}(x) & y_3^{(3)}(x)\end{vmatrix}f(t)dt+\begin{vmatrix} y_1(x) & y_2(x) & y_3(x)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\\ y_1^{(2)}(x) & y_2^{(2)}(x) & y_3^{(2)}(x)\end{vmatrix}f(x)$$

now to solve a given heterogeneous problem, we must examine Equation (1)and we $y(x)$ offer solutions

$$L(y)={y_0}^{(3)}(x)+P_1(x){y_0}^{(1)}(x)+P_0(x)y_0(x)$$

Solve the received function $y(x)$ Prove the following

$$L(y)=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(3)}(x) & y_2^{(3)}(x) & y_3^{(3)}(x)\end{vmatrix}f(t)dt+f(x)+P_1(x)\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(1)}(x) & y_2^{(1)}(x) & y_3^{(1)}(x)\end{vmatrix}f(t)dt+$$
$$+P_0(x)\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1(x) & y_2(x) & y_3(x)\end{vmatrix}f(t)dt$$

from here we have to add the matrices,

$$L(y)=$$
$$=\int\limits_0^x\begin{vmatrix} y_1(t) & y_2(t) & y_3(t)\\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t)\\ y_1^{(3)}(x)+P_1(x)y_1^{(1)}(x)+P_0(x)y_1(x) & y_2^{(3)}(x)+P_1(x)y_2^{(1)}(x)+P_0(x)y_2(x) & y_3^{(3)}(x)+P_1(x)y_3^{(1)}(x)+P_0(x)y_3(x)\end{vmatrix}$$
$$f(t)dt+f(x)$$

Homogeneous equation condition $L(y) = y_1^{(3)}(x) + P_1(x)y_1^{(1)}(x) + P_0(x)y_1(x) = 0$ to be so we can function $f(x)$ and as a result we can say that we have obtained the solution of the heterogeneous part.

we get the Green function for the proposed problem and prove it given the problem Equation (1), (2) Heterogeneous linear differential with boundary value problem can also be considered as follows

$$L(y) = f(x), \quad 0 < x < 1 \tag{5}$$

With border conditions

$$U_1(y) = 0, \; U_2(y) = 0, \; U_3(y) = 0. \tag{6}$$

the kind of frontier conditions defined for us in advance

$$U_1(y) = \alpha_1 y(0) - \beta_1 y(1) = 0$$

$$U_2(y) = \alpha_2 y(0) - \beta_2 y(1) = 0$$

$$U_3(y) = \alpha_3 y(0) - \beta_3 y(1) = 0$$

we can say that we can solve the equation and function of Green (5), (6) using differential operators as follows

$$y(x,t) = (L_0 - \lambda I)^{-1} f = \int\limits_0^1 G_0(x,t,\lambda) f(t) dt$$

where

$$G_0(x,t,\lambda) = -\frac{\begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) & y_3(x,\lambda) & g(x,t) \\ U_1(y_1) & U_1(y_2) & U_1(y_3) & U_1(g) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) & U_2(g) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) & U_3(g) \end{vmatrix}}{\begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix}}$$

$G_0(x,t,\lambda)$ – is a Green function.

If $x > t$ function $g(x,t)$ It has the following form that has been proven before

$$g(x,t) = \begin{vmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1^{(1)}(t) & y_2^{(1)}(t) & y_3^{(1)}(t) \\ y_1(x) & y_1(x) & y_1(x) \end{vmatrix}$$

If $x \leq t$ then$g(x,t) = 0$.

$$\Delta_0(\lambda) = \begin{vmatrix} U_1(y_1) & U_1(y_2) & U_1(y_3) \\ U_2(y_1) & U_2(y_2) & U_2(y_3) \\ U_3(y_1) & U_3(y_2) & U_3(y_3) \end{vmatrix}$$

5 Discussion

Since we have obtained the Green function of the solution of the third-order heterogeneous linear differential equation, everything in this system is technically solvable. To solve that, we proposed the proposed method and showed that the third-order heterogeneous linear differential equation with the boundary problem does not have a solution but has an infinite solution.

6 Conclusion

From the subject of the research, it is concluded that the problem we studied in the third-order heterogeneous linear differential equations is a set of Green's function. Every real function holds in the solution of the set of linear differential equations, and such equations not only a definite solution but also infinitely solvable. Its field of application in physics, for example, finding the temperature at all points of an iron rod with one end at absolute zero and the other at the freezing point of water, is a boundary value problem. If the problem depends on both place and time, the value of the problem can be specified at a certain point for all times or at a certain time for the whole space, and another example of a linear differential equation with boundary conditions can be given. The boundary condition that specifies the value of the function is the Dirichlet boundary condition. For example, if one end of an iron bar is kept at absolute zero, then the problem value at that point in space is specified.

7 Result

Based on our findings and analysis of our research we found that the proposed problem does not have a solution for each parameter λ a solution can be obtained. So the proposed problem for each parameter λ has an infinite Green function.

References

[1] Berezin F.A., Faddeev L.D. "Zamechanie ob uravnenii Shredingera s singulyarnyim potentsialom [Remark on the Scrodinger equation with singular potential]", *Izv. Vuzov. Dokl.* 2 (1961): 372–375.

[2] Berezin F.A. "O modeli Li [About Lee 's model]", *Matematicheskiy sbornik* 4 (60) (1963): 425–446.

[3] Goloschapova N.I., Zastavnyiy V.P., Malamud M.M. "Polozhitelno opredelennyie funktsii i spektralnyie svoystva operatora Shredingera s tochechnyimi vzaimodeystviyami [Positive definite functions and spectral properties of the Schrodinger operator with point interactions]", *Matematicheskie zametki* 1 (90) (2011): 149–154.

[4] Kanguzhin B.E., Dairbaeva G., Madibayulyi Zh. "Identifikatsiya granichnyih usloviy differentsialnogo operatora [Identification of boundary conditions of a differential operator]", *Vestnik KazNU, ser. mat., mech., inf.* 3 (103) (2019): 13-18

[5] Keselman G.M. "Bezuslovnaya shodimost razlozheniy po sobstvennyim funktsiyam nekotoryih differentsialnyih operatorov [Unconditional convergence of expansions in eigenfunctions of some differential operators]", *Izv. Universitetyi SSSR, ser. mat.* 2 (2017): 82-93.

[6] Kanguzhin B.E., Aniyarov A.A. "Korrektnaya zadacha dlya operatora Laplasa v prokolotom diske [Well-Posed Problem for the Laplace Operator in a Punctured Disk]", *Matem. zametki* 6 (89) (2011): 856–867.

[7] Kokebaev B.K., Otelbaev M., Shyinyibekov A.N. "K voprosam rasshireniy i ogranicheniy operatorov [On the questions of extensions and restrictions of operators]", *Dokl. Akademiya nauk SSSR* 6 (271) (1983): 1307-1313.

[8] Landau L.D., Lifshits E.M. *Kurs teoreticheskoy fiziki* [Course of theoretical physics] (M.: Nauka, 1989).

[9] Mihaylov V.P. "O bazah Rissa [About Riesz bases]", *Dokl. Akad. AN SSSR* 2 (144) (2018): 981-984.

[10] Minlos R.A., Faddeev L.D. "Tochechnoe vzaimodeystvie dlya trehchastichnoy sistemyi v kvantovoy mehanike [Point interaction for a three-particle system in quantum mechanics]", *Dokl. Akad. AN SSSR* 6 (141) (1961): 1335–1338.

[11] Neyman-Zade M.I., Shkalikov A.A. "Operatoryi Schrödingera s singulyarnyimi potentsialami iz prostranstv multiplikatorov [Schrödinger operators with singular potentials from spaces of multipliers]", *Matematicheskie zametki* 5 (66) (1999): 599–607.

[12] Naymark M.A. *Lineynyie differentsialnyie operatory* [Linear differential operators] (M., 2019): 528.

[13] Neyman-Zade M.I., Savchuk A.M. "Operatoryi Schrödingera s singulyarnyimi potentsialami [Schrödinger Operators with Singular Potentials]", *Tr. Mat. inst. im. V.A. Steklova* 236 (2002): 262–371.

[14] Shkalikov A.A. "O bazisnosti sobstvennyih funktsiy obyiknovennyih differentsialnyih operatorov s integralnyimi granichnyimi usloviyami [On the basis property of the eigenfunctions of ordinary differential operators with integral boundary conditions]", *Vestnik MGU. Ser. Mat. Meh.* 6 (2019): 12-21.

IRSTI 27.35.45 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.04

A. Shakir*, A. Kabidoldanova, Kh. Khompysh
Al-Farabi Kazakh National University, Kazakhstan, Almaty
*E-mail: ajdossakir@gmail.com

SOLVABILITY OF A NONLINEAR INVERSE PROBLEM FOR A PSEUDOPARABOLIC EQUATION WITH P-LAPLACIAN

Inverse problems of determining the right-hand side of a differential equation arise in the mathematical modeling of many physical phenomena, when an external source or some of its parameters acting to the motion of the process are unknown or unacceptable for measurement, for example, the source is in a high-temperature environment or underground, etc. This paper deals to study the solvability of an inverse problem for a nonlinear pseudoparabolic equation (sometimes they called Sobolev-type equations) with p-Laplacian and damping term with variable exponent. The inverse problem consists of determining a coefficient of the right hand side depending only on time. An additional information for this investigated inverse problem is given as an integral overdetermination condition. Under the suitable conditions on the exponents and on the data the global and local in time existence of a weak solutions to the inverse problem are established. The existence of weak solution proved by Faedo-Galerkin method. The global and local in time a priori estimates for the Galerkin approximations are obtained. On the basis of a priori estimates and by using compactness theorems and the monotonicity method, the convergence of the Galerkin approximations to the solution of the initial inverse problem is proved.
Key words: inverse problem, pseudoparabolic equations, existence, weak solution.

А. Шәкір*, А. Кабидолданова, Х. Хомпыш
Әл-Фараби атындағы қазақ ұлттық университеті, Қазақстан, Алматы қ.
*E-mail: ajdossakir@gmail.com

p-Лапласианды псевдопараболалық теңдеу үшін сызықты емес кері есептің шешімділігі

Дифференциалдық теңдеудің оң жағын анықтау кері есептер көп жағдайда физикалық құбылыстың қозғалысына әсер сыртқы күштер немесе олардың кейбір параметрлері белгісіз әлде өлшеуге қол жетімсіз болатын, мәселен, әсер етуші жылу көзі жоғары температуралы ортада немесе жердің астында болған үрдістерді математикалық тұрғыдан модельдеуде туындайды. Бұл ұсынылған мақалада сызықты емес p-Лапласианды және айнымалы көрсеткішті псевдопараболалық (кейбір жұмыстарда мұндай теңдеулер соболев типті теңдеулер деп аталады) теңдеу үшін кері есептің шешімділігі толыққанды зерттелінеді. Қарастырылатын кері есеп теңдеудің оң жақ бөлігіндегі тек қана уақытқа тәуелді коэффициентті анықтаудан тұрады. Бұл зерттелінетін кері есеп үшін қосымша ақпарат интегралдық қосымша шарт түрінде қойылды. Теңдеудегі көрсеткіштер мен есептің бастапқы берілгендері үшін қолайлы шарттар орындалған кездегі кері есептің әлсіз шешімдердің глобальды және локальды бар болуы көрсетілді. Шешімнің бар болуы Федо-Галеркин әдісі көмегімен дәлелденді. Галеркинге жуық шешімдер үшін уақыт бойынша глобальды және локальды априорлық бағалаулар алынды. Осы алынған априорлық бағалаулар негізінде компактылық теоремалар мен монотондық әдістерін қолдана отырып, галеркиннің жуық шешімдердің бастапқы есептің шешіміне жинақталуы дәлелденді.
Түйін сөздер: Кері есеп, псевдопараболалық теңдеу, шешімнің бар болуы, әлсіз шешім.

А. Шакир*, А. Кабидолданова, Х. Хомпыш
Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
*E-mail: ajdossakir@gmail.com

Разрешимость нелинейной обратной задачи для псевдопараболического уравнения с p-Лапласианом

Обратные задачи определения правой части дифференциального уравнения возникают при математическом моделировании многих физических явлений, когда внешний источник или некоторые его параметры, влияющие на движение процесса, неизвестны или неприемлемы для измерения, например, источник находится в высокотемпературной среде или под землей и т. д. В данной работе исследуется разрешимость обратной задачи для одного нелинейного псевдопараболического уравнения (в некоторых работах такие уравнения называются уравнениями типа соболева) с p-лапласианом и демпфирующим членом с переменным показателем степени. Исследуемая обратная задача состоит в определения коэффициента правой части, зависящего только от времени. Дополнительная информация для этой исследуемой обратной задачи задается в виде интегрального условия переопределения. При подходящих условиях на показатели и на данные задачи, установлены глобальное и локальное существование слабых решений. Существование решения доказано с помощью методом Фаэдо-Галеркина. Получены глобальные и локальные по времени априорные оценки для галеркинских приближений. На основе полученных этих априорных оценок и используя теоремы компактности а также метода монотонности, доказаны сходимости галеркинских приближенных решений к решению исходной задачи.
Ключевые слова: Обратная задача, псевдопараболические уравнения, существования решения, слабое решение.

1 Statement of the problem

In this work, we consider the following nonlinear inverse problem for the pseudoparabolic equation with $p-$Laplacian diffusion and damping term with variable exponents

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) + |u|^{m(x)-2} u = f(t) \cdot g(x,t), \text{ in } Q_T, \tag{1}$$

$$u(x,0) = u_0(x) \text{ in } \Omega, \tag{2}$$

$$u(x,t) = 0 \text{ on } \Gamma_T, \tag{3}$$

$$\int_\Omega (u \cdot \omega + \nabla u \cdot \nabla \omega)\, dx = e(t), \ \ t \geq 0. \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and $Q_T = \{(x,t) : \ x \in \Omega, \ 0 < t \leq T\}$ is a cylinder with lateral Γ_T. The functions $g(x,t), \ u_0(x), \ \omega(x)$, and $e(t)$ are given. The exponents p is given positive number and m is given function, such that

$$1 < p < \infty, \ \ 1 < m_- \leq m(x) \leq m_+ < \infty, \ \ \forall x \in \Omega; \tag{5}$$

where

$$m_- = \inf_x m(x) \ \text{ and } \ m_+ = \sup_x m(x)$$

The inverse problem (1)-(4) consists of determining the coefficient $f(t)$ of the right hand side and a solution $u(x,t)$ from (1) and additional information (4) which given by integral overdetermination condition, and the initial-boundary conditions (2)-(3).

2 Introduction

Inverse problems of determining the right-hand side of a differential equation arise in the mathematical modeling of many physical phenomena, when an external source or some of its parameters acting to the motion of the process are unknown or unacceptable for measurement, for example, the source is in a high-temperature environment or underground, etc.

Equations like (1) with a one time derivative appearing in the highest order term are called pseudo-parabolic or Sobolev equations, and arise in many areas of mathematics and physics. For instance, they have been used, to model thermodynamics processes [19], filtration in porous media [8], and nonsteady flow of second order fluids [10], the motion of non-Newtonian fluids [3], [21], and many other physical phenomena.

In the case $p = 2$ and $m = 2$, the equation (1) becomes the classical pseudoparabolic equation. To our knowledge, the inverse problems for pseudoparabolic equations have not been studied a lot, see for classical pseudoparabolic equations [1], [7], [9], [14], [11], [16], [17,18], and for pseudoparabolic equations with p-Laplacian and other related equations [4], [2], [12,13], [20], and references therein.

Recently, Antontsev and et. in [4] have been considered the inverse problem (1)-(4) with $m = const$ and with the right-hand side $F(x,t) = f(t) \cdot (\omega(x) - \Delta\omega(x))$, where $\omega(x)$ is the same function appearing also in the overdetermination condition (4). In this paper, we consider the inverse problem (1)-(4) with variable exponent $m = m(x)$ and with the right-hand side $F(x,t) = f(t)g(x,t)$, where $g(x,t)$ is an arbitrary function in $L^{\infty}(0,T;L^2(\Omega))$. Under suitable assumptions on the exponents and data, we prove the global and local existence theorems as analogical results in [4].

3 Preliminaries

Let $q : \Omega \to [1,\infty]$ be a measurable function. We define the Lebesgue space with variable exponent $q(.)$ by

$$L^{q(.)}(\Omega) := \{u : \Omega \to R \;\; measurable \;\; and \;\; \int_{\Omega} |\lambda u(x)|^{q(x)}dx < \infty \;\; for \; some \;\; \lambda > 0\}.$$

Equipped with the following Luxembourg-type norm([22]):

$$\|u(x)\|_{q(\cdot)} := inf\{\lambda > 0 : \int_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \leq 1\}$$

is a Banach space.

We use the classical and the following nonlinear Gronwall's inequality ([5]) to establish the first and second local estimates.

Lemma 1 *If $y : \mathbb{R}^+ \longrightarrow [0,\infty)$ is a continuous function such that*

$$y(t) \leq C_1 \int_0^t y^{\mu}(s)ds + C_2, \quad t \in \mathbb{R}^+, \quad \mu > 1$$

for some positive constants C_1 *and* C_2*, then*

$$y(t) \leq C_2 \left(1-(\mu-1)C_1C_2^{\mu-1}t\right)^{-\frac{1}{\mu-1}} \quad \text{for } \ 0 \leq t < t_{\max} := \frac{1}{(\mu-1)C_1C_2^{\mu-1}}.$$

4 Weak formulation

Assume that the data of the problem satisfy the following conditions

$$u_0(x) \in H_0^1(\Omega) \cap W^{1,p}(\Omega) \cap L^{m(x)}(\Omega); \tag{6}$$

$$|g_0(t)| := \left| \int\limits_\Omega g(x,t)\omega(x)dx \right| \geq l_0 > 0 \text{ for all } \ t \geq 0; \tag{7}$$

$$g(x,t) \in L^\infty(0,T;L^2(\Omega)); \tag{8}$$

$$\omega(x) \in W^{1,p}(\Omega) \cap L^{m(x)}(\Omega) \cap W_0^{1,2}(\Omega); \tag{9}$$

$$e(t) \in W^{1,2}([0,T]), \quad \text{and} \quad \int\limits_\Omega u_0 \cdot \omega dx = e(0). \tag{10}$$

Lemma 2 *Under the conditions (5) and (7)-(10), the inverse problem (1)-(4) is equivalent to the following problem for a nonlinear parabolic equation containing the nonlinear nonlocal operator of the function u*

$$u_t - \Delta u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{m(x)-2}u = f(t,u)g(x,t), \quad Q_T, \tag{11}$$

$$u(x,0) = u_0(x), \ \Omega, \tag{12}$$

$$u(x,t) = 0, \quad \Gamma_T. \tag{13}$$

Here

$$f(t,u) = \frac{1}{g_0(t)}\left(e'(t) + \int\limits_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\omega \ dx + \int\limits_\Omega |u|^{m(x)-2}u \cdot \omega dx\right). \tag{14}$$

The prove is analogical as in [4].

Definition 1 *A function $u(x,t)$ is a weak solution to the problem (11)-(14), if:*

1. $u \in L^{\infty}(0,T; W_0^{1,2} \cap W_p^1 \cap L^{m(x)}) \cap L^p(Q_T) \cap L^{m(x)}(Q_T), \quad u_t \in L^2(0,T; W_0^{1,2}(\Omega))$;
2. $u(0) = u_0$ *a.e. in* Ω;
3. *For every* $\varphi \in W_0^{1,2} \cap W_p^1 \cap L^{m(x)}(\Omega)$ *and for a.a.* $t \in (0,T)$ *holds*

$$\frac{d}{dt}\int\limits_{\Omega}(u\varphi + \nabla u \cdot \nabla\varphi)\,dx + \int\limits_{\Omega}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dx + \int\limits_{\Omega}|u|^{m(x)-2}u dx = \int\limits_{\Omega} f(t,u)g\varphi dx. \quad (15)$$

5 Main result

Let be

$$u_0 \in H_0^1(\Omega) \cap W^{1,p}(\Omega) \cap L^{m(x)}(\Omega) \quad (16)$$

The following theorems are valid.

Theorema 1 *Assume that the conditions (7)-(10) and* (16) *be fulfilled and*

$$1 < p \leq 2 \quad \text{and} \quad 1 < m_- \leq m(x) \leq m_+ \leq 2, \ \forall x \in \Omega. \quad (17)$$

Then the problem (11)-(14) has at least one weak solution in the sense of definition 1 global in time.

Theorema 2 *If instead of* (17) *holds the condition*

$$1 < p < \infty \quad \text{and} \quad 1 < m_- \leq m(x) \leq m_+ \leq 2^*, \ x \in \Omega \quad (18)$$

then the problem (11)-(14) has at least one weak solution in the sense of definition 1 local in time.

In order to prove these theorems it is enough to establish the first and second a priori estimates. After then by using these a priori estimates and the monotonicity method [6], we establish the passage to limit for Galerkin's approximation, and as a result, we get that the limit function is a weak solution to the our investigated problem.

5.1 Galerkin's approximations

Let $\{\psi_k\}_{k\in N}$ be an orthonormal family in $L^2(\Omega)$ and a linear combinations are dense in $V := W_0^{1,2} \cap W^{1,p} \cap L^{m(x)}(\Omega)$ [15]. Given $n \in N$, let us consider the n-dimensional space V^n spanned by $\psi_1, \ldots, \psi_n$. For each $n \in N$, we search for approximate solutions

$$u^n(x,t) = \sum_{j=1}^{n} c_j^n(t)\psi_j(x), \quad \psi_j \in V^n, \quad (19)$$

where the coefficients $c_1^n(t), \dots, c_n^n(t)$ are defined as the solutions of the following n ordinary differential equations derived from

$$\begin{aligned}\int\limits_\Omega (u_t^n\psi_k + \nabla u_t^n \nabla\psi_k)\,dx + \int\limits_\Omega |\nabla u^n|^{p-2}\nabla u^n\cdot\nabla\psi_k\,dx + \int\limits_\Omega |u^n|^{m(x)-2}u^n\cdot\psi_k\,dx = \\ f(t,u^n)\int\limits_\Omega g\psi_k dx,\end{aligned} \tag{20}$$

for $k = 1,\ 2,\ \dots,\ n.$

The system (20) of ODEs is supplemented with the following Cauchy data

$$u^n(0) = u_0^n \quad \text{in } \Omega. \tag{21}$$

and assume that

$$u_0^n \to u_0(x) \text{ as } n\to\infty \ \text{ in } \ W_0^{1,2}\cap W^{1,p}\cap L^{m(x)}(\Omega). \tag{22}$$

According to the general theory of nonlinear ODE, the problem (20)-(21) has a solution $c_j^n(t)$ in $[0, t_0]$, where $t_0 \in (0, T]$. The solution can be extended to $[0, T]$ by a priory estimate which we shall obtain below.

5.2 First and second a priory estimates

First priory estimate. Let be now $1 < p, \quad 2 \ge m_+ \ge m(x) \ge m_- > 1$. Multiplying (20) by $c_j^n(t)$ and summing with respect to j, from 1 to n, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2\right) + \|\nabla u^n\|_{L^p(\Omega)}^p + \int\limits_\Omega |u^n|^{m(x)}\,dx = R, \tag{23}$$

where

$$\begin{aligned}R =& \frac{1}{g_0(t)}\left(e'(t) + \int\limits_\Omega |\nabla u^n|^{p-2}\,\nabla u^n\cdot\nabla\omega\ dx\right. \\ &\left.+ \int\limits_\Omega |u^n|^{m(x)-2}u^n\cdot\omega dx\right)\int\limits_\Omega g(x,t)u^n(x)dx = \sum_{i=1}^3 R_{1i},\end{aligned} \tag{24}$$

Using the Holder's and Youn's inequalities and the assumptions (17), we estimate each term on the right hand side of (23)

$$|R_{11}| = \left|\frac{1}{g_0(t)}\int\limits_\Omega gu^n e'(t)dx\right| \le \frac{1}{8}\|u^n\|_{2,\Omega}^2 + \frac{1}{2l_0^2}\|g(t)\|_{2,\Omega}^2\,|e'(t)|^2. \tag{25}$$

$$|R_{12}| \leq \frac{1}{l_0} \|g(t)\|_{2,\Omega} \|u^n\|_{2,\Omega} \|\nabla\omega\|_{p,\Omega} \|\nabla u^n\|_{p,\Omega}^{p-1} \leq$$
$$\frac{1}{2} \|\nabla u^n\|_{p,\Omega}^p + M(p) \left(\frac{1}{l_0} \|g(t)\|_{2,\Omega} \|\nabla\omega\|_{p,\Omega} \|u^n\|_{2,\Omega} \right)^p \leq \tag{26}$$
$$\frac{1}{2} \|\nabla u^n\|_{p,\Omega}^p + \frac{1}{8} \|u^n\|_{2,\Omega}^2 + M'(t)$$

where $M'(t) = \left(M(p) \frac{1}{l_0} \|g(t)\|_{2,\Omega} \|\nabla\omega\|_{p,\Omega} \right)^{\frac{2}{2-p}}$

Estimate the term R_{13} by using Holder, Young inequalities and the Poincare's and the Sobelev inequality

$$\|u\|_{m_+,\Omega} \leq M(\Omega) \|\nabla u\|_{2,\Omega}, \text{which holds for } m_+ \leq 2^*,$$

where $2^* = \frac{2d}{d-2}$ if $d > 2$ and $2^* \in (1, \infty)$ if $d = 2$. Using algebraic inequality $(a_1 + ... + a_k^s) \leq L(a_1^s + ... + a_k^s)$, where L=const depending only k and s.

$$|R_{13}| \leq \frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left(\int_\Omega \left(|u^n|^{m(x)-1} \right)^{\frac{m_+}{m_+-1}} dx \right)^{\frac{m_+-1}{m_+}} \leq$$
$$\frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left[\int_{\Omega_-} |u^n|^{m_- -1 \cdot \frac{m_+}{m_+-1}} dx + \|u^n\|_{m_+,\Omega}^{m_+} \right]^{\frac{m_+-1}{m_+}} \leq \tag{27}$$
$$\frac{1}{l_0} \|u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \left[\|u^n\|_{m_+,\Omega}^{m_+} + M(\Omega, m_-, m_+) \|u^n\|_{m_+,\Omega}^{m_+ \cdot \frac{m_- -1}{m_+-1}} \right]^{\frac{m_+-1}{m_+}} \leq$$
$$M'' \|\nabla u^n\|_{2,\Omega} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega} \|\nabla u^n\|_{2,\Omega}^{m_+-1} \leq M''' \|\nabla u^n\|_{2,\Omega}^{m_+}$$

where

$$\Omega_+ := \left\{ x \in \Omega : \quad |u|^{m(x)} > 1 \right\}, \quad \Omega_- := \left\{ x \in \Omega : \quad |u|^{m(x)} \leq 1 \right\},$$

$$M'' := \frac{1}{l_0} L \left[1 + M'(\Omega, m_-, m_+) \right], \quad M''' := M'' \sup_{t\in[0,T]} \|g\|_{2,\Omega} \|\omega\|_{m_+,\Omega}.$$

Plugging (25), (26), (27) into (23) and using the assumption $m_+ \leq 2^*$, we obtain

$$\frac{1}{2} \frac{d}{dt} \left(1 + \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \right) + \|\nabla u^n\|_{L^p(\Omega)}^p + \int_\Omega |u^n|^{m(x)} dx \leq$$
$$M_1 \left[1 + \|u^n\|_{2,\Omega}^2 + \|\nabla u^n\|_{2,\Omega}^2 \right]^\theta + M_0, \tag{28}$$

where

$$\theta = \max\{1, \frac{p}{2}, \frac{m_+}{2}\}, \ M_0 = \frac{1}{2 l_0^2} \sup_{t\in[0,T]} \left(\|g(t)\|_{2,\Omega}^2 |e'(t)|^2 + M'(t) \right), \quad M_1 = \max\{M''', \frac{1}{4}\}.$$

Omitting second and third terms on left hand side, integrating by $\tau \in (0,t)$, we get and in case $p \le 2,\ m_+ \le 2$ applying Gronwall's lemma for $Y(t) := 1 + \|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega}$, we get

$$Y(t) \le M_0 e^{M_1 T} \tag{29}$$

In case $2 < m_+$ or $2 < p$ applying generalized Gronwall's lemma for $Y(t)$, we have

$$Y(t) \le M_0 \left(1 - (\theta - 1) M_1 M_0^{\theta-1} T\right)^{-\frac{1}{\theta-1}} \quad \text{for} \quad 0 \le t \le T_0 := \frac{1}{(\theta-1) M_1 M_0^{\theta-1}}. \tag{30}$$

Substituting (29) and (30) into (28), and taking supremum by t, we obtain the following first energy estimate

$$\sup_{t \in (0, T_{max}]} \left(\|u^n\|^2_{2,\Omega} + \|\nabla u^n\|^2_{2,\Omega} \right) + \|\nabla u^n\|^p_{L^p(Q_{T_{max}})} + \int\limits_\Omega |u^n|^{m(x)}\, dx \le K_0, \tag{31}$$

where $T_{max} = T$ if (17) holds, i.e. global in time, and $T_{max} = T_0$ (18) holds, i.e. local in time.

Second priory estimate. Let us now multiply (20) by $\frac{d c_j^n}{dt}$ and sum up the result from $j = 1$ to $j = n$, and integrate by τ from 0 to $t \in [0,T]$).

Then we have

$$\begin{aligned} &\int\limits_0^t \left(\|u_t^n(\tau)\|^2_{2,\Omega} + \|\nabla u_t^n(\tau)\|^2_{2,\Omega} \right) d\tau + \frac{1}{p} \|\nabla u^n\|^p_{L^p(\Omega)} + \int\limits_\Omega \frac{1}{m(x)} |u^n|^{m(x)} dx \le \\ &\frac{1}{p} \|\nabla u^n(0)\|^p_{L^p(\Omega)} + G, \end{aligned} \tag{32}$$

$$\begin{aligned} \text{where} \quad G = &\int\limits_0^t \frac{1}{g_0(\tau)} \left(e'(\tau) + \int\limits_\Omega |\nabla u^n|^{p-2} \nabla u^n \cdot \nabla \omega \ dx \right. \\ &\left. + \int\limits_\Omega |u^n|^{m(x)-2} u^n \cdot \omega dx \right) \int\limits_\Omega g(x,\tau) u_t^n(x,\tau) dx d\tau. \end{aligned} \tag{33}$$

Let we estimate G on the right side

$$\begin{aligned} &|G| \le \frac{1}{2} \int\limits_0^t \|u_t^n\|^2_{2,\Omega}\, d\tau + \frac{1}{2 l_0^2} \int\limits_0^t \|g(t)\|^2_{2,\Omega} \, (|e'(t)| + \\ &\|\nabla \omega\|_{p,\Omega} \cdot \|\nabla u^n\|^{p-1}_{p,\Omega} + \|\omega\|_{m_+,\Omega} \left(\int\limits_\Omega |u^n|^{m(x)-1 \cdot \frac{m_+}{m_+-1}}\, dx \right)^{\frac{m_+-1}{m_+}} \Bigg)^2 d\tau \le \\ &\frac{1}{2} \int\limits_0^t \|u_t^n\|^2_{2,\Omega}\, d\tau + \overline{M} \int\limits_0^t \Big(|e'(t)|^2 + \|\nabla \omega\|^2_{p,\Omega} \cdot \|\nabla u^n\|^{2(p-1)}_{p,\Omega} + \end{aligned}$$

$$\overline{M}(m_+,m_-,\Omega)\|\omega\|^2_{m_+,\Omega}\|\nabla u^n\|^{2(m_+-1)}_{2,\Omega}\Big)\,d\tau \le$$
$$\frac{1}{2}\int_0^t \|u^n_t\|^2_{2,\Omega}\,d\tau + \overline{M}\,\|\nabla\omega\|^2_{p,\Omega}\int_0^t \|\nabla u^n\|^{2(p-1)}_{p,\Omega}\,d\tau + \overline{M}' \tag{34}$$

where $\overline{M}' := \overline{M}\int\limits_0^t \left(|e'(t)|^2 + M(m_+,m_-,\Omega)\|\omega\|^2_{m_+,\Omega}K_0^{m_+-1}\right)d\tau$.

Plugging (34) into (32) we obtain

$$\int_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right)d\tau + \frac{1}{p}\|\nabla u^n\|^p_{p,\Omega} + \int_\Omega \frac{1}{m(x)}|u^n|^{m(x)}dx \le$$
$$\overline{M}_0\int_0^t\left(\frac{1}{p}\|\nabla u^n\|^p_{p,\Omega}\right)^{\frac{2(p-1)}{p}}d\tau + \overline{M}_1 \tag{35}$$

where $\overline{M}_1 := \overline{M}' + \frac{1}{p}\|\nabla u_0\|^p_{p,\omega}$. Omitting first and third terms on left hand side and in case $p \le 2$ applying classical Gronwall's lemma for $Z(t) := \frac{1}{p}\|\nabla u^n\|^p_{p,\Omega}$ we get

$$Z(t) \le \overline{M}_0 e^{\overline{M}_1 T}. \tag{36}$$

In the case $2 < p$, applying the generalized Gronwall's lemma for $Z(t) := \frac{1}{p}\|\nabla u^n\|^p_{p,\Omega}$ we get

$$Z(t) \le \overline{M}_0\left(1 - (\frac{2(p-1)}{p} - 1)\overline{M}_1\overline{M}_0^{\frac{2(p-1)}{p}-1}t\right)^{-\frac{1}{\frac{2(p-1)}{p}-1}} \tag{37}$$

for

$$0 \le t < T_1 := \frac{1}{(\frac{2(p-1)}{p} - 1)\overline{M}_1\overline{M}_0^{\frac{2(p-1)}{p}-1}} < T_0. \tag{38}$$

Substituting (36) and (37) into (35), and taking supremum by t, we obtain the following first energy estimate

$$\int_0^t \left(\|u^n_t(\tau)\|^2_{2,\Omega} + \|\nabla u^n_t(\tau)\|^2_{2,\Omega}\right)d\tau + \frac{1}{p}\|\nabla u^n\|^p_{p,\Omega} + \gamma\int_\Omega \frac{1}{m(x)}|u^n|^{m(x)}dx \le K_1, \tag{39}$$

where $T_{max} = T$ if (17) holds, i.e. global in time, and $T_{max} = T_1$ (18) holds, i.e. local in time.

6 Conclusion

In conclusion, we have established the unique solvability of the inverse problem pseudo-parabolic equation with p-Laplacian and damping term with variable exponents. Using the Galerkin method the approximate solutions are constructed. The global and local a priori estimates are obtained for approximate solutions. Using these a priori estimates and the compactness theorems and the monotonicity method the convergence of approximate solutions to the solution of the initial problem is proved. The uniqueness of the posed inverse problem is also obtained. The results of this work can be applied to solve various inverse problems for linear and nonlinear equations of mathematical physics.

7 Acknowledgment

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP08052425.

References

[1] Ablabekov B.S., *Obratnaiye zadachi dlya pseudoparabolicheskikh urabnenii* [Inverse Problems for Pseudoparabolic Equations]. (Bishkek:Ilim, 2001): 181.

[2] Abylkairov U.U., Khompysh Kh., *An inverse problem of identifying the coefficient in Kelvin-Voight equations.* Appl. Math. Sci.–2015. –103:9. –5079–5088.

[3] Al'shin A.B., Korpusov M.O., Sveshnikov A.G., *Blow-up in nonlinear Sobolev type equations.* (Berlin.: De Gruyter, 2011): 648.

[4] Antontsev S.N., Aitzhanov S.E., Ashurova G.R., *An inverse problem for the pseudo-parabolic equation with p-Laplacian*, Evol. Eq. and Cont. The. –2021.

[5] Antontsev S.N., Diaz J.I., Shmarev S., *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics.* Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, 2002.

[6] Antontsev S.N., Khompysh Kh., *An inverse problem for generalized Kelvin-Voigt equation with p-Laplacian and damping term.* Inverse problems (submitted).

[7] Asanov A., Atamanov E.R., Nonclassical and Inverse Problems for Pseudoparabolic Equations, (Berlin: De Gruyter, 1997): 152.

[8] Barenblatt G.I., Entov V.M., Ryzhik V.M., *Teoriya nestasionarnoi filtratsii jidkostei i gazov* [Theory of Unsteady Fluids.]. (Moscow:Nedra, 1972): 288.

[9] Fedorov V.E.,Urasaeva A.V., *An inverse problem for linear Sobolev type equation.* J. of Inverse and Ill-posed Prob. –2004. –12. –387-395.

[10] Huilgol R.,*A second order fluid of the differential type.* Int. J. Non-Linear Mech. –1968. -3. –471-482.

[11] Khompysh Kh., *Inverse Problem for 1D Pseudo-parabolic Equation.* Fun. Anal. in Interdisciplinary Appl. –2017. –216. –382-387.

[12] Khompysh Kh., *Inverse problem with integral overdetermination for system of equations of Kelvin-Voight fluids.* Advanced Materials Research. –2013. –705. –15-20.

[13] Khompysh Kh., Nugymanova N.K., *Inverse Problem For Integro-Differential Kelvin-Voigt Equation*, Jour. of Inverse and Ill-Posed Prob. (Submitted).

[14] Kozhanov A.I., *On the solvability of inverse coefficient problems for some Sobolev-type equations.* Nauchn. Vedom. Belgorod. Univ. –2010. –18:5. –88-97.

[15] Lions J.-L., Quelques methodes de resolution des problemes aux limites non liniaires, (Paris.: Dunod, 1969): 554.

[16] Lorenzi A., Paparoni E., *Identification problems for pseudoparabolic integrodifferential operator equations.* J. Inverse. Ill-Posed Probl. –1997. –5. –235-253.

[17] Lyubanova A.Sh., Tani A., *An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity.* Appl. Anal. –2011. –90. –1557-1568.

[18] Lyubanova A.Sh., Velisevich A.V., *Inverse problems for the stationary and pseudoparabolic equations of diffusion.* Applicable Anal. –2019. –98. –1997-2010.

[19] Ting T.W., *Certain nonsteady flows of second-order fluids.* Arch. Rational Mech. Anal. –1963. –14. –1-26.

[20] Yaman M., *Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation.* Jour. of Ineq. and App. –2012. –2012. –1-8.

[21] Zvyagin V.G., Turbin M.V., *The study of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids.* J. Math. Sci. –2010. –168. –157-308.

[22] Antontsev S.N., Shmarev S., Evolution PDEs with nonstandard growth conditions: existence, uniqueness, localization, blow-up, (Paris: Atlantis Press, 2015): 426.

Список литературы

[1] Аблабеков Б.С., *Обратные задачи для псевдопараболических уравнений.* Bishkek:Ilim, 2001. –181c.

[2] Abylkairov U.U., Khompysh Kh., *An inverse problem of identifying the coefficient in Kelvin-Voight equations.* Appl. Math. Sci.–2015. –103:9. –5079–5088.

[3] Al'shin A.B., Korpusov M.O., Sveshnikov A.G., *Blow-up in nonlinear Sobolev type equations.* (Berlin.: De Gruyter, 2011): 648.

[4] Antontsev S.N., Aitzhanov S.E., Ashurova G.R., *An inverse problem for the pseudo-parabolic equation with p-Laplacian*, Evol. Eq. and Cont. The. –2021.

[5] Antontsev S.N., Diaz J.I., Shmarev S., *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics.* (Berlin.: De Gruyter, 2011) Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, 2002.

[6] Antontsev S.N., Khompysh Kh., *An inverse problem for generalized Kelvin-Voigt equation with p-Laplacian and damping term.* Inverse problems (submitted).

[7] Asanov A., Atamanov E.R., Nonclassical and Inverse Problems for Pseudoparabolic Equations, (Berlin: De Gruyter, 1997): 152.

[8] Баренблатт Г.И., Ентов В.М., Рыжик В.М., *Теория нестационарной фильтрации жидкости и газа* . M.:Nedra, 1972. –288c.

[9] Fedorov V.E.,Urasaeva A.V., *An inverse problem for linear Sobolev type equation.* J. of Inverse and Ill-posed Prob. –2004. –12. –387-395.

[10] Huilgol R., *A second order fluid of the differential type.* Int. J. Non-Linear Mech. –1968. -3. –471-482.

[11] Khompysh Kh., *Inverse Problem for 1D Pseudo-parabolic Equation.* Fun. Anal. in Interdisciplinary Appl. –2017. –216. –382-387.

[12] Khompysh Kh., *Inverse problem with integral overdetermination for system of equations of Kelvin-Voight fluids.* Advanced Materials Research. –2013. –705. –15-20.

[13] Khompysh Kh., Nugymanova N.K., *Inverse Problem For Integro-Differential Kelvin-Voigt Equation*, Jour. of Inverse and Ill-Posed Prob. (Submitted).

[14] Kozhanov A.I., *On the solvability of inverse coefficient problems for some Sobolev-type equations.* Nauchn. Vedom. Belgorod. Univ. –2010. –18:5. –88-97.

[15] Lions J.-L., *Quelques methodes de resolution des problemes aux limites non liniaires.* (Paris.: Dunod, 1969): 554.

[16] Lorenzi A., Paparoni E., *Identification problems for pseudoparabolic integrodifferential operator equations.* J. Inverse. Ill-Posed Probl. –1997. –5. –235-253.

[17] Lyubanova A.Sh., Tani A., *An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity.* Appl. Anal. –2011. –90. –1557-1568.

[18] Lyubanova A.Sh., Velisevich A.V., *Inverse problems for the stationary and pseudoparabolic equations of diffusion.* Applicable Anal. –2019. –98. –1997-2010.

[19] Ting T.W., *Certain nonsteady flows of second-order fluids.* Arch. Rational Mech. Anal. –1963. –14. –1-26.

[20] Yaman M., *Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation.* Jour. of Ineq. and App. –2012. –2012. –1-8.

[21] Zvyagin V.G., Turbin M.V., *The study of initial-boundary value problems for mathematical models of the motion of Kelvin-Voigt fluids.* J. Math. Sci. –2010. –168. –157-308.

[22] Antontsev S.N., Shmarev S., *Evolution PDEs with nonstandard growth conditions: existence, uniqueness, localization, blow-up.* (Paris: Atlantis Press, 2015): 426.

IRSTI 27.31.17; 27.35.38; 27.35.41 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.05

S.E. Aitzhanov[1*], **G.R. Ashurova**[1], **K.A. Zhalgassova**[2]
[1]Al-Farabi Kazakh National University, Kazakhstan, Almaty
[2]M.Auezov South Kazakhstan University, Kazakhstan, Shymkent
*e-mail: Aitzhanov.Serik81@gmail.com

IDENTIFICATION OF THE RIGHT HAND SIDE OF A QUASILINEAR PSEUDOPARABOLIC EQUATION WITH MEMORY TERM

The study of equations of mathematical physics, including inverse problems, is relevant today. This work is devoted to the fundamental problem of studying the solvability and qualitative properties of the solution of the inverse problem for a quasilinear pseudoparabolic equation (also called Sobolev-type equations) with memory term. To date, studies of direct and inverse problems for a pseudoparabolic equations are rapidly developing in connection with the needs of modeling and control of processes in thermal physics, hydrodynamics, and mechanics of a continuous medium. The pseudoparabolic equations similar to those considered in this work arise in the description of heat and mass transfer processes, processes of non-Newtonian fluids motion, wave processes, and in many other areas.

The main types of the inverse problems are: boundary, retrospective, coefficient and geometric. The boundary and retrospective inverse problems lead to the study of linear problems. In turn, the statements related to the study of coefficient and geometric types bring to the nonlinear problems. Coefficient inverse problems are divided into two main groups: coefficient inverse problems, where the unknown is a function of one or several variables, and finite-dimensional coefficient inverse problems.

In this article the existence and uniqueness of a weak and strong solution of the inverse problem in a bounded domain are proved by the Galerkin method. Also we used Sobolev's embedding theorems, and obtained a priori estimates for the solution. Moreover, we get local and global theorems on the existence of the solution.

Key words: Pseudoparabolic equation, inverse problem, existence, uniqueness, local solvability, global solvability, non-local condition.

С.Е. Айтжанов[1*], Г.Р. Ашурова[1], К.Ә. Жалғасова[2]
[1]Әл-Фараби атындағы Қазақ Ұлттық университеті, Қазақстан, Алматы қ.
[2]М.Әуезов атындағы Оңтүстік Қазақстан мемлекеттік университеті,
Қазақстан, Шымкент қ.
*e-mail: Aitzhanov.Serik81@gmail.com

Жады бар квазисызықты псевдопараболалық теңдеудің оң жағын анықтау

Математикалық физика теңдеулерін, оның ішінде кері есептерді зерттеу бүгінгі күні өзекті болып табылады. Бұл жұмыс квазисызықтық псевдопараболалық теңдеу үшін кері есептің шешімділігі мен сапалық қасиеттерін зерттеудің іргелі мәселесіне арналған (Соболев типті теңдеулер деп те аталады). Бүгінгі таңда псевдопараболалық теңдеулер үшін тура және кері есептерді зерттеу жылу физикасы, гидродинамика және үздіксіз орта механикасындағы процестерді модельдеу және басқару қажеттіліктеріне байланысты тез дамып келеді. Осы жұмыста қарастырылған псевдопараболалық теңдеулер жылу-масса алмасу процестерін, ньютондық емес сұйықтықтардың қозғалыс процестерін, толқындық процестерді және басқа да көптеген салаларды сипаттау кезінде пайда болады.

Кері есептердің негізгі түрлеріне мыналар жатады: шекаралық, ретроспективті, коэффициенттік және геометриялық. Шекаралық және ретроспективті кері есептер -сызықтық есептерді зерттеуге, ал коэффициенттік және геометриялық есептер -сызықтық емес есептерді зерттеуге алып келеді. Коэффициенттік кері есептер екі негізгі түрге бөлінеді-коэффициенттік кері есептер, онда бір немесе бірнеше айнымалылардан тәуелді функциясы белгісіз және шекті өлшемді коэффициенттік кері есептер. Мақалада Галеркин әдісімен шенелген облыстағы кері есептің әлсіз және әлді шешімінің бар және жалғыздығы дәлелденеді. Соболевтің енгізу теоремалары қолданылып, шешімнің априорлық бағалаулары алынды. Шешімнің локалді және глобалді шешімділігі туралы теоремалар алынды.
Түйін сөздер: Псевдопараболалық теңдеу, кері есеп, шешімнің бар болуы, шешімнің жалғыздығы, локалді шешімділік, глобалді шешімділік, локалді емес шарт.

С.Е. Айтжанов[1*], Г.Р. Ашурова[1], К.Ә. Жалғасова[2]
[1]Казахский национальный университет имени аль-Фараби, Казахстан, г. Алматы
[2]Южно-Казахстанский государственный университет имени М. О. Ауэзова, Казахстан, г. Шымкент
[*]e-mail: Aitzhanov.Serik81@gmail.com

Идентификация правой части квазилинейного псевдопараболического уравнения с памятью

Исследование уравнений математической физики, в том числе обратных задач на сегодняшний день является актуальной. Эта работа посвящена фундаментальной проблеме исследованию разрешимости и качественных свойств решения обратной задачи для квазилинейного псевдопараболическогоуравнения (называемых также уравнениями соболевского типа) с памятью. На сегодняшний день исследования прямых и обратных задач для псевдопараболических уравнений бурно развиваются в связи с потребностями моделирования и управления процессами в теплофизике, гидродинамике и механике сплошной среды. Псевдопараболические уравнения подобные рассматриваемым в данной работе возникают при описании процессов тепломассопереноса, процессов движение неньютоновских жидкостей, волновых процессов и во многих других областях. К основным типам обратных задач относятся: граничные, ретроспективные, коэффициентные и геометрические. Граничные и ретроспективные обратные задачи приводят к исследованию линейных задач. В свою очередь, постановки, к которым приводит исследование коэффициентных и геометрических задач, являются нелинейными. Коэффициентные обратные задачи подразделяются на два основных вида — коэффициентные обратные задачи, в которых неизвестной является функция одной или нескольких переменных, и конечномерные коэффициентные обратные задачи. В статье методом Галеркина доказывается существование и единственность слабого и сильного решения обратной задачи в ограниченной области. Использование теорем вложения Соболева, получены априорные оценки решения. Получены локальная и глобальная теорема о существовании решения.
Ключевые слова: Псевдопараболическое уравнение, обратная задача, существования, единственность, локальная разрешимость, глобальная разрешимость, нелокальное условие.

1 Introduction

Let Ω is a bounded area of a space $R^N,\ N \geq 1$ with a sufficiently smooth boundary Γ, Q_T is a cylinder $\Omega \times (0,T)$ of finite height T, $S = \Gamma \times (0,T)$. Let $b(x,t),\ h(x,t),\ u_0(x),\ \varphi(t),\ \omega(x)$ are given functions, χ, a and β are positive constants. Consider an inverse problem in the cylinder Q_T for a pseudoparabolic equation with a nonlocal overdetermination condition. Find a pair of functions $\{u(x,t),\ f(t)\}$ that satisfy:

$$u_t - \chi\Delta u_t - a\Delta u - \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = b(x,t)|u|^{\beta-2}u + f(t)h(x,t), \tag{1}$$

$$u(x,0) = u_0(x), \tag{2}$$

$$u|_S = 0, \tag{3}$$

$$\int\limits_\Omega u(x,t)(\omega(x) - \chi\Delta\omega(x))dx = \varphi(t). \tag{4}$$

The monograph ([1] and see its references) considers a wide class of direct problems for nonlinear Sobolev-type equations. In papers [2-12] some inverse problems similar to our problem statement were studied.

Let us note some papers on inverse problems for Sobolev-type equations with the integral overdetermination condition. Yaman M. [7] obtained sufficient conditions for both destruction and stability of the solution. The work [8] establishes an existence theorem for regular solutions of the inverse problem of recovering coefficients in equations of composite type. A.I. Kozhanov and L.A. Teleshova [9] have proved the existence theorem for regular solutions of a nonlinear inverse problem for nonstationary differential equations of higher order is proved. Kozhanov A.I. and Namsaraeva G.V. [10] showed the existence and uniqueness of the regular solutions of linear inverse problem for equation of Sobolev type. The work [11] is devoted to the investigation of the inverse problem for the equation

$$u_t - \chi\Delta u_t - \Delta u = b(x,t)|u|^{\beta-2}u + f(t)h(x),$$

with the conditions (2)-(4). In this paper the existence of a weak solution is proved, the asymptotic behavior of the solutions is shown at $t \to \infty$. Moreover, sufficient conditions of finite time "blow up"of the solution are obtained. In the work [12] is dedicated to the inverse problem for a pseudoparabolic equation with p-Laplacian. In the present work, we proved the existence of a weak solution and showed the asymptotic behavior of the solutions at $t \to \infty$. We obtained sufficient conditions of finite time "blow up"of the solution, furthermore we get sufficient conditions for the disappearance (vanishing) of the solution in a finite time.

From a physical point of view, the considered initial-boundary value problem is a mathematical model of quasi-stationary processes in semiconductors and magnets allowing for a wide variety of physical factors.

Let the functions $h(x,t)$, $\omega(x)$, $\varphi(t)$, $u_0(x)$ satisfy the following conditions:

$$\begin{array}{l} h_1(t) \equiv \int\limits_\Omega h(x,t)\omega(x)dx \neq 0, \ \ \forall t \in [0,T], \\ h(x,t) \in L_\infty(0,T;\ L_2(\Omega)) \cap L_2(Q_T) \cap L_{\beta*}(Q_T), \ \ \beta* = \frac{\beta}{\beta-1}, \ \ \beta \geq 2. \end{array} \tag{5}$$

$$\omega \in L_2(\Omega) \cap L_\beta(\Omega) \cap \overset{0}{W}{}_2^2(\Omega), \ \ \beta \geq 2. \tag{6}$$

$$\int\limits_\Omega u_0(x)\omega(x)dx = \varphi(0), \ \ \varphi(t) \in W_2^1[0,T], \ \ |\varphi'(t)| \leq C, \ \ u_0 \in \overset{0}{W}{}_2^1(\Omega) \cap L_\beta(\Omega). \tag{7}$$

2 Main results

2.1 Reducing the inverse problem (1)-(4) to a direct problem

Lemma 1 *The problem* (1) - (4) *is equivalent to the following problem for a nonlinear pseudoparabolic equation containing a nonlinear nonlocal operator of the function* $u(x,t)$

$$u_t - \chi\Delta u_t - a\Delta u - \int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau = b(x,t)|u|^{\beta-2}u + F(t,u)h(x,t), \ \ x \in \Omega, \ \ \ t > 0, \tag{8}$$

$$u(x,0) = u_0(x), \ \ x \in \Omega, \ \ \ \ u|_S = 0. \tag{9}$$

Here

$$F(t,u) = \frac{1}{h_1(t)}\left(\varphi'(t) + a\int\limits_\Omega \nabla u\nabla\omega dx + \int\limits_0^t g(t-\tau)\int\limits_\Omega \nabla u(\tau)\nabla\omega dxd\tau - \int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right). \tag{10}$$

Proof. Indeed, from the equation (1) follows that

$$\begin{aligned}&\int\limits_\Omega u_t\omega dx-\chi\int\limits_\Omega \Delta u_t\omega dx-a\int\limits_\Omega \Delta u\omega dx-\int\limits_\Omega\int\limits_0^t g(t-\tau)\Delta u(\tau)d\tau\omega dx=\\&=\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx+\int\limits_\Omega f(t)h(x,t)\omega dx,\end{aligned}\tag{11}$$

next if conditions (4) and (5) are satisfied, then

$$F(t,u)=\frac{1}{h_1(t)}\left(\varphi'(t)+a\int\limits_\Omega \nabla u\nabla\omega dx+\int\limits_0^t g(t-\tau)\int\limits_\Omega \nabla u(\tau)\nabla\omega dxd\tau-\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right).\tag{12}$$

Therefore the relation (10) is fulfilled. Now consider the problem (8)-(9). If the relation (10) is performed, then it obviously leads to the equality (12) . Then

$$\begin{aligned}&F(t,u)=\tfrac{1}{h_1}\left(\varphi'(t)+a\int\limits_\Omega \nabla u\nabla\omega dx+\int\limits_0^t g(t-\tau)\int\limits_\Omega \nabla u(\tau)\nabla\omega dxd\tau-\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right)=\\&=\tfrac{1}{h_1}\left(\varphi'(t)-a\int\limits_\Omega \Delta u\omega dx-\int\limits_0^t g(t-\tau)\int\limits_\Omega \Delta u(\tau)\omega dxd\tau-\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right).\end{aligned}$$

By virtue of (11) we obtain that

$$\begin{aligned}&F(t,u)=\tfrac{1}{h_1}\left(\varphi'(t)+a\int\limits_\Omega \nabla u\nabla\omega dx+\int\limits_0^t g(t-\tau)\int\limits_\Omega \nabla u(\tau)\nabla\omega dxd\tau-\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right)=\\&=\tfrac{1}{h_1}\left(\varphi'(t)-\int\limits_\Omega (u_t-\chi\Delta u_t)\omega dx+\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx+\right.\\&\left.+\int\limits_\Omega f(t)h(x,t)\omega dx+\int\limits_\Omega b(x,t)|u|^{\beta-2}u\omega dx\right),\end{aligned}$$

$$\varphi'(t)-\int\limits_\Omega u_t(\omega-\chi\Delta\omega)dx=0.$$

In this way, $\frac{d}{dt}\left(\varphi(t)-\int\limits_\Omega u(\omega-\chi\Delta\omega)dx\right)=0$. We denote by $v(t)=\varphi(t)-\int\limits_\Omega u(\omega-\chi\Delta\omega)dx$. Then the function $v(t)$ can be found as a solution of the Cauchy problem: $v'(t)=0,\ \ v(0)=0$. ($v(0)=0$ follows from the agreement condition (7)). The unique solution of the problem is the function $v(t)=0$, consequently, $\int_\Omega u(\omega-\chi\Delta\omega)dx=\varphi(t)$.

Definition 1 *A function $u(x,t)$ from the space $W_2^1(0,T;\overset{0}{W}{}_2^1(\Omega))$ is called a weak solution of the problem* (8)-(9) *which satisfies the integral identity*

$$\begin{aligned}&\int\limits_0^T\int\limits_\Omega\left(\tfrac{\partial u}{\partial t}v+\chi\nabla u_t\nabla v+a\nabla u\nabla v\right)dxdt+\int\limits_0^T\int\limits_0^t g(t-\tau)\int\limits_\Omega \nabla u(\tau)\nabla v(t)dxd\tau dt-\\&-\int\limits_0^T\int\limits_\Omega b(x,t)|u|^{\beta-2}uvdxdt=\int\limits_0^T\int\limits_\Omega F(t,u)hvdxdt,\end{aligned}\tag{13}$$

for all $v(x,t)\in L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$.

Definition 2 *A function $u(x,t)$ from the space u_t, $\triangle u$, $\triangle u_t$, $\in L_2(Q_T)$, is called a strong solution of the problem* (8)-(9) *which satisfies the integral identity*

$$\begin{aligned}&\int\limits_0^T\int\limits_\Omega\left(\tfrac{\partial u}{\partial t}v-\chi\triangle u_t v-a\triangle uv\right)dxdt+\int\limits_0^T\int\limits_0^t g(t-\tau)\int\limits_\Omega \triangle u(\tau)v(t)dxd\tau dt-\\&-\int\limits_0^T\int\limits_\Omega b(x,t)|u|^{\beta-2}uvdxdt=\int\limits_0^T\int\limits_\Omega F(t,u)hvdxdt,\end{aligned}\tag{14}$$

for all $v(x,t)\in L_2(Q_t)$.

2.2 Existence of a weak solution

Theorem 1 *Let the conditions* (5)-(7) *are performed, and also* $2 < \beta < \frac{2N}{N-2}$, $N \geq 3$. *Then there is a weak solution* $u(x,t)$ *of the problem* (8)-(9) *on the interval* $(0,T)$, $T < T_0$, *and besides accepts the following inclusions:*

$$u \in L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)), \nabla u \in L_2(Q_T),\ Q_T = \Omega \times (0,T),$$

$$u_t \in L_2(0,T;\overset{0}{W}{}_2^1(\Omega)), |u|^{\beta-2}u \in L_\infty(0,T;\ L_{\frac{\beta}{\beta-1}}(\Gamma)).$$

Proof. Let us choose in $\overset{0}{W}{}_2^1(\Omega)$ some system of functions $\{\Psi_j(x)\}$ forming a basis in a given space $(\Delta\Psi + \lambda\Psi = 0,\ \Psi|_\Gamma = 0)$. We will look for an approximate solution of the problem (8)-(9) in the form

$$u_m(x,t) = \sum_{k=1}^{m} C_{mk}(t)\Psi_k(x) \tag{15}$$

where the coefficients $C_{mk}(t)$ are searched out from the conditions

$$\begin{aligned} &\sum_{k=1}^{m} C'_{mk}(t) \int_\Omega \left[\Psi_k\Psi_j + \chi \sum_{i=1}^{m} \frac{\partial\Psi_k}{\partial x_i}\cdot\frac{\partial\Psi_j}{\partial x_i}\right] dx + a\sum_{k=1}^{m} C_{mk}(t)\int_\Omega \nabla\Psi_k\nabla\Psi_j dx + \\ &+ \sum_{k=1}^{m}\int_0^t g(t-\tau)C_{mk}(\tau)\int_\Omega \nabla\Psi_k\nabla\Psi_j dx d\tau - \\ &- \sum_{k=1}^{m} C_{mk}(t)\int_\Omega b(x,t)|u_m|^{\beta-2}\Psi_k\Psi_j dx = \int_\Omega F(t,u_m)h(x,t)\Psi_j dx. \end{aligned} \tag{16}$$

$$u_{m0} = u_m(0) = \sum_{k=1}^{m} C_{mk}(0)\Psi_k = \sum_{k=1}^{m} \alpha_k\Psi_k \tag{17}$$

and besides

$$u_{m0} \to u_0 \quad strongly \quad in \quad \overset{0}{W}{}_2^1(\Omega) \quad at \quad m \to \infty \tag{18}$$

We introduce a notation

$$\vec{C}_m \equiv \{C_{1m}(t), ..., C_{mm}(t)\}^T, \vec{\alpha} \equiv \{\alpha_1, ..., \alpha_m\}^T,$$

$$a_{kj} = \int_\Omega \left[\Psi_k\Psi_j + \chi\left(\nabla\Psi_k, \nabla\Psi_j\right)\right] dx, b_{kj} = \int_\Omega \nabla\Psi_k\nabla\Psi_j dx,$$

$$f_{kj} = -a\int_\Omega \nabla\Psi_k\nabla\Psi_j dx + \int_\Omega b\left(x,t\right)\left|u_m\right|^{\beta-2}\Psi_k\Psi_j dx + \int_\Omega F(t,u_m)h(x,t)\Psi_j dx,$$

$$A_m\left(\vec{C}_m\right) \equiv \left\{a_{jk}\left(\vec{C}_m\right)\right\}, B_m\left(\vec{C}_m\right) \equiv \left\{b_{jk}\left(\vec{C}_m\right)\right\}, \vec{F}_m\left(\vec{C}_m\right) \equiv \left\{f_{jk}\left(\vec{C}_m\right)\right\}\vec{C}_m.$$

Then the system of equations (16) and condition (17) take the matrix form.

$$A_m\vec{C}'_m + \int_0^t g(t-\tau)B_m C_m(\tau)d\tau \equiv \vec{F}_m\left(\vec{C}_m\right),\quad \vec{C}_m(0) = \vec{\alpha}. \tag{19}$$

According to the Cauchy theorem, the problem (19) has at least one solution $\vec{C}_m$ on a certain time interval $t \in (0,T_m)$, $T_m > 0$. Below we obtain a priori estimates for u_m which is independent of m and, in some cases, valid for any finite t.

2.3 A priori estimates

To get the first estimate, we multiply both sides of the equality (16) by $C_{mj}(t)$ and summarize both sides of the obtained equality over $j=\overline{1,m}$. As a result, we get the equality

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}[|u_m|^2+\chi|\nabla u_m|^2]dx+a\int\limits_{\Omega}|\nabla u_m|^2dx+\\+\int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla u_m(t)dxd\tau=\int\limits_{\Omega}b(x,t)|u_m|^{\beta}dx+\int\limits_{\Omega}F(t,u_m)hu_mdx.\end{gathered} \tag{20}$$

Lemma 2 *If* $u\in \overset{0}{W}{}_2^1(\Omega),\ 2<\beta<\frac{2N}{N-2},\ \ N\geq 3,$ *then the next inequality is performed*

$$\|u\|_{\beta,\Omega}^2\leq C_0^2\,\|\nabla u\|_{2,\Omega}^{2\alpha}\,\|u\|_{2,\Omega}^{2(1-\alpha)}\leq \chi\,\|\nabla u\|_{2,\Omega}^2+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_0^{\frac{2}{1-\alpha}}}{\chi^{\frac{\alpha}{1-\alpha}}}\,\|u\|_{2,\Omega}^2\,,$$

where $C_0=\left(\frac{2(N-1)}{N-2}\right)^{\alpha},\ \ \alpha=\frac{(\beta-2)N}{2\beta},\ \ 0<\alpha<1.$

From the lemma 2 follows the inequality

$$\|u\|_{\beta,\Omega}^{\beta}\leq C_1\left(\|u\|_{2,\Omega}^2+\chi\,\|\nabla u\|_{2,\Omega}^2\right)^{\frac{\beta}{2}},$$

where $C_1=\left(\max\left\{1;\ \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}C_0^{\frac{2}{1-\alpha}}}{\chi^{\frac{\alpha}{1-\alpha}}}\right\}\right)^{\frac{\beta}{2}}.$

We estimate the right hand side of (20)

$$\left|\int\limits_{\Omega}b(x,t)|u_m|^{\beta}dx\right|\leq b_0\,\|u_m\|_{\beta,\Omega}^{\beta}\leq C_1b_0\left(\|u_m\|_{2,\Omega}^2+\chi\,\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{\beta}{2}}, \tag{21}$$

$$\left|\frac{\varphi'(t)}{h_1(t)}\int\limits_{\Omega}hu_mdx\right|\leq\frac{|\varphi'(t)|}{|h_1(t)|}\,\|h\|_{2,\Omega}\,\|u_m\|_{2,\Omega}\leq\frac{1}{4}\sup_{0\leq t\leq T}\frac{|\varphi'(t)|^2}{|h_1(t)|^2}\,\|h\|_{2,\Omega}^2+\|u_m\|_{2,\Omega}^2\,, \tag{22}$$

$$\begin{gathered}\left|\frac{a}{h_1(t)}\int\limits_{\Omega}hu_mdx\int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla\omega dxd\tau\right|\leq\\ \leq\frac{a}{|h_1(t)|}\,\|h(x,t)\|_{2,\Omega}\,\|u_m\|_{2,\Omega}\int\limits_0^t g(t-\tau)\,\|\nabla u_m(\tau)\|_{2,\Omega}\,\|\nabla\omega\|_{2,\Omega}\,d\tau\leq\\ \leq\frac{a^2}{4}\left(\|\nabla\omega\|_{2,\Omega}\sup_{0\leq t\leq T}\frac{1}{|h_1(t)|}\,\|h(x,t)\|_{2,\Omega}\right)^2\int\limits_0^t\|\nabla u_m(\tau)\|_{2,\Omega}^2\,d\tau+\|u_m\|_{2,\Omega}^2\,.\end{gathered} \tag{23}$$

$$\begin{gathered}\left|\int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla u_m(t)dxd\tau\right|\leq\int\limits_0^t g(t-\tau)\,\|\nabla u_m(\tau)\|_{2,\Omega}\,\|\nabla u_m(t)\|_{2,\Omega}\,d\tau\leq\\ \leq\frac{g_0}{2a}\int\limits_0^t\|\nabla u_m(\tau)\|_{2,\Omega}^2\,d\tau+\frac{a}{2}\,\|\nabla u_m(t)\|_{2,\Omega}^2\,.\end{gathered}$$

$$\begin{gathered}\left|\frac{1}{h_1(t)}\int\limits_{\Omega}hu_mdx\int\limits_{\Omega}b(x,t)|u_m|^{\beta-2}u_m\cdot\omega dx\right|\leq\frac{b_0}{|h_1(t)|}\,\|u_m\|_{\beta,\Omega}^{\beta}\,\|\omega\|_{\beta,\Omega}\,\|h\|_{\frac{\beta}{\beta-1},\Omega}\leq\\ \leq C_1b_0\,\|\omega\|_{\beta,\Omega}\sup_{0\leq t\leq T}\frac{1}{h_1^2(t)}\,\|h(x,t)\|_{\frac{\beta}{\beta-1},\Omega}\left(\|\nabla u_m\|_{2,\Omega}^2+C_1\,\|u_m\|_{2,\Omega}^2\right)^{\frac{\beta}{2}}.\end{gathered} \tag{24}$$

$$\begin{gathered}\frac{d}{dt}\int\limits_{\Omega}[|u_m|^2+\chi|\nabla u_m|^2]dx+a\int\limits_{\Omega}|\nabla u_m|^2dx\le\\ \le C_2\left(\|u_m\|_{2,\Omega}^2+\chi\|\nabla u_m\|_{2,\Omega}^2\right)+C_3\left(\|u_m\|_{2,\Omega}^2+\chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{\beta}{2}}+\\ +C_4\int\limits_0^t\|\nabla u_m(\tau)\|_{2,\Omega}^2d\tau+\frac{1}{4}\sup_{0\le t\le T}\frac{|\varphi'(t)|^2}{|h_1(t)|^2}\|h\|_{2,\Omega}^2.\end{gathered}\tag{25}$$

We denote by $y(t)\equiv\chi\|\nabla u\|_{2,\Omega}^2+\|u\|_{2,\Omega}^2$, then (25) takes the form

$$\frac{dy(t)}{dt}\le C_4\int\limits_0^t y(\tau)d\tau+C_3[y(t)]^{\frac{\beta}{2}}+C_2y(t)+\frac{1}{4}\sup_{0\le t\le T}\frac{|\varphi'(t)|^2}{|h_1(t)|^2}\|h\|_{2,\Omega}^2.$$

By integrating from 0 to t, we get

$$y(t)\le y(0)+\frac{1}{4}\sup_{0\le t\le T}\frac{|\varphi'(t)|^2}{|h_1(t)|^2}\int\limits_0^t\|h\|_{2,\Omega}^2d\tau+C_5(t^{\frac{2(\beta-1)}{\beta-2}}+t)+C_6\int\limits_0^t[y(\tau)]^{\frac{\beta}{2}}d\tau,$$

Applying the Bihari lemma to (25), if

$$\frac{C_4}{C_3}\left(e^{C_3\frac{p-2}{2}t}-1\right)<\frac{1}{\left(z(0)+\frac{C_2}{C_3}+\frac{C_0}{2(C_3-\gamma)}\right)^{\frac{p-2}{2}}},\quad 0\le t<T,$$

then the next inequality is true

$$z(t)\le\frac{z(0)+\frac{C_2}{C_3}+\frac{C_0}{2(C_3-\gamma)}}{\left[1-\left(z(0)+\frac{C_2}{C_3}+\frac{C_0}{2(C_3-\gamma)}\right)^{\frac{p-2}{2}}\frac{C_4}{C_3}\left(e^{C_3\frac{p-2}{2}t}-1\right)\right]^{\frac{2}{p-2}}}.$$

$$\begin{gathered}\chi\|\nabla u_m\|_{2,\Omega}^2+\|u_m\|_{2,\Omega}^2\le\\ \le\frac{\left(\chi\|\nabla u_m(x,0)\|_{2,\Omega}^2+\|u_m(x,0)\|_{2,\Omega}^2+C\right)e^{C_3t}}{\left[1-\left(\chi\|\nabla u_m(x,0)\|_{2,\Omega}^2+\|u_m(x,0)\|_{2,\Omega}^2+C\right)^{\frac{\beta-2}{2}}\frac{C_4}{C_3}\left(e^{C_3\frac{\beta-2}{2}t}-1\right)\right]^{\frac{2}{\beta-2}}}.\end{gathered}\tag{26}$$

From this estimate, we can conclude that there is $T_0>0$ such that

$$\|u_m\|_{2,\Omega}^2+\chi\|\nabla u_m\|_{2,\Omega}^2\le C_5,\quad for\quad all\quad t\in[0,T],\ T<T_0,\tag{27}$$

where the constant C_5 does not depend on $m\in\mathrm{N}$.

Returning to (25) and taking into account (27), we obtain one more inequality:

$$\int\limits_0^t\int\limits_{\Omega}|\nabla u_m|^2dxdt\le C_5.\tag{28}$$

Now we multiply equality (16) by $C'_{mj}(t)$ and summarize over $j=\overline{1,m}$. As a result, we get

$$\begin{gathered}\|\partial_\tau u_m\|_{2,\Omega}^2+\chi\|\nabla\partial_\tau u_m\|_{2,\Omega}^2+\frac{a}{2}\frac{d}{dt}\int\limits_{\Omega}|\nabla u_m|^2dx+\\ +\int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla\partial_t u_m(t)dxd\tau=\frac{1}{\beta}\frac{d}{dt}\int\limits_{\Omega}b(x,t)|u_m|^\beta dx-\\ -\frac{1}{\beta}\int\limits_{\Omega}b_t(x,t)|u_m|^\beta dx+\int\limits_{\Omega}F(t,u_m)h(x,t)\partial_t u_m dx.\end{gathered}\tag{29}$$

We integrate with respect to τ from 0 to t, then we get the relation

$$\begin{aligned}&\int\limits_0^t \left(\|\partial_\tau u_m\|_{2,\Omega}^2 + \chi \|\nabla \partial_\tau u_m\|_{2,\Omega}^2\right) d\tau + \frac{a}{2}\int\limits_\Omega |\nabla u_m|^2 dx = \frac{1}{2}\int\limits_\Omega |\nabla u_m(x,0)|^2 dx-\\ &-\frac{1}{\beta}\int\limits_\Omega b(x,0)|u_m(x,0)|^\beta dx + \frac{1}{\beta}\int\limits_\Omega b(x,t)|u_m|^\beta dx-\\ &-\frac{1}{\beta}\int\limits_0^t\int\limits_\Omega b_\tau(x,\tau)|u_m|^\beta dx d\tau - \int\limits_0^t\int\limits_0^\tau g(\tau-s)\int_\Gamma \nabla u_m(s)\nabla\partial_\tau u_m(\tau) d\Gamma ds d\tau+\\ &+\int\limits_0^t\int\limits_\Omega F(t,u_m)h(x,t)\partial_\tau u_m dx d\tau.\end{aligned} \tag{30}$$

We estimate the right hand side of (30):

$$\begin{aligned}&\left|\frac{1}{\beta}\int\limits_\Omega b(x,t)|u_m|^\beta dx\right| \le \frac{b_0}{\beta}\|u_m\|_{\beta,\Omega}^\beta \le\\ &\le \frac{C_1 b_0}{\beta}\left(\|u_m\|_{2,\Omega}^2 + \chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{\beta}{2}} \le \frac{C_1 b_0}{\beta} C_5^{\frac{\beta}{2}},\end{aligned} \tag{31}$$

$$\begin{aligned}&\left|\frac{1}{\beta}\int\limits_0^t\int\limits_\Omega b_\tau(x,\tau)|u_m|^\beta dx d\tau\right| \le \frac{b_0}{\beta}\int\limits_0^t \|u_m\|_{\beta,\Omega}^\beta d\tau \le\\ &\le \frac{C_1 b_0}{\beta}\int\limits_0^t \left(\|u_m\|_{2,\Omega}^2 + \chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{\beta}{2}} d\tau \le \frac{C_1 b_0}{\beta} C_5^{\frac{\beta}{2}} t,\end{aligned} \tag{32}$$

$$\begin{aligned}&\left|\int\limits_0^t\int\limits_\Omega \frac{\varphi'(\tau)}{h_1(\tau)} h(x,\tau)\partial_\tau u_m dx d\tau\right| \le\\ &\le \frac{3}{2}\int\limits_0^t\int\limits_\Omega \left|\frac{\varphi'(\tau)h(x,\tau)}{h_1(\tau)}\right|^2 dx d\tau + \frac{1}{6}\int\limits_0^t\int\limits_\Omega |\partial_\tau u_m|^2 dx d\tau.\end{aligned} \tag{33}$$

$$\begin{aligned}&\left|\int\limits_0^t\int\limits_\Omega \frac{h(x,\tau)}{h_1(\tau)}\partial_\tau u_m dx \int\limits_\Omega b(x,\tau)|u|^{\beta-2}u\omega dx d\tau\right| \le\\ &\le \frac{2}{3}C_1^{\frac{2\beta-2}{\beta}}\left(b_0\|\omega\|_{\beta,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\right)^2 \int\limits_0^t \left(\|u_m\|_{2,\Omega}^2 + \chi\|\nabla u_m\|_{2,\Omega}^2\right)^{\frac{2\beta-2}{\beta}} d\tau+\\ &+\frac{1}{6}\int\limits_0^t \|\partial_\tau u_m\|_{2,\Omega}^2 d\tau \le \frac{1}{6}\int\limits_0^t \|\partial_\tau u_m\|_{2,\Omega}^2 d\tau+\\ &+\frac{2}{3}C_1^{\frac{2\beta-2}{\beta}}\left(b_0\|\omega\|_{\beta,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{h_1(t)}\right)^2 C_5^{\frac{2\beta-2}{\beta}} t.\end{aligned} \tag{34}$$

$$\begin{aligned}&\left|\int\limits_0^t\int\limits_0^\tau g(\tau-s)\int_\Gamma \nabla u_m(s)\nabla\partial_\tau u_m(\tau) d\Gamma ds d\tau\right| \le \int\limits_0^t\int\limits_0^\tau g(\tau-s)\|\nabla u_m(s)\|_{2,\Omega}\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega} d\tau \le\\ &\le \frac{g_0}{2}\int\limits_0^t (t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}^2 d\tau + \frac{1}{2}\int\limits_0^t \|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}^2 d\tau.\end{aligned} \tag{35}$$

$$\begin{aligned}&\left|a\int\limits_0^t\int\limits_\Omega \frac{h(x,\tau)}{h_1(\tau)}\partial_\tau u_m dx \int\limits_0^\tau g(\tau-s)\int\limits_\Omega \nabla u_m(s)\nabla\omega dx ds d\tau\right| \le\\ &\le \frac{1}{6}\int\limits_0^t \|\partial_\tau u_m\|_{2,\Omega}^2 d\tau + \frac{2g_0}{3}\left(a\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{|h_1(t)|}\right)^2 \int\limits_0^t (t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}^2 d\tau.\end{aligned} \tag{36}$$

We substitute the obtained inequalities into the identity (30), and from the estimate (27), we get the second estimate

$$\int\limits_0^t \left(\|\partial_\tau u_m\|_{2,\Omega}^2 + \chi\|\nabla\partial_\tau u_m\|_{2,\Omega}^2\right) d\tau \le C_6. \tag{37}$$

2.4 Passage to the limit

From the obtained estimates (27), (28), (31) imply the next assertions respectively:

$$u_m \quad is \quad bounded \quad in \quad L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega)), \tag{38}$$

$$\nabla u_m \quad is \quad bounded \quad in \quad L_2(Q_T),\ Q_T=\Omega\times(0,T), \tag{39}$$

$$\partial_t u_m \quad is \quad bounded \quad in \quad L_2(0,T;\overset{0}{W}{}_2^1(\Omega)), \tag{40}$$

In addition, by virtue of the conditions which set for β:

$$|u_m|^{\beta-2}u_m \quad is \quad bounded \quad in \quad L_\infty(0,T;L_{\frac{\beta}{\beta-1}}(\Omega)), 2<\beta<\frac{2N}{N-2},\ \ N\geq 3. \tag{41}$$

From (38) follows that there exists a subsequence u_{m_k} of the sequence u_m, *-weakly converging to some element $u\in L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega))$, that is $u_{m_k} \ \to\ u$ *-weakly in $L_\infty(0,T;\overset{0}{W}{}_2^1(\Omega))$. Similarly, from (39)-(41) follows that there exists a sequence such that $\{u_{m_k}\}\subset\{u_m\}$, that $u_{m_k}\ \to\ u$ weakly in $L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$. By virtue of the Rellich-Kondrashov theorem, the embedding $W_2^1(Q_T)$ into $L_2(Q_T)$ is a compact. This means that we can choose the sequence u_{m_k} in such way that $u_{m_k}\ \to\ u$ in the norm of $L_2(Q_T)$, therefore it converges almost everywhere [13].

The reasoning above allows us to pass to the limit in (16). But first, we multiply each of the equality (16) by $d_j(t)\in C[0,T]$ and summarize both sides of the obtained equality over $j=\overline{1,m}$. Then we integrate with respect to t from 0 to T, and get

$$\begin{gathered}\int_0^t\int_\Omega\left(\frac{\partial u_m}{\partial t}\mu+\chi\nabla u_{mt}\nabla\mu+a\nabla u_m\nabla\mu\right)dxdt+\int_0^t\int_0^t g(t-\tau)\int_\Omega\nabla u_m(\tau)\nabla\mu(t)dxd\tau dt-\\ -\int_0^t\int_\Omega b(x,t)|u_m|^{\beta-2}u_m\mu dxdt=\int_0^t\int_\Omega F(t,u_m)h\mu dxdt,\end{gathered} \tag{42}$$

where $\mu(x,t)=\sum\limits_{j=1}^m d_j(t)\Psi_j(x)$.

Considering the obtained inclusions and convergences we pass to the limit in (42) at $m\to\infty$ and get (14) for $v=\mu$. Since the set of all functions $\mu(x,t)$ is dense in $W_2^1(0,T;\overset{0}{W}{}_2^1(\Omega))$, then the limit relation is performed for all $v(x,t)\in L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$.

3 Uniqueness of the weak solution

Theorem 2 *Let* $u_0(x)\in\overset{0}{W}{}_2^1(\Omega)$, $2<\beta\leq\frac{2(N-1)}{N-2}$, $N\geq 3$ *are fulfilled .Then the weak solution of the problem* (8)-(9) *on the interval* $(0,T)$ *is unique (in the sense of Definition 1).*

Proof. Suppose that the problem (8)-(9) has two solutions: $u_1(x,t)$ and $u_2(x,t)$. Then their difference $u(x,t)=u_1(x,t)-u_2(x,t)$ satisfies the condition $u(x,0)=0$ and the identity

$$\begin{gathered}\int_0^t\int_\Omega(u_\tau v+\chi\nabla u_\tau\nabla v+a\nabla u\nabla v)\,dxd\tau+\int_0^t\int_0^\tau g(\tau-s)\int_\Omega\nabla u(s)\nabla v(\tau)dxdsd\tau=\\ =\int_0^t\int_\Omega b(x,\tau)\left(|u_1|^{\beta-2}u_1-|u_1|^{\beta-2}u_1\right)vdxd\tau+\\ +\int_0^t\int_\Omega(F(\tau,u_1)-F(\tau,u_2))hvdxd\tau,\end{gathered}$$

By virtue of $v(x,t)\in L_2(0,T;\overset{0}{W}{}_2^1(\Omega))$, then as $v(x,t)$ we may take $u(x,t)$, that is we put $v(x,t)=u(x,t)$

$$\begin{aligned}&\int\limits_0^t\int\limits_\Omega\left(u_\tau u+\chi\nabla u_\tau\nabla u+a|\nabla u|^2\right)dxd\tau+\int\limits_0^t\int\limits_0^\tau g(\tau-s)\int\limits_\Omega\nabla u(s)\nabla u(\tau)dxdsd\tau=\\&=\int\limits_0^t\int\limits_\Omega b(x,\tau)\left(|u_1|^{\beta-2}u_1-|u_1|^{\beta-2}u_1\right)udxd\tau+\\&+\int\limits_0^t\int\limits_\Omega(F(\tau,u_1)-F(\tau,u_2))hudxd\tau.\end{aligned}\tag{43}$$

We estimate the right hand side of the inequality (43), using the following inequality $\left||u_1|^qu_1-|u_2|^qu_2\right|\leq(q+1)\left(|u_1|^q+|u_2|^q\right)|u_1-u_2|$, $q>0$.

$$\begin{aligned}&\left|\int\limits_\Omega b(x,\tau)\left(|u_1|^{\beta-2}u_1-|u_2|^{\beta-2}u_2\right)udx\right|\leq b_1(\beta-1)\int\limits_\Omega\left(|u_1|^{\beta-2}+|u_2|^{\beta-2}\right)u^2dx\leq\\&\leq b_1(\beta-1)\left(\int\limits_\Omega\left(|u_1|^{\beta-2}+|u_2|^{\beta-2}\right)^2u^2dx\right)^{\frac12}\left(\int\limits_\Omega u^2dx\right)^{\frac12}\leq\\&\leq b_1(\beta-1)\left(\int\limits_\Omega\left(|u_1|^{\beta-2}+|u_2|^{\beta-2}\right)^{\frac{2r}{r-2}}dx\right)^{\frac{r-2}{2r}}\left(\int\limits_\Omega u^rdx\right)^{\frac1r}\left(\int\limits_\Omega u^2dx\right)^{\frac12}\leq\\&\leq b_1(\beta-1)\left(\left(\int\limits_\Omega|u_1|^{\frac{2r(\beta-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}+\left(\int\limits_\Omega|u_2|^{\frac{2r(\beta-2)}{r-2}}dx\right)^{\frac{r-2}{2r}}\right)\times\\&\times\left(\int\limits_\Omega u^rdx\right)^{\frac1r}\left(\int\limits_\Omega u^2dx\right)^{\frac12}.\end{aligned}$$

We put $r=\frac{2N}{N-2},\ 2<\beta\leq\frac{2(N-1)}{N-2},\ \ N\geq3$. Then by the Sobolev embedding theorem $\overset{0}{W}{}_2^1(\Omega)\subset L_r(\Omega)$ and $\overset{0}{W}{}_2^1(\Omega)\subset L_{2r(\beta-2)/(r-2)}(\Omega)$.

In this case, taking into account the smoothness class of the solutions $u_1(x,t)$ and $u_2(x,t)$, we arrive at the estimate

$$\begin{aligned}&\left|\int\limits_\Omega b(x,\tau)\left(|u_1|^{\beta-2}u_1-|u_2|^{\beta-2}u_2\right)udx\right|\leq\\&\leq\tfrac a8\left\|\nabla u\right\|_{2,\Omega}^2+C'_{16}\left\|u\right\|_{2,\Omega}^2\leq\tfrac a8\left\|\nabla u\right\|_{2,\Omega}^2+C_{16}\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right).\end{aligned}\tag{44}$$

Analogically,

$$\begin{aligned}&\left|\tfrac{1}{h_1}\int\limits_\Omega b(x,\tau)\left(|u_1|^{\beta-2}u_1-|u_2|^{\beta-2}u_2\right)\omega dx\int\limits_\Omega hudx\right|\leq\\&\leq\tfrac a8\left\|\nabla u\right\|_{2,\Omega}^2+C_{17}\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right).\end{aligned}$$

$$\begin{aligned}&\left|\tfrac{1}{h_1}a\int\limits_\Omega\nabla u\nabla\omega dx\int\limits_\Omega hudx\right|\leq\tfrac{1}{|h_1|}\left\|\nabla u\right\|_{2,\Omega}\left\|\nabla\omega\right\|_{2,\Omega}\left\|u\right\|_{2,\Omega}\left\|h\right\|_{2,\Omega}\leq\\&\leq\tfrac a8\left\|\nabla u\right\|_{2,\Omega}^2+C_{18}\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right).\end{aligned}$$

$$\begin{aligned}&\left|\tfrac{a}{h_1(t)}\int\limits_\Omega hu_mdx\int\limits_0^tg(t-\tau)\int\limits_\Omega\nabla u_m(\tau)\nabla\omega dxd\tau\right|\leq\\&\leq C_{19}\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)+C_{20}\int\limits_0^t\left(\left\|u\right\|_{2,\Omega}^2+\chi\left\|\nabla u\right\|_{2,\Omega}^2\right)d\tau.\end{aligned}$$

$$
\begin{aligned}
&\left|\int_0^t g(t-\tau)\int_\Omega \nabla u_m(\tau)\nabla u_m(t)dxd\tau\right| \le \int_0^t g(t-\tau)\left\|\nabla u_m(\tau)\right\|_{2,\Omega}\left\|\nabla u_m(t)\right\|_{2,\Omega} d\tau \le \\
&\le \left\|\nabla u_m(t)\right\|_{2,\Omega}\int_0^t g(t-\tau)\left\|\nabla u_m(\tau)\right\|_{2,\Omega} d\tau \le \\
&\le \tfrac{2}{a}\int_0^t g^2(t-\tau)d\tau\int_0^t \left\|\nabla u_m(\tau)\right\|^2_{2,\Omega} d\tau + \tfrac{a}{8}\left\|\nabla u_m(t)\right\|^2_{2,\Omega} \le \\
&\le \tfrac{2g_0}{a}\int_0^t \left\|\nabla u_m(\tau)\right\|^2_{2,\Omega} d\tau + \tfrac{a}{8}\left\|\nabla u_m(t)\right\|^2_{2,\Omega} \le \\
&\le C_{21}\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right) + C_{22}\int_0^t\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right)d\tau.
\end{aligned}
\tag{45}
$$

By virtue of (44)-(45), we obtain

$$
\begin{aligned}
&\int_\Omega |u|^2dx + \chi\int_\Omega|\nabla u|^2dx + \tfrac{a}{2}\int_0^t\int_\Omega|\nabla u|^2dxd\tau \le \\
&\le C_{23}\int_0^t\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right)d\tau + C_{24}\int_0^t\int_0^\tau\left(\|u(s)\|^2_{2,\Omega}+\chi\|\nabla u(s)\|^2_{2,\Omega}\right)dsd\tau.
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&\int_\Omega |u|^2dx + \chi\int_\Omega|\nabla u|^2dx \le \\
&\le C_{25}\int_0^t(t-\tau+1)\left(\|u\|^2_{2,\Omega}+\chi\|\nabla u\|^2_{2,\Omega}\right)d\tau.
\end{aligned}
\tag{47}
$$

By virtue of Gronwall's lemma from the inequality (47), we get $\int_\Omega|u|^2dx+\chi\int_\Omega|\nabla u|^2dx = 0$ almost everywhere on the time interval $(0,T)$, which shows the uniqueness of the weak solution.From Lemma 1 we can establish the solvability of the inverse problem (1)-(4). Let $u(x,t)$ be a solution of the initial-boundary value problem (8)-(9) from the space (Theorem 1) $u\in L_\infty(0,T;\overset{0}{W}{}^1_2(\Omega))$,$\nabla u\in L_2(Q_T)$, $Q_T=\Omega\times(0,T)$, $u_t\in L_2(0,T;\overset{0}{W}{}^1_2(\Omega))$,$|u|^{\beta-2}u\in L_\infty(0,T;L_{\frac{\beta}{\beta-1}}(\Gamma))$. Obviously, the function $f(t)$ from the relation (10) belongs to the space $L_\infty(0,T)$. What was proved above means that the found functions $u(x,t)$ and $f(t)$ give a weak solution of the inverse problem.

4 Global solvability of the problem (8)-(9).

Consider the case when $1<\beta\le 2$. Let the conditions

$$
\begin{aligned}
&h_1(t)\equiv\int_\Omega h(x,t)\omega(x)dx\ne 0,\ \ \forall t\in[0,T],\\
&h(x,t)\in L_\infty(0,T;\ L_2(\Omega))\cap L_2(Q_T),\ \ 1<\beta\le 2,\\
&\omega\in L_2(\Omega)\cap L_\beta(\Omega)\cap\overset{0}{W}{}^2_2(\Omega),\ \ 1<\beta\le 2,\\
&\int_\Omega u_0(x)\omega(x)dx=\varphi(0),\ \ \varphi(t)\in W^1_2[0,T],\ \ |\varphi'(t)|\le C,\ \ u_0\in\overset{0}{W}{}^1_2(\Omega)\cap L_\beta(\Omega).
\end{aligned}
\tag{48}
$$

are fulfilled.

Lemma 3 *If $u\in\overset{0}{W}{}^1_2(\Omega)$, $1<\beta\le 2$, then the next inequality is performed*

$$
\int_\Omega|u_m|^\beta dx\le\left(\int_\Omega|u|^2dx\right)^{\frac{\beta}{2}}|\Omega|^{\frac{2-\beta}{2}}\le C_1\left(1+\int_\Omega|u|^2dx+\chi\int_\Omega|\nabla u|^2dx\right).
$$

For the case when $1 < \beta \le 2$, we estimate the right hand side of (20), applying Lemma 3, as well as the Cauchy and Young inequalities, we obtain

$$\left|\int\limits_{\Omega} b(x,t)|u_m|^{\beta}dx\right| \le b_1 \|u_m\|^{\beta}_{\beta,\Omega} \le C_1 b_1 \left(1 + \|u_m\|^2_{2,\Omega} + \chi \|\nabla u_m\|^2_{2,\Omega}\right).$$

$$\left|\frac{1}{h_1}\int\limits_{\Omega} \varphi'(t)h(x,t)u_m dx\right| \le \frac{|\varphi'|}{|h_1|}\|h\|_{2,\Omega}\|u_m\|_{2,\Omega} \le \|u_m\|^2_{2,\Omega} + \frac{|\varphi'|^2}{4h_1^2}\|h\|^2_{2,\Omega}.$$

$$\begin{aligned}
&\left|\tfrac{a}{h_1(t)}\int\limits_{\Omega} h u_m dx \int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla\omega dx d\tau\right| \le \\
&\le \tfrac{a}{|h_1(t)|}\|h(x,t)\|_{2,\Omega}\|u_m\|_{2,\Omega}\int\limits_0^t g(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}d\tau \le \\
&\le \tfrac{a}{|h_1(t)|}\|h(x,t)\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}\|u_m\|_{2,\Omega}\int\limits_0^t g(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}d\tau \le \\
&\le \tfrac{a^2}{4}\left(\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\tfrac{1}{|h_1(t)|}\|h(x,t)\|_{2,\Omega}\right)^2\int\limits_0^t g^2(t-\tau)d\tau\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \|u_m\|^2_{2,\Omega} \le \\
&\le \tfrac{a^2}{4}\left(\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\tfrac{1}{|h_1(t)|}\|h(x,t)\|_{2,\Omega}\right)^2\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \|u_m\|^2_{2,\Omega}.
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\left|\int\limits_0^t g(t-\tau)\int\limits_{\Omega}\nabla u_m(\tau)\nabla u_m(t)dx d\tau\right| \le \int\limits_0^t g(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}\|\nabla u_m(t)\|_{2,\Omega}d\tau \le \\
&\le \|\nabla u_m(t)\|_{2,\Omega}\int\limits_0^t g(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}d\tau \le \\
&\le \tfrac{1}{2a}\int\limits_0^t g^2(t-\tau)d\tau\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \tfrac{a}{2}\|\nabla u_m(t)\|^2_{2,\Omega} \le \\
&\le \tfrac{g_0}{a}\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \tfrac{a}{4}\|\nabla u_m(t)\|^2_{2,\Omega}.
\end{aligned}$$

$$\begin{aligned}
&\left|\tfrac{1}{h_1}a\int\limits_{\Omega}\nabla u_m\nabla\omega dx\int\limits_{\Omega}h(x,t)u_m dx\right| \le \tfrac{1}{|h_1|}\|\nabla u_m\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}\|u_m\|_{2,\Omega}\|h\|_{2,\Omega} \le \\
&\le \tfrac{a}{h_1^2}\|\nabla\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}\|u_m\|^2_{2,\Omega} + \tfrac{a}{4}\|\nabla u_m\|^2_{2,\Omega}.
\end{aligned}$$

$$\begin{aligned}
&\left|\tfrac{1}{h_1}\int\limits_{\Omega} b(x,t)|u_m|^{\beta-2}u_m\omega dx\int\limits_{\Omega}h(x,t)u_m dx\right| \le \tfrac{b_1}{|h_1|}\|u_m\|^{\beta}_{\beta,\Omega}\|\omega\|_{\beta,\Omega}\|h\|_{\frac{\beta}{\beta-1},\Omega} \le \\
&\le C_1\tfrac{b_1}{|h_1|}\|\omega\|_{\beta,\Omega}\|h\|_{\frac{\beta}{\beta-1},\Omega}\left(1 + \|u\|^2_{2,\Omega} + \chi\|\nabla u\|^2_{2,\Omega}\right).
\end{aligned}$$

Then from the obtained estimates follows that

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\int\limits_{\Omega}[|u_m|^2 + \chi|\nabla u_m|^2]dx + a\int\limits_{\Omega}|\nabla u_m|^2 dx \le \\
&\le C_1 b_1\left(1 + \|u_m\|^2_{2,\Omega} + \chi\|\nabla u_m\|^2_{2,\Omega}\right) + \|u_m\|^2_{2,\Omega} + \tfrac{|\varphi'|^2}{4h_1^2}\|h\|^2_{2,\Omega} + \\
&\tfrac{a^2}{4}\left(\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\tfrac{1}{|h_1(t)|}\|h(x,t)\|_{2,\Omega}\right)^2\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \|u_m\|^2_{2,\Omega} + \\
&+\tfrac{g_0}{a}\int\limits_0^t\|\nabla u_m(\tau)\|^2_{2,\Omega}d\tau + \tfrac{a}{4}\|\nabla u_m(t)\|^2_{2,\Omega} + \\
&+\tfrac{a}{h_1^2}\|\nabla\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}\|u_m\|^2_{2,\Omega} + \tfrac{a}{4}\|\nabla u_m\|^2_{2,\Omega} + \\
&+C_1\tfrac{b_1}{|h_1|}\|\omega\|_{\beta,\Omega}\|h\|_{\frac{\beta}{\beta-1},\Omega}\left(1 + \|u\|^2_{2,\Omega} + \chi\|\nabla u\|^2_{2,\Omega}\right).
\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\int\limits_{\Omega}[1+|u_m|^2+\chi|\nabla u_m|^2]dx+a\int\limits_{\Omega}|\nabla u_m|^2dx\le\\&\le C_2\left(1+\|u_m\|^2_{2,\Omega}+\chi\|\nabla u_m\|^2_{2,\Omega}\right)+\\&+C_3\int\limits_0^t\left(1+\|u_m\|^2_{2,\Omega}+\chi\|\nabla u_m\|^2_{2,\Omega}\right)d\tau+\frac{|\varphi'|^2}{4h_1^2}\|h(x,t)\|^2_{2,\Omega}.\end{aligned}\tag{50}$$

We denote by $y(t)\equiv 1+\|u_m\|^2_{2,\Omega}+\chi\|\nabla u_m\|^2_{2,\Omega}$, then (50) takes the form

$$\frac{dy(t)}{dt}\le C_2y(t)+C_2\int\limits_0^t y(\tau)d\tau+\frac{|\varphi'|^2}{4h_1^2}\|h(x,t)\|^2_{2,\Omega}.\tag{51}$$

Using the Gronwall's inequality, we obtain the required estimate

$$\|u_m\|^2_{2,\Omega}+\chi\|\nabla u_m\|^2_{2,\Omega}+\int\limits_0^t\int\limits_{\Omega}|\nabla u_m|^2dxdt\le C_5.$$

The derivatives $\partial_t u_m$ and $\partial_t\nabla u_m$ are estimated in a similar way, as well as (37). We multiply the equality (16) by $C'_{mj}(t)$ and summarize over $j=\overline{1,m}$. As a result, we get

$$\begin{aligned}&\|\partial_\tau u_m\|^2_{2,\Omega}+\chi\|\nabla\partial_\tau u_m\|^2_{2,\Omega}+\frac{a}{2}\frac{d}{dt}\int\limits_{\Omega}|\nabla u_m|^2dx=\frac{1}{\beta}\int\limits_{\Omega}b(x,t)|u_m|^{\beta-2}u_m\partial_t u_mdx+\\&+\int\limits_{\Omega}F(t,u_m)h(x,t)\partial_t u_mdx.\end{aligned}\tag{52}$$

We integrate with respect to τ from 0 to t, then get the relation

$$\begin{aligned}&\int\limits_0^t\left(\|\partial_\tau u_m\|^2_{2,\Omega}+\chi\|\nabla\partial_\tau u_m\|^2_{2,\Omega}\right)d\tau+\frac{a}{2}\int\limits_{\Omega}|\nabla u_m|^2dx=\int\limits_0^t\int\limits_{\Omega}b(x,t)|u_m|^{\beta-2}u_m\partial_\tau u_mdxd\tau+\\&+\int\limits_0^t\int\limits_{\Omega}F(t,u_m)h(x,t)\partial_\tau u_mdxd\tau.\end{aligned}\tag{53}$$

We estimate the right hand side of (53),

$$\begin{aligned}&\left|\int\limits_{\Omega}b(x,t)|u_m|^{\beta-2}u_m\partial_t u_mdx\right|\le b_1\int\limits_{\Omega}|u_m|^{\beta-1}|\partial_t u_m|dx\le\\&\le b_1\left(\int\limits_{\Omega}|u_m|^{\beta}dx\right)^{\frac{\beta-1}{\beta}}\left(\int\limits_{\Omega}dx\right)^{\frac{2-\beta}{2\beta}}\left(\int\limits_{\Omega}|\partial_t u_m|^2dx\right)^{\frac{1}{2}}\le\\&\le b_1\left(\int\limits_{\Omega}|u|^2dx\right)^{\beta-1}|\Omega|^{\frac{(2-\beta)(1+\beta)}{2\beta}}\left(\int\limits_{\Omega}|\partial_t u_m|^2dx\right)^{\frac{1}{2}}\le\\&\le\frac{5}{2}b_1^2|\Omega|^{\frac{(2-\beta)(1+\beta)}{\beta}}C_5^{2\beta-2}+\frac{1}{10}\|\partial_t u_m\|^2_{2,\Omega},\end{aligned}$$

$$\begin{aligned}&\left|\frac{1}{h_1}\int\limits_{\Omega}\varphi'(t)h(x,t)\partial_t u_mdx\right|\le\\&\le\frac{|\varphi'|}{|h_1|}\|h\|_{2,\Omega}\|\partial_t u_m\|_{2,\Omega}\le\frac{1}{10}\|\partial_t u_m\|^2_{2,\Omega}+\frac{5|\varphi'|^2}{2h_1^2}\|h\|^2_{2,\Omega}.\end{aligned}$$

$$\begin{aligned}&\left|\frac{1}{h_1}a\int\limits_{\Omega}\nabla u_m\nabla\omega dx\int\limits_{\Omega}h(x,t)\partial_t u_mdx\right|\le\\&\le\frac{1}{|h_1|}\|\nabla u_m\|_{2,\Omega}\|\nabla\omega\|_{2,\Omega}\|\partial_t u_m\|_{2,\Omega}\|h\|_{2,\Omega}\le\\&\le\frac{1}{12}\|\partial_t u_m\|^2_{2,\Omega}+\frac{3\|\nabla\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}}{h_1^2}\|\nabla u_m\|^2_{2,\Omega}\le\\&\le\frac{1}{10}\|\partial_t u_m\|^2_{2,\Omega}+\frac{5\|\nabla\omega\|^2_{2,\Omega}\|h\|^2_{2,\Omega}}{2h_1^2}C_5.\end{aligned}$$

$$\begin{aligned}
&\left|\frac{1}{h_1}\int\limits_{\Omega} b(x,t)|u_m|^{\beta-2}u_m\omega dx\int\limits_{\Omega} h(x,t)\partial_t u_m dx\right| \le \\
&\le \frac{b_1}{|h_1|}\|u_m\|_{\beta,\Omega}^{\beta-1}\|\omega\|_{\beta,\Omega}\|\partial_t u_m\|_{2,\Omega}\|h\|_{2,\Omega} \le \\
&\le C_1^{\frac{\beta-1}{\beta}}\frac{b_1}{|h_1|}\|\omega\|_{\beta,\Omega}\|h\|_{2,\Omega}\left(1+\|u\|_{2,\Omega}^2+\chi\|\nabla u\|_{2,\Omega}^2\right)^{\frac{\beta-1}{2}}\|\partial_t u_m\|_{2,\Omega} \le \\
&\le \frac{1}{10}\|\partial_t u_m\|_{2,\Omega}^2+\frac{5}{2}C_1^{\frac{2\beta-2}{\beta}}C_5^{\beta-1}\frac{b_1^2}{h_1^2}\|\omega\|_{\beta,\Omega}^2\|h\|_{2,\Omega}^2.
\end{aligned}$$

$$\begin{aligned}
&\left|\int\limits_0^t\int\limits_0^\tau g(\tau-s)\int_\Gamma \nabla u_m(s)\nabla\partial_\tau u_m(\tau)d\Gamma ds d\tau\right| \le \int\limits_0^t\int\limits_0^\tau g(\tau-s)\|\nabla u_m(s)\|_{2,\Omega}\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}d\tau \le \\
&\le \int\limits_0^t\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}\int\limits_0^\tau g(\tau-s)\|\nabla u_m(s)\|_{2,\Omega}d\tau \le \\
&\le \frac{1}{2}\int\limits_0^t\int\limits_0^\tau g^2(\tau-s)ds\int\limits_0^t\|\nabla u_m(s)\|_{2,\Omega}^2 ds d\tau+\frac{1}{2}\int\limits_0^t\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}^2 d\tau \le \\
&\le \frac{g_0}{2}\int\limits_0^t\int\limits_0^\tau\|\nabla u_m(s)\|_{2,\Omega}^2 ds d\tau+\frac{1}{2}\int\limits_0^t\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}^2 d\tau \le \\
&\le \frac{g_0}{2}\int\limits_0^t(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}^2 d\tau+\frac{1}{2}\int\limits_0^t\|\nabla\partial_\tau u_m(\tau)\|_{2,\Omega}^2 d\tau.
\end{aligned}$$

$$\begin{aligned}
&\left|a\int\limits_0^t\int\limits_\Omega\frac{h(x,\tau)}{h_1(\tau)}\partial_\tau u_m dx\int\limits_0^\tau g(\tau-s)\int\limits_\Omega\nabla u_m(s)\nabla\omega dx ds d\tau\right| \le \\
&\le a\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{|h_1(t)|}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}\int\limits_0^\tau g(\tau-s)\|\nabla u_m(s)\|_{2,\Omega}ds d\tau \le \\
&\le \frac{1}{10}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}^2 d\tau+\frac{5}{2}\left(a\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{|h_1(t)|}\right)^2\int\limits_0^t\int\limits_0^\tau g^2(\tau-s)ds\int\limits_0^\tau\|\nabla u_m(s)\|_{2,\Omega}^2 ds d\tau \le \\
&\le \frac{1}{10}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}^2 d\tau+\frac{5}{2}\left(a\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{|h_1(t)|}\right)^2 g_0\int\limits_0^t\int\limits_0^\tau\|\nabla u_m(s)\|_{2,\Omega}^2 ds d\tau \le \\
&\le \frac{1}{10}\int\limits_0^t\|\partial_\tau u_m\|_{2,\Omega}^2 d\tau+\frac{5g_0}{2}\left(a\|\nabla\omega\|_{2,\Omega}\sup_{0\le t\le T}\frac{\|h(x,t)\|_{2,\Omega}}{|h_1(t)|}\right)^2\int\limits_0^t(t-\tau)\|\nabla u_m(\tau)\|_{2,\Omega}^2 d\tau
\end{aligned} \tag{54}$$

Substituting the obtained inequality into (53), we get the required estimate

$$\int\limits_0^t\left(\|\partial_\tau u_m\|_{2,\Omega}^2+\chi\|\nabla\partial_\tau u_m\|_{2,\Omega}^2\right)d\tau \le C_6. \tag{55}$$

Theorem 3 *Let the conditions* (48) *be satisfied, and also* $0 < b_0 \le b(x,t) \le b_1 < \infty$, $1 < \beta \le 2$, $N \ge 3$. *Then there exists a weak solution* $u(x,t)$ *of the problem* (8)-(9) *on the interval* $(0,T)$.

The uniqueness is proved in an analogical way, and for the case when $1 < \beta \le 2$. For this case, we formulate the assertion in the form of the theorem.

Theorem 4 *Let the conditions* (48) *be satisfied,* $1 < \beta \le 2$, $N \ge 3$. *Then the weak solution of the problem* (8)-(9) *on the interval* $(0,T)$ *is unique (in the sense of Definition 1).*

We assume that the next conditions are valid for the case when $b(x,t) = -q(x,t)$:

$$\begin{aligned}
&h_1(t) \equiv \int\limits_\Omega h(x,t)\omega(x)dx \ne 0,\ \ \forall t\in[0,T],\\
&h(x,t) \in L_\infty(0,T;\ L_2(\Omega))\cap L_2(Q_T)\cap L_{\frac{\beta}{\beta-1}}(Q_T),\ \ \beta\ge 2,\\
&\omega \in L_2(\Omega)\cap L_\beta(\Omega)\cap \overset{0}{W}{}_2^2(\Omega),\ \ \beta\ge 2,\\
&\int\limits_\Omega u_0(x)\omega(x)dx = \varphi(0),\ \ \varphi(t)\in W_2^1[0,T],\ \ |\varphi'(t)|\le C,\ \ u_0\in \overset{0}{W}{}_2^1(\Omega)\cap L_\beta(\Omega).
\end{aligned} \tag{56}$$

$$
\begin{aligned}
&0 < q_0 \le q(x,t) \le q_1,\\
&2 < \beta < \tfrac{2N}{N-2},\ N \ge 3,\\
&q_0 - q_1 \left\|\omega\right\|_{\beta,\Omega} \sup_{0\le t\le T} \tfrac{1}{|h_1(t)|} \left\|h(x,t)\right\|_{\frac{\beta}{\beta-1},\Omega} > 0.
\end{aligned}
\tag{57}
$$

Theorem 5 *Let the conditions* (56) *and* (57) *be satisfied, then there exists a weak solution* $u(x,t)$ *of the problem* (8)-(9) *(in the sense of Definition 1) on the interval* $(0,T)$.

4.1 Existence and uniqueness of a strong solution

Theorem 6 *Let the conditions* (5)-(7) *are performed, and also* $2 < \beta < \frac{2N}{N-2},\ N \ge 3$*, additional conditions* $u_0(x) \in \overset{0}{W}{}_2^2(\Omega)$*. Then there is a strong solution* $u(x,t)$ *(in the sense of Definition 2) of the problem* (8)-(9) *on the interval* $(0,T),\ T < T_0$.

Theorem 7 *Let* $u_0(x) \in \overset{0}{W}{}_2^2(\Omega)$, $2 < \beta \le \frac{2(N-1)}{N-2},\ N \ge 3$ *are fulfilled .Then the strong solution (in the sense of Definition 2) of the problem* (8)-(9) *on the interval* $(0,T)$ *is unique.*

Theorem 8 *Let the conditions*

1) (48) *be satisfied, and also* $0 < b_0 \le b(x,t) \le b_1 < \infty,\ 1 < \beta \le 2,\ N \ge 3$;

2) (56) *and* (57).

Then there exists and unique a strong solution $u(x,t)$ *of the problem* (8)-(9) *on the interval* $(0,T)$.

5 Conclusion

In this article we investigated the solvability of the inverse problem for a quasilinear pseudoparabolic equation with memory. As a result, we proved the existence and uniqueness of a weak and strong solutions of the inverse problem in a bounded domain, and obtained local and global theorems on the existence of the solution. The field of application of the obtained results are inverse problems for pseudoparabolic equations and theory of boundary value problems. These results allow to proceed to the new inverse problems, generalize and systemize research in the given area.

6 Acknowledgment

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan, Grant No. AP09057950.

References

[1] *Al'shin A.B., Korpusov M.O., Sveshnikov A.G.* Blow-up in nonlinear Sobolev type equations. - Berlin: De Gruyter Series in Nonlinear Analysis and Applications, 15. Walter de Gruyter Co., 2011. -648 p.

[2] *Asanov A., Atamanov E.R.* Nonclassical and Inverse Problems for Pseudoparabolic Equations. - Utrecht, The Netherlands: Inverse and ill Posed Problems Series, VSP, 1997. -152 p.

[3] *Lyubanova A.Sh., Tani A.* An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity // Applicable Analysis. - 2011. - № 90. - P.1557-1571.

[4] *Шергин С.Н., Пятков С.Г.* О некоторых классах обратных задач для псевдопараболических уравнений // Матем. заметки СВФУ. -2014. -Т.21, №2. -С. 106-116.

[5] *Pyatkov S.G., Shergin S.N.* On some mathematical models of filtration theory // Вестник ЮУрГУ. Сер. Матем. моделирование и программирование. -2015. -Т.8, №2. -Р. 105-116.

[6] *Abylkairov U.U., Khompysh K.* An inverse problem of identifying the coefficient in kelvin-voight equations // Applied Mathematical Sciences. -2015. -V.9, №101-104. -P. 5079-5088.

[7] *Yaman M.* Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation // Journal of Inequalities and Applications. -2012. -V.2012, №274. -P. 1-8.

[8] *Кожанов А.И.* О разрешимости обратных задач восстановления коэффициентов в уравнениях составного типа // Вестник НГУ серия математика, механика, информатика. -2008. -Т.8, №3. -С. 81-99.

[9] *Кожанов А.И., Телешева Л.А.* Нелинейные обратные задачи с интегральным переопределнием для некоторых нестационарных дифференциальных уравнений высокого порядка // Вестник ЮУрГУ. Серия Математическое моделирование и программирование. -2017. -Т.10, №2. -Р. 24-37.

[10] *Кожанов А.И., Намсараева Г.В.* Линейные обратные задачи для одного класса соболевского типа // Челябинский физико-математический журнал. -2018. -Т.3, №2. -С. 153-171.

[11] *Айтжанов С.Е., Ашурова Г.Р.* Разрешимость обратной задачи Соболевского типа // Вестник КазНПУ имени Абая. Серия физико-математические науки. -2020. №2(70). -С. 23-32.

[12] *Antontsev S.N., Aitzhanov S.E., Ashurova G.R.* An inverse problem for the pseudo-parabolic equation with p-Laplacian // Evolution Equations and Control Theory. -2021. doi: 10.3934/eect.2021005

References

[1] *Al'shin A.B., Korpusov M.O., Sveshnikov A.G.* Blow-up in nonlinear Sobolev type equations. - Berlin: De Gruyter Series in Nonlinear Analysis and Applications, 15. Walter de Gruyter Co., 2011. -648 p.

[2] *Asanov A., Atamanov E.R.* Nonclassical and Inverse Problems for Pseudoparabolic Equations. - Utrecht, The Netherlands: Inverse and ill Posed Problems Series, VSP, 1997. -152 p.

[3] *Lyubanova A.Sh., Tani A.* An inverse problem for pseudoparabolic equation of filtration. The existence, uniqueness and regularity // Applicable Analysis. - 2011. - № 90. - P.1557-1571.

[4] *Shergin S.N., Pyatkov S.G.* (2014) *O nekotory'kh klassakh obratny'kh zadach dlya psevdoparabolicheskikh uravnenij* [On some classes of inverse problems for pseudoparabolic equations]. Matematicheskie zametki SVFU. -2014. -T.21, №2. -P. 106-116.

[5] *Pyatkov S.G., Shergin S.N.* (2015) *On some mathematical models of filtration theory* [On some mathematical models of filtration theory]. Vestnik YuUrGU. Seriya Matematicheskoe modelirovanie i programmirovanie. -2015. -T.8, №2. -P. 105-116.

[6] *Abylkairov U.U., Khompysh K.* An inverse problem of identifying the coefficient in kelvin-voight equations // Applied Mathematical Sciences. -2015. -V.9, №101-104. -P. 5079-5088.

[7] *Yaman M.* Blow-up solution and stability to an inverse problem for a pseudo-parabolic equation // Journal of Inequalities and Applications. -2012. -V.2012, №274. -P. 1-8.

[8] *Kozhanov A.I.* (2008) *O razreshimosti obratny'kh zadach vosstanovleniya koe'fficzientov v uravneniyakh sostavnogo tipa* [On the solvability of inverse problems of recovering coefficients in equations of composite type]. Vestnik NGU. Seriya matematika, mekhanika, informatika. -2008. -T.8, №3. -P. 81-99.

[9] *Kozhanov A.I., Telesheva L.A.* (2017) *Nelinejny'e obratny'e zadachi s integral'ny'm pereopredeleniem dlya nekotory'kh nestaczionarny'kh differenczial'ny'kh uravnenij vy'sokogo poryadka* [Nonlinear inverse problems with integral overdetermination for some nonstationary higher order differential equations]. Vestnik YuUrGU. Seriya Matematicheskoe modelirovanie i programmirovanie. -2017. -T.10, №2. -P. 24-37.

[10] *Kozhanov A.I., Namsaraeva G.V.* (2018) *Linejny'e obratny'e zadachi dlya odnogo klassa sobolevskogo tipa* [Linear inverse problems for one class of Sobolev type]. Chelyabinskij fiziko-matematicheskij zhurnal. -2018. -T.3, №2. -P. 153-171.

[11] *Aitzhanov S.E., Ashurova G.R.* (2020) *Razreshimost' obratnoj zadachi Sobolevskogo tipa* [Solvability of the Sobolev-Type Inverse Problem]. Vestnik KazNPU imeni Abaya. Seriya fiziko-matematicheskie nauki. -2020. №2(70). -P. 23-32.

[12] *Antontsev S.N., Aitzhanov S.E., Ashurova G.R.* An inverse problem for the pseudo-parabolic equation with p-Laplacian // Evolution Equations and Control Theory. -2021. doi: 10.3934/eect.2021005

2-бөлім | Раздел 2 | Section 2

Механика | Механика | Mechanics

IRSTI 30.19.25 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.06

A.M. Alimzhanov, K.Zh. Shetiyeva*, D.D. Bekmukambetova
Al-Farabi Kazakh National University, Kazakhstan, Almaty
*E-mail: karlygash.shetiyeva@gmail.com,

RESEARCH OF THE STRESS STATE OF A PIPELINE ELEMENT WITH OVALIZATION UNDER CORROSION-POWER EFFECT

The stress state of an element of a thick-walled pipeline when ovalizing a cross section is studied under conditions of power and corrosion effect in the statement of plane deformation. The material of the element under the influence of external loads goes into an elastic-plastic state. The corrosive effect of a pumped medium leads to softening of the material in the plastic zone. This softening of the material is taken into account by a special inhomogeneity function in the Tresca-Saint-Venant plasticity condition. The elastic-plastic problem for an thick-walled elliptical element under uniform external and internal pressure is considered in non-axisymmetric setting. The problem is solved by the method of sharing static and physical equations for the considered elastoplastic material and the perturbation method in the theory of an elastoplastic body. An assessment of the strength and bearing capacity of a thick-walled pipeline element with ovalization under presence and absence of corrosion damage is given.
Key words: thick-walled pipeline element with ovalization, elastoplastic state, corrosion damage to the material, plastic inhomogeneity, softening function.

А.М. Алимжанов, Қ.Ж. Шетиева*, Д.Д. Бекмукамбетова
Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
*E-mail: karlygash.shetiyeva@gmail.com

Коррозиялық-күштік әсер ету жағдайында қалың қабырғалы құбыр элементінің көлденең қимасының сопақталу кезіндегі кернеулі күйін зерттеу

Жазық деформация қойылымында коррозиялық-күштік әсер ету жағдайында қалың қабырғалы құбыр элементінің оның көлденең қимасының сопақталу кезіндегі кернеулі күйі зерттелді. Сыртқы қысымның әсерінен элемент материалы серпімді пластикалық күйге өтеді. Айдалатын ортаның коррозиялық әсері иілгіш аймақта материалдың жұмсаруына әкеледі. Материалдың жұмсаруы Треск-Сен-Венанның иілгіштік шартында біртексіздіктің арнайы функциясымен ескеріледі. Асимметриялық есеп қойылымында бірқалыпты сыртқы және ішкі қысымның әсерінен қалың қабырғалы эллиптикалық элемент үшін серпімді пластикалық есеп қарастырылды. Есеп қарастырылып отырған серпімді иілгіш материал үшін статикалық және физикалық теңдеулерді бірлесіп пайдалану әдісімен және серпімді иілгіш дене теориясында ұйытқулар әдісімен шешілді. Коррозиялық-күштік әсер ету жағдайында қалың қабырғалы құбыр элементінің көлденең қимасының сопақталуы кезіндегі беріктігі мен көтергіштік қабілеттілігіне баға берілді.
Түйін сөздер: қалың қабырғалы эллиптикалық элемент, серпімді иілгіш күй, материалдың коррозиялық зақымдануы, беріктендіру функциясы. анықтамасы.

А.М. Алимжанов, К.Ж. Шетиева*, Д.Д. Бекмукамбетова
Казахский национальный университет им.аль-Фараби, Казахстан, г.Алматы
*E-mail: karlygash.shetiyeva@gmail.com

Исследование напряженного состояния элемента трубопровода с овализацией при коррозионно-силовом воздействии

Исследовано НС элемента толстостенного трубопровода при овализации его поперечного сечения в условиях коррозионно-силового воздействия в постановке плоской деформации. Материал элемента под действием внешнего давления переходит в упругопластическое состояние. Коррозионное воздействие перекачиваемой среды приводит к разупрочнению материала в пластической зоне. Это разупрочнение материала учитывается специальной функцией неоднородности в условии пластичности Треска-Сен-Венана. Рассмотрена упругопластическая задача для толстостенного эллиптического элемента под действием равномерного наружного и внутреннего давления в неосесимметричной постановке. Задача решена методом совместного использования статических и физических уравнений для рассматриваемого упругопластического материала элемента и методом возмущений в теории упругопластического тела. Дана оценка прочности и несущей способности толстостенного элемента трубопровода с овализацией при коррозионно-силовом воздействии.

Ключевые слова: толстостенный элемент трубопровода с овализацией, упругопластическое состояние, коррозионные повреждения материала, функция разупрочнения.

1 Introduction

We studied the stress state, strength and carrying capacity of an element of a thick-walled pipeline with the circular cross-section under the action of uniform and non-uniform external pressure along the contour [1] with accounting of softening of material in the plastic zone due to corrosion effect of a pumping medium.

During operation of the elements of pipeline structures are subjected to various exorbitant loads, leading to ovalization of pipes. Ovalization of pipes significantly affects their strength and bearing capacity [2,3]. Moreover, during operation, corrosive wear of the inner surface of a thick-walled element occurs in aggressive working environments, which further reduces its bearing capacity.

In this regard, this work is devoted to the study of the stress state, strength and bearing capacity of an elastoplastic element of a thick-walled pipeline with ovalization under conditions of corrosion-force action, leading to weakening of the material in the plastic zone.

2 State of the problem

Ovalization is a factor that is taken into account in regulatory documents at the stage of delivery and installation of pipes, during design and construction [2,3], however there are no standards for the limiting value of ovality of pipelines in operation, despite the large amount of diagnostic data on their ovalization [4]. Meanwhile, the coincidence of the zone of increased stresses caused by ovalization of pipes with places of rupture and corrosion damage indicates that ovalization should be taken into account and be able to evaluate from the point of view of the operability of the pipeline. The ovalization of the cross section of an infinitely long elastic pipe in pure bending was studied by Brazier [5]. Elastoplastic ovalization of the profile of an initially straight cylindrical pipe bent by external moments along the mandrel is considered in [6]. Since in [6] only a part of the pipe is subject to bending, the study of the deformation of the pipe section is carried out in the framework of the plane stress state. In [7], the reliability and residual life of gas pipeline sections with defects such as ovalization and wall thinning due to corrosion and erosion processes are considered on the basis of a probabilistic approach. The strength characteristics of steel pipelines with geometric defects of the "dint" type were studied in [8]. In this case, a dint leads to a stress concentration in the defect zone under the action of internal pressure and can cause the appearance of other surface defects, including corrosion.

Ovality of the section is a geometric defect in the section of the pipe resulting from the transformation of the initial annular section of the pipe into an elliptical. Ovalization of the section is considered by us as a result of significant external transverse (radial) loads on the pipeline. This allows us to research the elastoplastic stress state, strength and bearing capacity of a pipeline element under conditions of corrosion-force action in the formulation of plane deformation. At the same time, the decrease in the strength properties of the pipe material during loading due to the accumulation of damage and defects can be taken into account by introducing a special softening function (radial inhomogeneity of strength characteristics) in the known criteria of material plasticity for axisymmetric and plane problems [9-11]. In [12], modified plasticity criteria were used that can take into account the accumulation of material damage under difficult boundary conditions, when the plastic inhomogeneity changes in accordance with the change in the elastoplastic boundary. In this work, we use just such plasticity criteria.

3 Solution of the problem

The element of a thick-walled underground pipeline with ovalization is in plane de-formation [14]. The equations of the cross-section of the oval pipeline element in the polar coordinate system r, θ are written for the inner contour in the form $a_0 + f_1(r,\theta)$, and for the outer contour in the form $1 + f_2(r,\theta)$. Here, $f_1(r,\theta)$, $f_2(r,\theta)$ are some functions of coordinates, $a_0 < 1$.

The pipeline material is taken to be ideally elastoplastic, obeying the Prandtl loading diagram [15].

Equilibrium equations in general form are written as $\sigma_{ij,j} = 0$.

In the considered formulation, the equilibrium equations of the pipeline in the polar coordinate system r, θ take the form :

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \qquad \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + 2\frac{\tau_{r\theta}}{r} = 0 \tag{1}$$

Here $\sigma_r, \sigma_\theta, \tau_{r\theta}$ are the components of the stress tensor.

In the elastic region, Hooke's law is valid for a homogeneous, isotropic linear elastic material:

$$\varepsilon_{ij} = \frac{1}{E}\left((1+\mu)\sigma_{ij} - \mu\delta_{ij}\sigma_{kk}\right), \tag{2}$$

where σ_{ij} and ε_{ij} are the components of the stress and strain tensors, E is the modulus of elasticity, μ is the Poisson's ratio, and δ_{ij} is the Kronecker symbol.

The stress function in the elastic region $\Phi(r,\theta)$ must satisfy the biharmonic equation (here ∇^2 is the Laplace operator)

$$\nabla^2\nabla^2\Phi = 0. \tag{3}$$

The solution of equation (3) at $r, m\theta$ can be presented in the general form

$$\Phi_m = (C_1 r^m + C_2 r^{-m} + C_3 r^{m+2} + C_4 \varphi_m(r))\cos m\theta,\ \ m = 0, 1, 2, \ldots, \tag{4}$$

where $\varphi_m(r) = r^m \ln r$ at $m = 0, 1$; $\varphi_m(r) = r^{-m+2}$ at $m \geq 2$. The constants $C_1 - C_4$ in (4) are found in the course of the solution from the boundary conditions.

The plasticity condition is generally written as follows:

$$f_* \left(\sigma_{ij}, \sigma_{s*}\left(x_i, \chi_j\right)\right) = 0. \tag{5}$$

In condition (5), σ_{ij} are the components of the stress tensor, $\sigma_{s*}(x_i, \chi_j)$ are the strength characteristics of the material in the plastic zone, which are continuous and differentiable functions of the coordinates x_i and loading parameters χ_j [12].

As a condition for the transition of a material into a plastic state, we take the Tresca-Saint-Venant condition, which is widely used in calculations of plastically deformable metal structures and constructions:

$$(\sigma_\theta - \sigma_r)^2 + 4\tau_{r\theta}^2 = 4K_*^2 \tag{6}$$

where K_* is the adhesion coefficient of material.

The material strength parameter K_* in condition (6) characterizes the plastic inhomogeneity formed as a result of varying degrees of damage to the material (the presence of many defects and microcracks in it) due to the force-corrosion effect and distributed over the thickness of the plastic zone, similar to the outline of its boundary. $K_* = K_1$ At the border of the plastic zone, the value K_* is constant: $K_* = K_1$. The quantity K_* is a special softening function that depends on the coordinates r, θ and loading parameters r_O, δ [12]:

$$K_* = K_*(r, r_0, \theta, \delta) \tag{7}$$

Here r_0, δ are the axisymmetric and non-axisymmetric loading parameters: $r_0 = r_0(P_0, P_1 + P_2)$, $\delta = \delta(P_1 - P_2)$.

The problem is solved by the method of joint use of static and physical equations for the considered elastoplastic material.

The problem also uses the perturbation method in the theory of an elastoplastic body [13]. The solution by the perturbation method is determined near the known "zero" solution at $\delta = 0$. As an initial state, we will take the solution we obtained earlier for a circular thick-walled element in an axisymmetric formulation [1]:

$$\begin{gathered} \sigma^0_{[r]} = P_0 + 2\int\limits_{a_o}^{r_o} r^{-1} K^0_* dr, \quad \sigma^0_{[\theta]} = \sigma^0_{[r]} + 2K^0_*, \quad \tau^0_{[r\theta]} = 0, \\ \left.\begin{matrix} \sigma^0_{(r)} \\ \sigma^0_{(\theta)} \end{matrix}\right\} = P + K_1 r_o^2 \left(1 \mp \frac{1}{r^2}\right), \quad \tau^0_{(r\theta)} = 0. \end{gathered} \tag{8}$$

Axisymmetric boundary conditions on the inner and outer contour a_0 and 1 of thick-walled circular element and the conjugation conditions on the contour r_0 have the form (Figure 1):

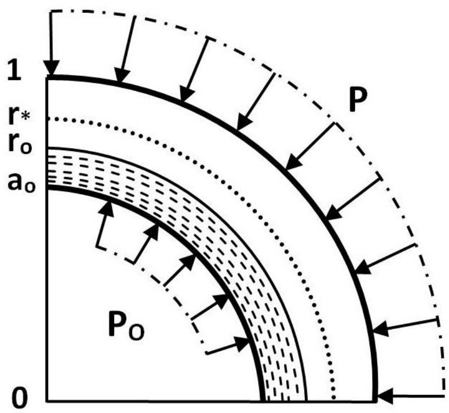


Figure 1: Design scheme for a circular thick-walled pipeline element

$$\sigma^0_{[r]} = P_0 \text{ at } r = a_0;\ \sigma^0_{(r)} = P$$

$$\text{at } r = 1; [\sigma^0_r] = [\sigma^0_\theta] = 0 \text{ at } r = r_0$$

In expressions (8), square (round) brackets at the indices mean belonging to the plastic (elastic) zone. The symbol K^0_* denotes the softening function (7) in the axisymmetric case, which depends only on the current radius r and the boundary radius r_0 :

$$K^0_* = K_*(r, r_0) = (K_0 - K_1)\overline{f}(r, r_0) + K_1. \tag{9}$$

Here K_0 and K_1 is the value of the strength of the material on the inner contour a_0 and on the elastoplastic radius r_0, $\overline{f}(r, r_0)$ – some kernel with the properties $\overline{f}(a_0,\, 1) = 1$, $\overline{f}(r_0, r_0) = 0$. In [10], the kernel $\overline{f}(r, r_0)$ was taken as a kernel that describes well the decrease in the value of K^0_* during loading both along the radius r and depending on the position of the boundary radius r_0 (n is the nonlinearity parameter): $\overline{f}(r, r_0) = \dfrac{a_0^n\,(r_0^n - r^n)}{r^n\,(1 - a_0^n)}$.

Elastoplastic radius r_0 in our "zero" solution is implicitly determined from the transcendental equation

$$P_0 - P + 2\int\limits_{a_o}^{r_o} r^{-1} K^0_* dr + K_1(1 - r_0^2) = 0 \tag{10}$$

In the absence of corrosion damage, the parameter $K^0_* = K_1$ and the radius of the plastic zone r_0 are found from the equation

$$P_0 - P + 2K_1\left(\ln\left(\frac{r_0}{a_0}\right) + \frac{1}{2}\left(1 - r_o^2\right)\right) = 0. \tag{11}$$

According to the perturbation method, the solution is sought in the form of rows of the sought components in powers of a small parameter, which is δ

$$\begin{aligned}
\sigma_{ij} &= \sum_0^\nu \delta^\nu \sigma_{ij}^{(\nu)} = \sigma_{ij}^0 + \sum_1^\nu \delta^\nu \sigma_{ij}^{(\nu)},\\
K_* &= \sum_0^\nu \delta^\nu K_*^{(\nu)} = K_*^0 + \sum_1^\nu \delta^\nu K_*^{(\nu)},\\
r_s &= \sum_0^\nu \delta^\nu r_\nu = r_0 + \sum_1^\nu \delta^\nu r_\nu
\end{aligned} \tag{12}$$

where r_s is the sought elastoplastic boundary.

For this, the initial equations, boundary conditions and conjugation conditions are linearized. Equilibrium equations (1) retain their form for any approximation

$$\frac{\partial\sigma_r^{(\nu)}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}^{(\nu)}}{\partial\theta}+\frac{\sigma_r^{(\nu)}-\sigma_\theta^{(\nu)}}{r}=0,\qquad \frac{\partial\tau_{r\theta}^{(\nu)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta^{(\nu)}}{\partial\theta}+2\frac{\tau_{r\theta}^{(\nu)}}{r}=0. \tag{13}$$

The linearization of the relations of the theory of ideal plasticity consists in the combined use of equations (1), (6). Introducing the stress function $F=F(r,\theta)$ according to (1)

$$\sigma_r^{(\nu)}=\frac{1}{r}\frac{\partial F^{(\nu)}}{\partial r}+\frac{1}{r^2}\frac{\partial^2F^{(\nu)}}{\partial\theta^2},\quad \sigma_\theta^{(\nu)}=\frac{\partial^2F^{(\nu)}}{\partial r^2},\quad \tau_{r\theta}^{(\nu)}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F^{(\nu)}}{\partial\theta}\right),\quad \nu=0,1,2,\dots \tag{14}$$

and linearizing equation (6) we obtain in the plastic zone an inhomogeneous differential equation in partial derivatives for the function $F^{(\nu)}(r,\theta)$:

$$r^2\frac{\partial^2F^{(\nu)}}{\partial r^2}-r\frac{\partial F^{(\nu)}}{\partial r}-\frac{\partial^2F^{(\nu)}}{\partial\theta^2}=r^2f^{(\nu)}(r,\theta),\qquad \nu\geq 0. \tag{15}$$

Here $f^{(\nu)}$ is the right side of the corresponding linearized relation: $f^0=2K_*^0$, $f^{(I)}=2K_*^{(I)}$, $f^{(II)}=-\dfrac{1}{K_O}(\tau_{r\theta}^{(I)})^2+2K_*^{(II)}$.

The solution to equation (15) $F^{(\nu)}$ is determined taking into account static or geometric boundary conditions.

The linearization of the boundary conditions depends on the given forces on the initial contour, and the linearization of the conjugation conditions on the elastoplastic boundary is determined by the nature of the initial conjugation conditions.

4 Thick-walled pipeline element with ovalization under uniform external and internal pressure

Consider a thick-walled pipeline element with ovalization, loaded with uniform internal P_0 and external P pressures, under conditions of plane deformation (Fig. 2)

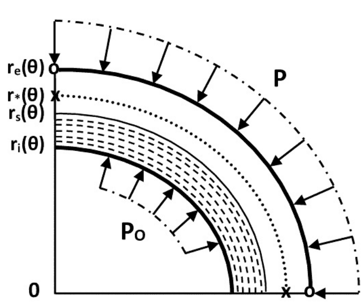


Figure 2: Design scheme for a thick-walled pipeline element with ovalization

The equations of the outer and inner contours of the pipeline element have the form

$$\begin{aligned} r_e&=1+\delta d_1\cos 2\theta,\\ r_i&=a_0(1+\delta d_2\cos 2\theta), \end{aligned} \tag{16}$$

where δ is the parameter of deviation of the oval contour from the circular one, d_1 and d_2 are geometric coefficients.

We construct a solution to the problem in the form (12) at $\nu\geq 0$. The zero solution for $\nu=0$ is given in the previous part of the work. Let us find a solution for $\nu=1$.

Consider the plastic zone of a thick-walled element. We represent the stress function $F^{(1)}$ in equation (15) based on the geometric boundary conditions of the outer and inner contours (16): $F^{(1)} = R(r)\cos 2\theta$.

Solving equation (15), we find the function $F^{(1)}$ in the plastic zone:

$$F^{(1)} = \left(A_i R_i + R_i \int\limits_{a_0}^{r} \frac{V_i(r)}{V(r)} dr \right) \cos 2\theta, \ , \quad i = \overline{1,2}, \tag{17}$$

where $A_i R_i = A_1 R_1 + A_2 R_2 = r(A_1 \cos(\sqrt{3}\ln r) + A_2 \sin(\sqrt{3}\ln r))$, $V(r)$ is the Wronskian of the system of solutions R_i, $V_i(r)$ is the determinant obtained from the Wronskian by replacing the i^{th} column with a column with a single nonzero element $2K_*^{(I)} \cos^{-1} 2\theta$ located at its end.

The stress components in the plastic zone under the Tresque-Saint-Venant condition based on (14), (17) are written as:

$$\begin{aligned}
&\sigma_{[r]}^{(I)} = \frac{1}{r}[A_1 \cos(\sqrt{3}\ln r) + A_2 \sin(\sqrt{3}\ln r) - 2(\sqrt{3}\cos(\sqrt{3}\ln r) + \sin(\sqrt{3}\ln r))\times \\
&\times \int\limits_{a_0}^{r} \frac{-\sin(\sqrt{3}\ln r) K_*^{(I)}}{\cos 2\theta} dr + 2(\cos(\sqrt{3}\ln r) - \sqrt{3}\sin(\sqrt{3}\ln r))\times \\
&\times \int\limits_{a_0}^{r} \frac{\cos(\sqrt{3}\ln r) K_*^{(I)}}{\cos 2\theta} dr] \cos 2\theta, \quad \sigma_{[\theta]}^{(I)} = \sigma_{[r]}^{(I)} + 2K_*^{(I)}, \\
&\tau_{[r\theta]}^{(I)} = \frac{1}{2r}[(A_1 - \sqrt{3}A_2)\cos(\sqrt{3}\ln r) + (A_2 - \sqrt{3}A_1)\sin(\sqrt{3}\ln r) - \\
&-8\sin(\sqrt{3}\ln r) \cdot \int\limits_{a_0}^{r} \frac{-\sin(\sqrt{3}\ln r) K_*^{(I)}}{\cos 2\theta} dr + 8\cos(\sqrt{3}\ln r)\times \\
&\times \int\limits_{a_0}^{r} \frac{\cos(\sqrt{3}\ln r) K_*^{(I)}}{\cos 2\theta} dr] \sin 2\theta.
\end{aligned} \tag{18}$$

Find constants A_1 and A_2. For this, we have linearized boundary conditions on the inner contour

$$\sigma_{[r]}^{(I)} + a_0 d_2 \frac{d\sigma_{[r]}^0}{dr} \cos 2\theta = 0, \quad \tau_{[r\theta]}^{(I)} + 2d_2(\sigma_{[\theta]}^0 - \sigma_{[r]}^0)\sin 2\theta = 0, \quad \text{at} \quad r = a_0. \tag{19}$$

Substituting (18) into (19), we obtain

$$A_1 = -2(\sqrt{3}\sin(\sqrt{3}\ln a_0) + \cos(\sqrt{3}\ln a_0)) a_0 d_2 K_*(a_0, r_0),$$

$$A_2 = 2(\sqrt{3}\cos(\sqrt{3}\ln a_0) - \sin(\sqrt{3}\ln a_0)) a_0 d_2 K_*(a_0, r_0).$$

Substituting A_1 and A_2 in (18), we can obtain expressions for the components $\sigma_{[r]}^{(I)}$, $\sigma_{[\theta]}^{(I)}$, $\tau_{[r\theta]}^{(I)}$ in their final form.

Consider the elastic region of a thick-walled element. The components $\sigma_{(r)}^{(I)}$, $\sigma_{(\theta)}^{(I)}$, $\tau_{(r\theta)}^{(I)}$ of the stress tensor in the elastic region are found from the stress function $\Phi^{(1)}$ in equations (4), (14):

$$\begin{aligned}
\sigma_{(r)}^{(I)} &= (-2C_1 - 6C_2 r^{-4} - 4C_4 r^{-2})\cos 2\theta = 0,\\
\sigma_{(\theta)}^{(I)} &= (2C_1 + 6C_2 r^{-4} + 12C_3 r^2)\cos 2\theta = 0,\\
\tau_{(r\theta)}^{(I)} &= (2C_1 - 6C_2 r^{-4} + 6C_3 r^2 - 2C_4 r^{-2})\sin 2\theta
\end{aligned} \tag{20}$$

To determine C_1, C_2, C_3, C_4 we have linearized boundary conditions on the outer contour of the element

$$\sigma_{(r)}^{(I)} + d_1 \frac{d\sigma_{(r)}^0}{dr}\cos 2\theta = 0, \quad \tau_{(r\theta)}^{(I)} + 2d_1(\sigma_{(\theta)}^0 - \sigma_{(r)}^0)\sin 2\theta = 0, \quad \text{at} \quad r = 1 \tag{21}$$

and two linearized conditions for conjugation of stresses at the boundary of the plastic zone

$$[\sigma_r^{(I)}] = 0, \;\; [\tau_{r\theta}^{(I)}] = 0 \qquad \text{at} \quad r = r_0 \tag{22}$$

Then, from conditions (21), (22), we obtain the boundary value problem for the elastic region of a thick-walled oval element:

$$\begin{aligned}
&\sigma_{(r)}^{(I)} = M_1(r)\cos 2\theta, \quad \tau_{(r\theta)}^{(I)} = M_2(r)\sin 2\theta, \quad \text{at } r = r_0,\\
&\sigma_{(r)}^{(I)} = -M_0(r)\cos 2\theta, \quad \tau_{(r\theta)}^{(I)} = -2M_0(r)\sin 2\theta, \text{ at } r = 1.
\end{aligned} \tag{23}$$

Here under the condition of Tresque-Saint-Venant

$$M_0 = 2r_0^2 d_1 K_1,$$

$$M_1 = \frac{2a_0 d_2 K_*(a_0, r_0)}{r}(\sqrt{3}\sin(\sqrt{3}\ln\frac{r_0}{a_0}) + \cos(\sqrt{3}\ln\frac{r_0}{a_0})) -$$

$$-\frac{2}{r_0}(\sqrt{3}\cos(\sqrt{3}\ln r_0) + \sin(\sqrt{3}\ln r_0)) \cdot \int_{a_0}^{r_0} \frac{-\sin(\sqrt{3}\ln r)K_*^{(I)}}{\cos 2\theta}dr +$$

$$+\frac{2}{r_0}(\cos(\sqrt{3}\ln r_0) - \sqrt{3}\sin(\sqrt{3}\ln r_0)) \cdot \int_{a_0}^{r_0} \frac{\cos(\sqrt{3}\ln r)K_*^{(I)}}{\cos 2\theta}dr,$$

$$M_2 = \frac{4a_0 d_2 K_*(a_0, r_0)}{r}\cos(\sqrt{3}\ln\frac{r_0}{a_0}) - \frac{8}{r_0}\sin(\sqrt{3}\ln r_0) \cdot \int_{a_0}^{r_0} \frac{-\sin(\sqrt{3}\ln r)K_*^{(I)}}{\cos 2\theta}dr +$$

$$+\frac{8}{r_0}\cos(\sqrt{3}\ln r_0)\int_{a_0}^{r_0} \frac{\cos(\sqrt{3}\ln r)K_*^{(I)}}{\cos 2\theta}dr$$

Solving the boundary value problem (23), we find all constants $C_1 - C_4$:

$$C_1 = \frac{1}{2}(1 - r_0^{-2})^{-3}\left(M_0 r_0^{-2}(2 - r_0^{-2} - r_0^{-4}) - M_1(1 + r_0^{-2} + 2r_0^{-4}) + 2M_2 r_0^{-4}\right),$$

$$C_2 = \frac{1}{2}(1 - r_0^{-2})^{-3}\left(3M_0(1 - r_0^{-2}) - M_1(3 + r_0^{-2}) + 2M_2 r_0^{-2}\right),$$

$$C_3 = \frac{1}{6}r_0^{-2}(1 - r_0^{-2})^{-3}\left(3M_0(r_0^{-4} - r_0^{-2}) + M_1(1 + 3r_0^{-2}) + M_2(1 - 3r_0^{-2})\right),$$

$$C_4 = \frac{1}{2}(1 - r_0^{-2})^{-3}\left(-M_0(1 + r_0^{-2} - 2r_0^{-4}) + M_1(2 + r_0^{-2} + r_0^{-4}) - M_2(r_0^{-2} + r_0^{-4})\right)$$

Substituting C_1, C_2, C_3, C_4 in (20), we obtain the stress components in the elastic region of the thick-walled element $\sigma_{(r)}^{(I)}$, $\sigma_{(\theta)}^{(I)}$, $\tau_{(r\theta)}^{(I)}$ in the final form.

We seek the equation for the boundary of the plastic zone r_s in the form $r_s = r_0 + \delta r_1$. To determine the value of r_1, we use the linearized conditions for the conjugation of the components σ_θ and K_* on r_0 :

$$\left[\sigma_\theta^{(I)} + \frac{d\sigma_\theta^0}{dr}r_1\right] = 0, \qquad \left[K_*^{(I)} + \frac{dK_*^0}{dr}r_1\right] = 0 \quad \text{at} \quad r = r_0, \tag{24}$$

From conditions (24) we obtain

$$r_1 = (\sigma_{(\theta)}^{(I)} - \sigma_{[\theta]}^{(I)})/\left(\frac{d\sigma_{[\theta]}^0}{dr} - \frac{d\sigma_{(\theta)}^0}{dr}\right), \quad K_*^{(I)} = -\frac{dK_*^0}{dr}r_1 \text{ at } r = r_0$$

Then we finally have

$$r_1 = \psi(r_0) r_0 \cos 2\theta,$$

where $\psi(r_0)$ is some function of $r_0 : \psi(r_0) = \dfrac{Y_1 + Y_2 + Y_3}{Y_4 + Y_5 + Z}$.

Here under the condition of Tresque-Saint-Venant

$$Y_1 = -4(2 - 7r_0^{-2} + 5r_0^{-4})(1 - r_0^{-2})^{-3} d_1 K_1,$$

$$Y_2 = (1 + 5r_0^{-2} - 11r_0^{-4} - 3r_0^{-6} - (1 - r_0^{-2})^3)\frac{a_0 d_2 K_*(a_0, r_0))}{r_0(1 - r_0^{-2})^3} \times$$

$$\times\left(\sqrt{3}\sin\left(\sqrt{3}\ln\frac{r_0}{a_0}\right) - \cos\left(\sqrt{A}3\ln\frac{r_0}{a_0}\right)\right),$$

$$Y_3 = -8\left(1 - 3r_0^{-2} + r_0^{-4} + 3r_0^{-6}\right)\frac{a_0 d_2 K_*(a_0, r_0))}{r_0(1 - r_0^{-2})^3} \times \cos\left(\sqrt{3}\ln\frac{r_0}{a_0}\right),$$

$$Y_4 = -2(1 + 5r_0^{-2} - 11r_0^{-4} - 3r_0^{-6} - (1 - r_0^{-2})^3)\frac{2\widetilde{K}_*}{(1 - r_0^{-2})^3} \times$$

$$\times((\sqrt{3}\cos(\sqrt{3}\ln r_0) + \sin(\sqrt{3}\ln r_0))B_1 + (\cos(\sqrt{3}\ln r_0) - \sqrt{3}\sin(\sqrt{3}\ln r_0))B_2),$$

$$Y_5 = -4(1 - 3r_0^{-2} + r_0^{-4} + 3r_0^{-6})\frac{\widetilde{K}_*}{(1 - r_0^{-2})^3} \times (\sin(\sqrt{3}\ln r_0)B_1 + \cos(\sqrt{3}\ln r_0)B_2),$$

$$Z = 4K_1, \quad \widetilde{K}_* = (K_0 - K_1)\frac{a_0^n}{1 - a_0^n}\frac{n}{r_0},$$
$$B_1 = \frac{r_0}{2}\sin(\sqrt{3}\ln r_0 - \frac{\pi}{3}) - \frac{a_0}{2}\sin(\sqrt{3}\ln a_0 - \frac{\pi}{3}),$$
$$B_2 = \frac{r_0}{2}\cos(\sqrt{3}\ln r_0 - \frac{\pi}{3}) - \frac{a_0}{2}\cos(\sqrt{3}\ln a_0 - \frac{\pi}{3})$$

In the homogeneous case, the expression Y_1 is the same, the expressions Y_2 and Y_3 are preserved, but instead of $K_*(a_0, r_0)$, K_1, $Y_4 = Y_5 = 0$ should be written. In this case, the radius r_0 corresponds to the homogeneous case.

The equation for the boundary of the plastic zone r_s is written in the form:

$$r_s = r_0(1 + \delta\psi(r_0)\cos 2\theta).$$

The obtained solution area is as follows

$$r_0(1 - \delta\psi(r_0)) \geq a_0(1 - \delta d_2).$$

The bearing capacity of a thick-walled pipeline element with ovalization is determined as follows: If $\psi(r_0) = d_1$, then the boundary equation r_s has the form $r_s = r_0(1 + \delta d_1 \cos 2\theta)$. In this case, the ellipses bounding the outer contour of the element and the plastic zone will be similar. Consequently, the plastic zone will reach the critical contour at once in all its points and the bearing capacity is determined from a simple condition $r_* = r_0$.

In the absence of corrosion damage ($r_* = 1$), the condition for determining the bearing capacity of the element will take the form from $r_0 = 1$. Here r_* is the numerically determined critical radius of the element ($a_0 < r_* \leq 1$), corresponding to the maximum point on the loading diagram $\Delta P = \Delta P(r_0)$, at which a thick-walled element is destroyed [1].

In all other cases ($\psi(r_0) \neq d_1$), the bearing capacity of a pipeline element with ovalization can be researched as follows.

From the equation

$$r_*(1 \pm \delta d_1) = r_0(1 \pm \delta\psi(r_0)), \tag{25}$$

at $\theta = 0$ or $\theta = \pi/2$, we find the value r_0, and then from (10) we obtain the critical value of external loads, at which the plastic zone will reach some "critical" points of the thick-walled element.

In the absence of corrosion damage to the element, equation (11) should be adopted and $r_* = 1$ used in equation (25). In this case, critical points (marked with zeros in Fig. 2) are located on the outer contour of the element $1 + \delta d_1 \cos 2\theta$ at the points of its greatest ($\theta = 0$) or least ($\theta = \pi/2$) curvature.

In the presence of corrosion damage of the element, the critical points are located inside the element (marked with crosses in Fig. 2) on the contour $r_*(1 + \delta d_1 \cos 2\theta)$ in the directions of its greatest ($\theta = 0$ or least ($\theta = \pi/2$) curvature (this is determined through the coefficients d_1 and d_2). Reaching these crosses by any point of the plastic zone will lead to the destruction of the element. Note that in an oval element, the plastic zone in the presence of damage to the material becomes larger in size and somewhat elongated in the directions of its greatest curvature. Wherein, in the above directions, the element acquires the greatest damage.

In this case $d_1 = 0$, $d_2 = 1$, we have a thick-walled circular element with an oval hole. The bearing capacity of such an element is determined from the equation $r_* = r_0(1 + \delta\psi(r_0))$ at $Y_1 = 0$. Consequently, the "critical" points are located inside the element on the contour r_* in the directions of the greatest curvature ($\theta = \pi/2$) of the hole.

In this case $d_1 = 1$, $d_2 = 0$, we have a thick-walled oval element with a circular hole. The bearing capacity of such element is determined from equation (25) at $Y_2 = Y_3 = 0$. "Critical" points are located inside the element on the contour $r_*(1 + \delta d_1 \cos 2\theta)$ in the directions of the greatest ($\theta = 0$) or least ($\theta = \pi/2$) curvature of its outer contour.

All the obtained solutions at $\gamma = 1$ go to the homogeneous case, and at $\delta = 0$ go to the axisymmetric case.

5 Conclusion

The stress state of an elastoplastic element of a thick-walled pipeline when ovalizing a cross section is studied under conditions of power and corrosion effect using a special softening function (plastic inhomogeneity) in the plasticity condition of Tresca-Saint-Venant. An elastoplastic problem is considered for a thick-walled pipeline element with ovalization under the action of uniform external and internal pressure in a nonaxisymmetric formulation. The problem is solved by the method of joint use of equilibrium equations and physical equations in each zone and their "sewing" through the conjugation conditions on the elastoplastic boundary, as well as by the method of perturbations in the theory of an elastoplastic body.

An assessment of the strength and bearing capacity of a loaded thick-walled pipeline element with ovalization in the presence and absence of corrosion damage is given.

References

[1] Alimzhanov A.M., Shetiyeva K.Zh. "Research of the stress state of an element of a thick-wall pipeline under conditions of power and corrosion effect", *Journal of Mathematics, Mechanics and Computer Science* 1 (105) (2020): 109-119.

[2] СП 33.13330.2012. "Raschet na prochnost' stal'nyh truboprovodov [Strength analysis of steel pipelines].

[3] GOST 32388-2013. *Truboprovody tekhnologicheskiye. Normy i metody rascheta na prochnost', vibratsiyu i seysmicheskiye vozdeystviya* [Technological pipelines. Norms and methods of calculation for strength, vibration and seismic effects] (M.: Standartinform, 2016): 114.

[4] Kharionovskiy V.V. *Nadezhnost' i resurs konstruktsiy gazoprovodov* [Reliability and resource of gas pipeline structures] (M.: Nedra, 2000): 467.

[5] Brazier L.G. "On flexure of thin cylindrical shells and other sections", *R. Soc. Lond. Ser. A.* (1927): 104–114.

[6] Bukashkina O.S. "Ob uprugoplasticheskoy deformatsii truby izognutoy vneshnimi momentami po opravke [Bukashkina O.S. Elastoplastic deformation of a pipe bent by external moments along a mandrel]", *Vestnik SPbGU. Ser.* 1 (1) (2003): S.87-92.

[7] Pimenov S.A. "Otsenka konstruktivnoy nadezhnosti i riska otkaza defektnykh uchastkov gazoprovodov [Pimenov S.A. Assessment of the structural reliability and the risk of failure of defective sections of gas pipelines", *Tezisy dokladov region. molodezhn. nauchno-tekhn. foruma, N.Novgorod* (2002): 220-22.

[8] Mustafin T.R. "Vliyaniye vmyatin na prochnost' stal'nykh gazonefteprovodov [Mustafin T.R. Influence of dents on the strength of steel gas and oil pipelines]", *Transport i khraneniye nefteproduktov i uglevodorodnogo syr'ya.* 4 (2012): 62-64.

[9] Alimzhanov M.T. "Uprugoplasticheskaya zadacha, uchityvayushchaya neodnorodnost' mekhanicheskih svojstv materiala [Elastic-plastic problem, taking into account the heterogeneity of the mechanical properties of the material]", *Reports of the USSR Academy of Sciences* 6 (1978): 1281-1284.

[10] Alimzhanov M.T. "O nakoplenii povrezhdenij i nesushchej sposobnosti elementov tolstostennyh konstrukcij [On the accumulation of damage and bearing capacity of elements of thick-walled structures]" , *International journal Problems of mechanical engineering and automation* 1 (1992): 58-64.

[11] Alimzhanov A.M. "Dvumernaya uprugoplasticheskaya zadacha s lokal'noj plasticheskoj neodnorodnost'yu [Two-dimensional elastic-plastic problem from the local plastic heterogeneity]" , *Mechanics and modeling of technology processes* 1 (1997): 3-17.

[12] Alimzhanov A.M. "Ploskaya uprugoplasticheskaya zadacha dlya neodnorodnogo tela s otverstiem [Flat elastic-plastic problem for a non-uniform body with a hole]" , *Bulletin of the Russian Academy of Sciences. Mechanics of solids* 2 (1998): 119-138.

[13] Ivlev D.D. and Ershov L.V. *Metod vozmushchenij v teorii uprugoplasticheskogo tela* [The perturbation method in the theory of an elastoplastic body] (Moscow: Science, 1978).

[14] Sedov L.I. *Mekhanika sploshnoy sredy*[Continuum Mechanics] 2 (M.: Nauka, 1984): 568.

[15] Bykovtsev G.I., Ivlev D.D. *Teoriya plastichnosti* [Plasticity theory] (Vladivostok: Dal'nauka, 1998): 528.

IRSTI 53.37.29 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.07

Rakhmatulina A.[1*], **Imanbaeva N.**[1] **Nurmaganbetova A.**[1], **Sakenova A.**[2] **Smagulova N.**[1]

[1]Almaty Technological University, Institute of Mechanics and Engineering Science named after acad. U.A. Dzholdasbekov, Kazakhstan, Almaty

[2]Satpayev University, Kazakhstan, Almaty

E-mail: kazrah@mail.ru*

DETERMINATION OF THE BALANCING MOMENT OF THE SIX-LINK STRAIGHT-LINE CONVERSION MECHANISM OF THE BEAMLESS ROD PUMP DRIVE

The paper considers a six-link straight-line conversion scissor mechanism, which is used as a new design of the conversion mechanism of the beamless rod pump drive.

The purpose of balancing the conversion mechanism of rod pump drive (RPD) is to reduce the required engine power and its uniform load per cycle of movement. The task of optimal dynamic balancing of the conversion mechanism of the RPD is to determine the optimal values of the weight of the counterweight G_{CW} and the distance $l = OL$ from the crank axis at which the minimum peak value of the balancing moment on the crank shaft is provided. In practice, the determination of these values is carried out empirically by comparing two peak values – the torque on the crank shaft for the cycle of the mechanism movement. The result kinetostatics analysis, solving the equilibrium equations of the six-link scissor mechanism, determined reactions of mechanism hinges and values – the torque on the shaft of the crank shaft per cycle of movement of the mechanism. Also, for the reliability of the results, according to the principle of possible movements through the power of the acting forces, values – the torque on the crank shaft were determined.

Key words: Drive, transforming mechanism, crank, connecting rod, balancer, poise, analysis.

Рахматулина А.[1*], Иманбаева Н.[1], Нурмаганбетова А.[1], Сакенова А.[2], Смагулова Н.[1]

[1]Алматы технологиялық университеті, Ө.А. Жолдасбеков атындағы механика және машинатану институты, Қазақстан, Алматы қ.

[2]Сәтбаев университеті, Қазақстан, Алматы қ.

E-mail: kazrah@mail.ru*

Штангалы сорғыш қондырғыларының теңгерімсіз жетегінің алтызвенолы түзу-сызықты бағыттауыш түрлендіргіш механизмінің теңдестіру моментін анықтау

Бұл мақалада алтызвенолы түзу-сызықты бағыттауыш түрлендіргіш механизмі туралы айтылады, оны біз штангалық сорғы қондырғыларының теңгерімсіз жетегінің түрлендіргіш механизмінің жаңа конструкциясы ретінде қолданамыз.

Штангалы сорғыш қондырғыларының (ШСҚ) түрлендіргіш механизмін теңестірудің мақсаты қозғалтқыштың қажетті қуатын және оның қозғалыс циклі кезінде оның біркелкі жүктемесін азайту болып табылады. $G_{\prod}$ түрлендіру механизмін оңтайлы динамикалық теңдестірудің міндеті қарсы салмақтың GP оңтайлы мәндерін және иінді $l_{\prod} = OL$ білікте теңдестіру сәтінің минималды мәні қамтамасыз етілетін иінді біліктен қашықтықты анықтау болып табылады. Іс жүзінде бұл шамаларды анықтау $_{\text{т}}$ екі мәні – механизмнің бір қозғалыс циклына иінді біліктің айналу моментін салыстыру арқылы эмпирикалық түрде жүзеге асырылады.Кинетостатикалық талдау нәтижесінде алты звенолы топсалы-иінтіректі механизмнің буындарының тепе-теңдік теңдеулерін, сонымен қатар алғанда механизм ілмектерінің реакция күш және $_{\text{т}}$ мәнінің – механизмнің қозғалу циклі үшін иінді біліктің айналу моментінің бірлесіп шешілуі нәтижесінде анықталды.

Сондай-ақ, нәтижелердің сенімділігі тексеру үшін әрекет етуші күштердің қуаттары арқылы ықтимал орын ауыстыру қағидасына сәйкес $_{\text{т}}$ мәндері – иінді біліктің айналу моменті анықталды.
Түйін сөздер: Жетек, түрлендіруші механизм, айналшақ, бұлғақ, балансир, тепе-теңдік, талдау.

Рахматулина А.[1*], Иманбаева Н.[1], Нурмаганбетова А.[1], Сакенова А.[2], Смагулова Н.[1]
[1]Алматинский технологический университет, Институт механики и машиноведения им. акад. У.А. Джолдасбекова, Казахстан, г. Алматы
[2]Satbayev University, Казахстан, г. Алматы
E-mail: kazrah@mail.ru*

Определение уравновешивающего момента шестизвенного прямолинейно-направляющего преобразующего механизма безбалансирного привода штанговых насосных установок

В данной статье рассматривается шестизвенный прямолинейно-направляющий шарнирно-рычажный механизма, которого используем в качестве новой конструкции перобразующего механизма безбалансирного привода штанговых насосных установок.
Цель уравновешивания преобразующего механизма штанговых насосных установок (ШНУ) заключается в уменьшении необходимой мощности двигателя и равномерной его нагрузки за цикл движения. Задача оптимального динамического уравновешивания преобразующего механизма ШНУ заключается в определении оптимальных значений веса противовеса $G_{\prod}$ и расстояние $l_{\prod} = OL$ от оси кривошипа при котором обеспечивается минимальное пиковое значение уравновешивающего момента на валу кривошипа. На практике определение этих величин осуществляется эмпирически путем сравнения двух пиковых значений $_{\text{ур}}$ – крутящего момента на валу кривошипа за цикл движения механизма. В результате кинетостатического анализа, решая совместно уравнения равновесия звеньев шестизвенного шарнирно-рычажного механизма, определены силы реакции шарниров механизма и значения $_{\text{ур}}$ – крутящего момента на валу кривошипа за цикл движения механизма. Так же для проверки достоверности результатов, по принципу возможных перемещений через мощности действующих сил определили значения $_{\text{ур}}$ – крутящего момента на валу кривошипа

Ключевые слова: Привод, преобразующий механизм, кривошип, шатун, балансир, уравновешенность, анализ.

1 Introduction

Consider the dynamic modes of operation of the rod pump drive. When rod string goes up, the balancing devices give the power body the energy accumulated during the motion of the rods down. In the device under consideration, the counterweights are attached by special devices to the 3rd link of the mechanism. It worth to note that the weight of the 3rd link also plays the role of a counterweight, i.e. performs useful work.

In the problem of dynamic synthesis of mechanisms, the structural diagram of the mechanism and, as a rule, metric parameters (lengths of links) are considered known. When designing the mechanism, the given dynamic characteristics and mass-inertial characteristics are found, and in some cases constant geometric parameters are sought, at which the required dynamic characteristics of the movement are provided. As a rule, the problem of dynamic balancing of mechanisms is considered for mechanisms for which $a_{\max}/g > 1$. The following tasks are most common [4, 5, 6].

1. balance (balance the masses) the impact of the mechanism on the support (the foundation on which the mechanism is located);

2. balance (balance the moment or power) the impact on the engine of the mechanism (or on the prime mover);

3. balance the effect on the kinematic pairs.

It is clear that to reduce the value of the inertia forces of moving links, it is necessary to reduce the mass of these links. Due to the complexity of solving the problem using this feature, such problems are solved by special methods [1, 5, 7].

It is customary to distinguish between static and dynamic balancing of the mechanism; their elimination in the designed mechanism will correspond to its static and dynamic balancing [1]. In this case, according to the degree of equilibrium, you can get an exact or approximate solution (balancing).

Traditionally, the criterion for accurate static balancing of the mechanism is the condition that the main vector of inertia forces of its links is equal to zero.

$$\overline{F}^{IN} = 0 \tag{1}$$

which corresponds to the immobility of the general center of mass of the mechanism. With precise dynamic balancing, simultaneously with the specified condition, it is still necessary to zero out the main moment of the inertia forces of the links, i.e.

$$\overline{F}^{IN} = 0, \quad M_0 = 0. \tag{2}$$

If the mechanism has managed to achieve the exact balancing conditions in any way, then these conditions will be preserved under any law of motion of the input link and, consequently, the balance of the mechanism (both static and dynamic) becomes an integral quality of the mechanism.

Approximate balancing of the mechanism can be considered as an approximation to the exact one, when in solving a specific problem some minor conditions can be neglected.

2 Analysis of literature data and problem statement

As a drive for rod pumps, beam-pumping units are traditionally used, which have a simple studied scheme, and in comparison with other drives, an economical, repair-suitable design [1, 2, 3-7, 8, 9,10].

Figure 6a shows the straight-line lambda-shaped Chebyshev mechanism, and Figures 1b-e show its modifications and related mechanisms, which can be obtained by applying the Roberts-Chebyshev theorem. The mechanisms shown in Figures 1a and 1b provide a fairly high accuracy of reproducing a rectilinear trajectory with equal lengths of the connecting rod-rocker group links and with strict observance of symmetry in the size of the connecting rod (the connecting rod is an isosceles triangle).

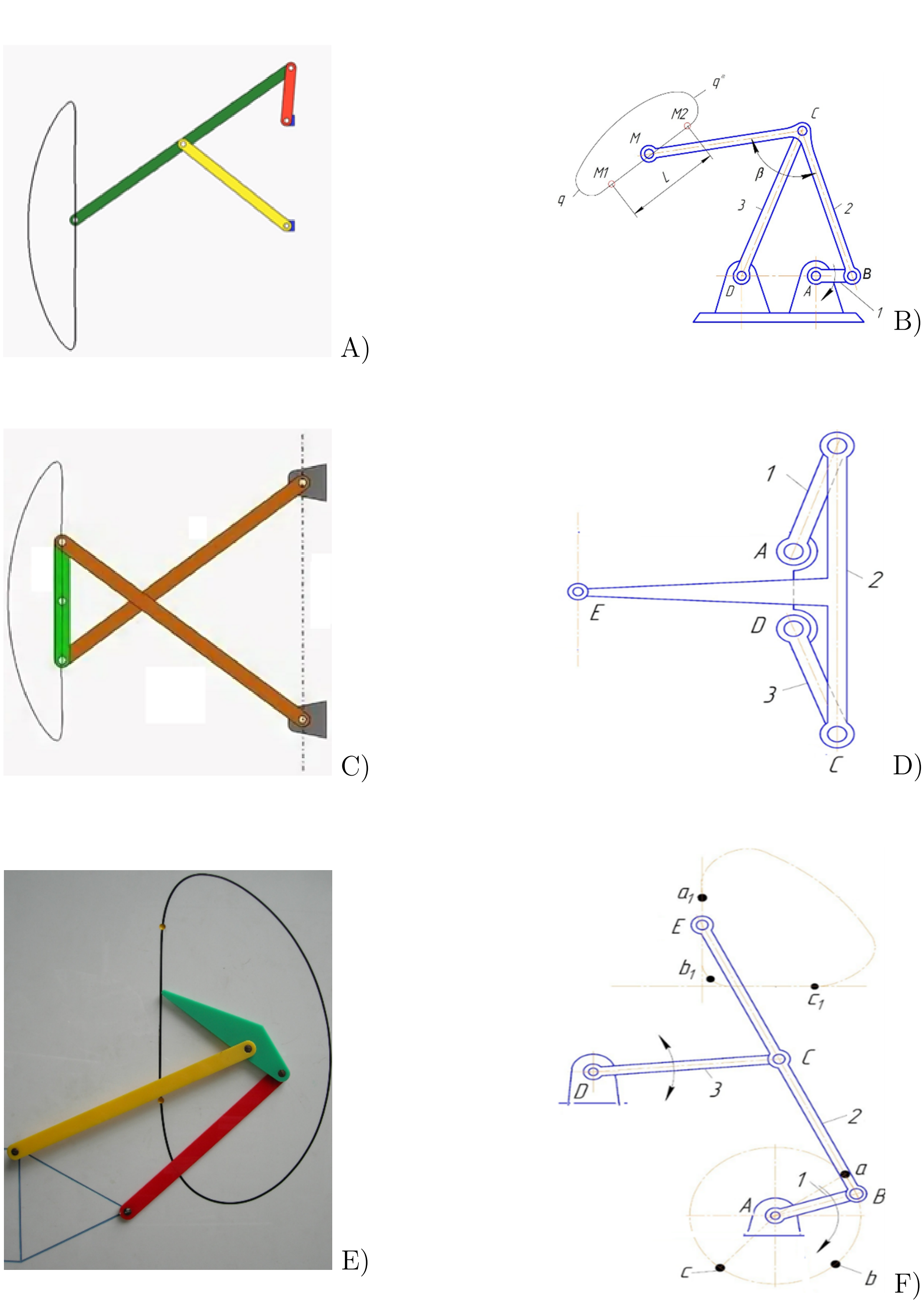


Figure 1: Chebyshev Mechanisms

For further research, we selected the mechanisms shown in Figures 1a and 1g, since the use of the remaining schemes is difficult due to the lack of access to the wellhead. However, a more detailed study of the functionality of the selected schemes showed that it is not possible to achieve a reduction in overall dimensions here. Thus, the gain is only in eliminating the arched horsehead.

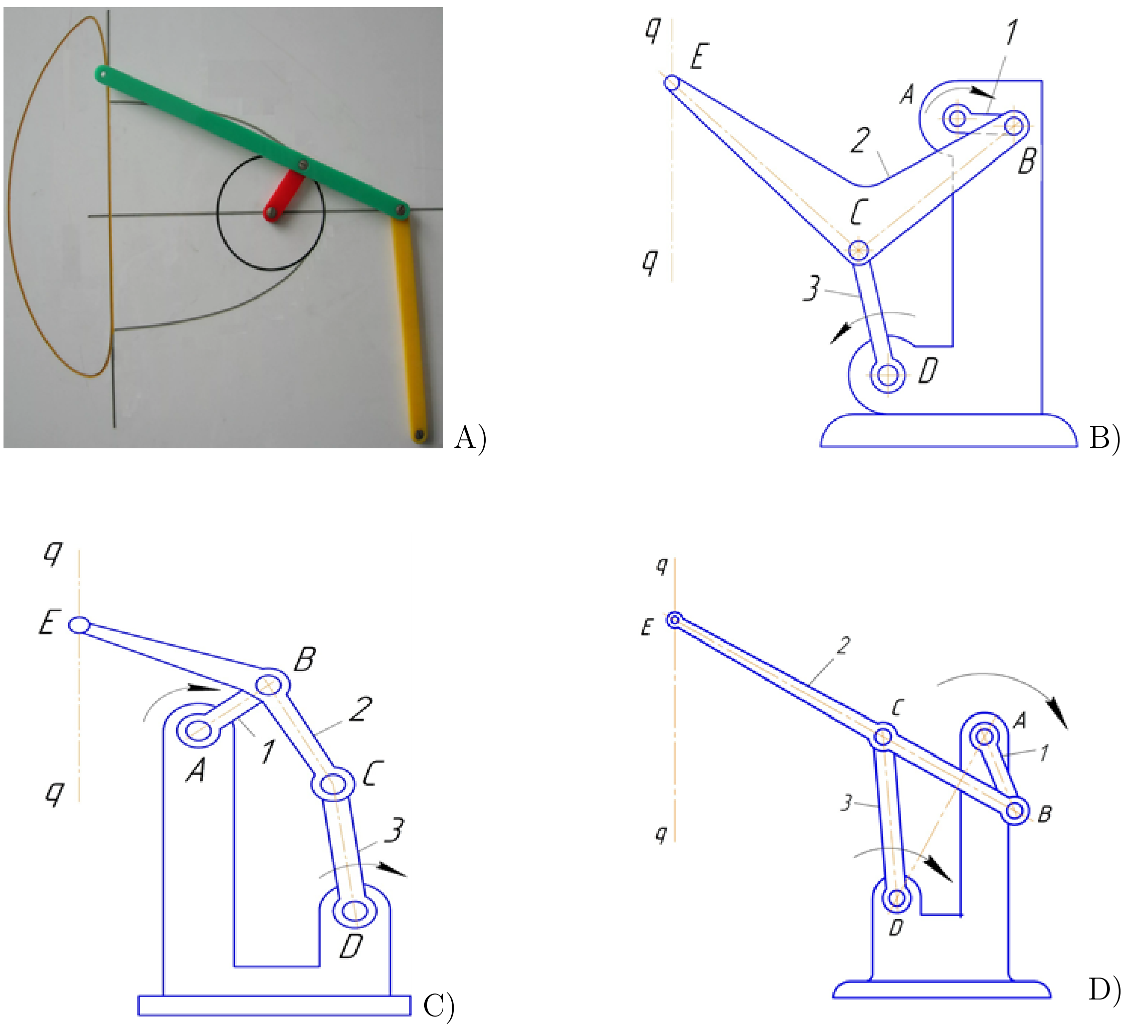


Figure 2: The Evans mechanism and its variants

Much more interesting were the mechanisms of the Evans type (Figure 2) and Evans – de Jonge type (Figure 3), which are characterized by good access to the well, as well as a small ratio of the length of the straight section of the rod curve trajectory to the lengths of the mechanism links. Upon closer examination, the schemes in Figures 2 were the most promising. The diagram in Figure 7a is the most compact.

However, to ensure the maximum ratio of the length of the straight segment to the length of the connecting rod, an additional synthesis must be carried out taking into account this additional restriction. This mechanism, therefore, is a competing scheme with respect to the Roberts scheme, and the position of the upper rack was significantly lower than in the Roberts mechanism.

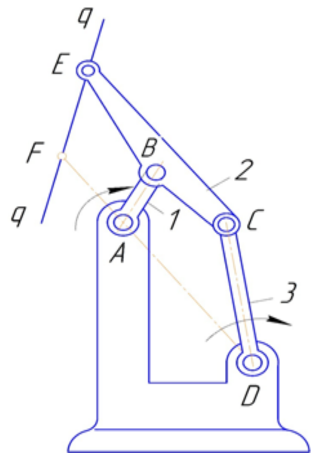


Figure 3: Evans-de Jonge Mechanisms

In [11, 12], a methodology, algorithms, and software for the kinematic and kinetostatic calculation and optimal balancing of the conversion mechanisms of the RPD with a two-arm beam and rotary equalizer were developed. The methodology and software were used to calculate the pumping units RPD6-2,5-3500 and RPD8-3-4500 with maximum loads in the wellhead gland of 6t and 8t.

An alternative option is to use straight-line mechanisms as a conversion mechanism. Thus, the advantages of the PU (pumping unit) "with a floating beam" were confirmed by the experience of developing and operating the 2CKM7 type, created on the basis of the CKH70-3012 pumping unit [2, 11]. The Evans lemniscate straight line is used here as a converting mechanism. Another example of the use of straight-line mechanisms is the recent "Minnesota" development [13], in which the reciprocating motion of the rod suspension is provided by the Roberts mechanism. The goal of the development is initially to eliminate the massive and complex head ("horse head") in typical installations [6]. Moreover, the overall dimensions in both cases were almost two times smaller than the prototypes.

3 Solution of the problem

There are other types of straight-line mechanisms that could also be used effectively [11, 12, 13, 14, 15]. But a systematic study of them in relation to the problem under consideration has never been conducted.

Next, we will conduct a kinetostatic analysis of the six-link scissor mechanism of the RPD. The mechanism is affected by the load in the wellhead oil seal and the gravity of the links and loads (Figure 4). The forces of friction in the joints and the forces of inertia are not taken into account, since the values of these forces are insignificant.

For the equilibrium of the system of forces, the main vector of the force and the main moment of the force are equal to zero

$$\sum \vec{F_i} = 0 \tag{3}$$

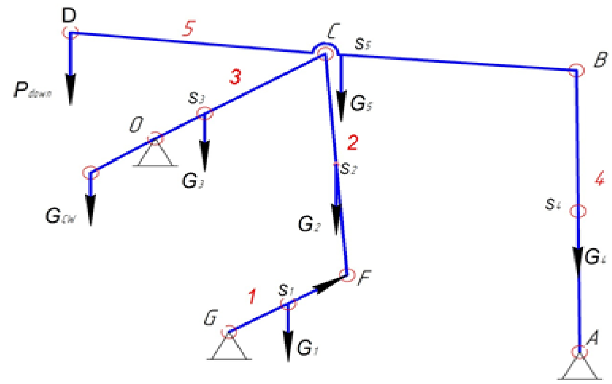


Figure 4: Power analysis of the six-link scissor mechanism conversion mechanism of pumping units

and

$$\sum \vec{M}_i = 0 \tag{4}$$

We consider the equilibrium of each link. G_5 – weight of the fifth link operates on the 5th link at the center of mass, as well as P load (the weight of the rod string and pumped liquids) in the suspension point of the D rod string, and reactive power R_{54} and R_{52}. Then the equilibrium equations of the 5th link

$$\begin{gathered} -R_{54}^x - R_{52}^x = 0 \\ -G_5 - R_{54}^y - R_{52}^y - P = 0 \\ G_5(X_{S5} - X_B) + R_{52}^y(X_C - X_B) + R_{52}^x(Y_B - Y_C) + P(X_D - X_B) = 0 \end{gathered} \tag{5}$$

Consider the equilibrium of the 4th link, the connecting rod. The connecting rod is affected at the center of mass of the connecting rod by G_4 – the weight of the connecting rod and R_{04}, R_{54} reaction forces. Then we make the equilibrium equations of the connecting rod – the 4th link, taking into account $R_{45} = -R_{54}$.

$$\begin{gathered} R_{04}^x + R_{54}^x = 0 \\ -G_4 + R_{04}^y + R_{54}^y = 0 \\ G_4(X_{S4} - X_A) + R_{54}^y(X_B - X_A) + R_{54}^x(Y_A - Y_B) = 0 \end{gathered} \tag{6}$$

Now consider the equilibrium of the 3rd link. The third link is affected at the point S_3 by G_3 force – the weight of the third link, at point by G_{CW} force – the weight of the counterweight and R_{03} and R_{32} reaction forces.

Then for the 3rd link

$$\begin{gathered} R_{04}^x - R_{32}^x = 0 \\ -G_3 + R_{03}^y - R_{32}^y - G_{CW} = 0 \\ G_3(X_{S3} - X_O) + R_{32}^y(X_C - X_O) + R_{32}^x(Y_O - Y_C) + G_{CW}(X_{CW} - X_O) = 0 \end{gathered} \tag{7}$$

Now consider the balance of the 2nd link, the connecting rod. The connecting rod is affected in the center of mass of the connecting rod by G_2 – the weight of the connecting rod and R_{21}, R_{32}, R_{52} reaction forces.

Considering that $R_{52} = -R_{25}$ and $R_{32} = -R_{23}$, we compose the equilibrium equations of the connecting rod - the 2nd link

$$\begin{gathered}
-R_{21}^x + R_{32}^x + R_{52}^x = 0 \\
-G_2 - R_{21}^y + R_{32}^y + R_{52}^y = 0 \\
G_2(X_{S2} - X_F) + R_{32}^y(X_C - X_F) + R_{32}^x(Y_F - Y_C) + R_{52}^y(X_C - X_F) + R_{52}^x(Y_F - Y_C) = 0
\end{gathered} \tag{8}$$

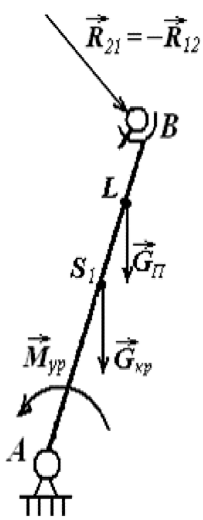


Figure 5: The equilibrium of the crank

The crank is affected at S_1 point in the center of mass of the crank by G_1 – the weight of the crank and $_{et}$ engine torque, also at F point R_{21} reaction and at G point reaction. We compose the equilibrium equations for the 1st link, where $R_{21} = -R_{12}$.

$$\begin{gathered}
R_{01}^x + R_{21}^x = 0 \\
-G_1 + R_{01}^y + R_{21}^y = 0 \\
G_1(X_{S1} - X_A) + R_{21}^y(X_F - X_G) + R_{21}^x(Y_F - Y_G) + M_{et} = 0
\end{gathered} \tag{9}$$

By solving equilibrium equations simultaneously, we find unknown components of reactions and an unknown moment.

The purpose of the balancing task is to minimize input torque M on the crank shaft. To do this, one needs to properly pick up the mass counterbalance and the distance of the center counterweight from the axis of rotation. In the case of rotary balancing, the counterweight is set on the crank. And here the counterweight is set on the 3rd link.

The distance of the center of the mass counterweight from the axis of the rotation of the link is defined in the first approximation as [15]:

$$OL = k \cdot H_s \left(P_{up} + P_{down}\right)/4 \cdot F_{cw} \tag{10}$$

where H_s – is the length of the rod string, P_{up}, P_{down} are loads at the point of rod suspension at the motion up and down, F_{cw} is the total weight of counterweights, the correcting coefficient that is manually entered by the user until the two peaks of values - torque on the shaft of the crank will not be equal.

Let's use the well-known principle of possible movements

$$\sum \delta_i = 0 \quad or \quad \sum N_i = 0. \tag{11}$$

According to the principle of possible movements, power of these forces should be zero. Let's write this down for our problem:

$$F\hat{V}_{S_1} + \hat{F}_2\hat{V}_{S_2} + \hat{F}_3\hat{V}_{S_3} + \hat{F}_4\hat{V}_{S_4} + \hat{F}_5\hat{V}_{S_5} + F_{CW}\hat{V}_L + +\hat{F}_P\hat{V}_D + M_{GF} = 0 \tag{12}$$

Here, are the velocities of the corresponding points of gravity forces application;
ω_{GF} – is angular speed of the crank;
M – is the torque on the crank shaft.

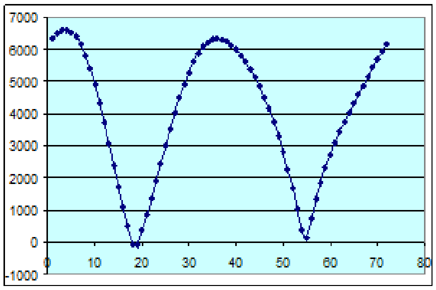
4 Discussion of experimental results

The results of the study of the balancing modes are presented in table 1. The last two columns show and values, which were found in such a way as to minimize torque on the crank shaft. In the calculation program, this is achieved by manually specifying correcting coefficient in formula (10).

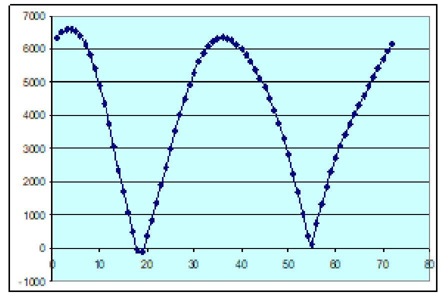
It was also found that when the direction of the crank rotation is changed (counterclockwise), the moment on the crank shaft practically does not change, but the balancing mode is somewhat worse (the counterweights are removed from the axis of rotation).

Table 1 – Results of the study of balancing modes

Pumping mode				The maximum torque on the crank shaft	Weight of the counterweight with optimal balancing	Counterweight distance from the crank rotation axis
P_{up}	P_{down}	Angular velocity	The length of the crank			
kN	kN	rpm	mm	kNm	kg	mm
60	30	6,9	550	10,7	478	0,525m
60	40	4,3	1000	8,412	478	0,753m
60	40	6,9	1000	8,365	478	0,753m



A) Graph of the change in torque as a result of kinetostatic analysis



B) Graph of torque changes based on the principle of possible displacements

Figure 6: The Graphs of torque changes

5 Conclusion

According to the results of the kinetostatic analysis, a graph of the change in the torque is obtained. Figure (5) shows the graphs of the torque change obtained in various ways.

The results of the work carried out show that the goal of the study has been achieved. Since a detailed kinetostatic analysis confirms the possibility of using the studied six-link straight-line mechanism as a converting mechanism for the rod pumping drive.

1. A kinetostatic analysis was performed and a mathematical model of the kinetostatic analysis of the six-link conversion mechanism in the Maple environment was developed in order to test the performance of the new design.

2. Numerical results obtained by various methods confirm that the results are reliable.

6 Acknowledgments

This research has been funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08052127).

References

[1] Wang D., Liu H., "Dynamic Modeling and Analysis of Sucker Rod Pumping System in a Directional Well" , *Mechanism and Machine Science, Asian MMS* (2016).

[2] Reges GD, Schnitman L, Reis R et. al. "A new approach to diagnosis of sucker rod pump systems by analyzing segments of downhole dynamometer cards" , *In: SPE artificial lift conference-Latin America and Caribbean, Society of Petroleum Engineers, Salvador* (2015): 1-13.

[3] Chen G.S., Liu X., *Friction Dynamics of Oil-Well Drill Strings and Sucker Rods* (Friction Dynamics, 2016).

[4] Mezrina N.M., "Algorithm for economic standard calculation while planning equipment maintenance and repair and determination of the equipment market value" , *Kalashnikov ISTU* 1(20) (2017): 64-69.

[5] *Passport of the innovative development program of PJSC NK Rosneft* (Moscow, 2016): 30.

[6] Niladri Kumar Mitra, *Principles of Artificial Lift* (Allied Publishers, 2012): 464.

[7] Tan Ch., Qu Y., Yan X., Banger P., *Predicting the Dynamometer Card of a Rod Pump* (Algorithmica Technologies, 2018).

[8] Guo B., Liu X., Tan X. *Petroleum Production Engineering* (Houston: Gulf Professional Publishing, 2017): 780.

[9] Ibraev S.M., *Dynamic synthesis and optimization of the linkage drive* (Almaty, 2014): 303.

[10] Volokhin A.V., Arsibekov D.V., Volokhin E.A., Volokhin V.A. "Pat. 2688615 Russian Federation, IPC F 04 B 47/02. Balancing drive of sucker rod pump installation" , No. 2018130381, announced on August 20, 2018; publ. 05/21/2019, Bulletin No. 15. 2 p.

[11] Hand A., "Cost Cutting's Effect on Oil and Gas Safety" , *Automation World Journal* (2018): 52-57.

[12] Askarov E.S. Baizhanova S.B. and others. "Patent RK 31911, Machine-rocking chair with increased stroke" , *MPK* F04B47 / 02, publ. 15.03.17, bull. No. 5.

[13] Ibraev S., Imanbaeva N., Nurmaganbetova A., Zhauyt A., "Computerized modeling of kinematics and kinetostatics of sucker-rod pump power units" , *16th International Scientific Conference: Engineering for Rural Development* (2017).

[14] Imanbaeva N.S., Nurmaganbetova A.T., Isametova M.E., Rakhmatulina A.B., Sakenova A.M., "Study mode converts trim mechanism sucker rod pumping units (SRPU), to determine the distance from the rotational axis of the counterweight crank" , *Vestnik KazNRTU* №1 (2017): 328-332.

[15] Chicherov L.G., Molchanov G.V., Rabinovich A.M, i dr. *Raschyot i konstruirovanie neftepromyslovogo oborudovaniya* [Calculation and design of oilfield equipment] (M.: Nedra, 1987): 422.

3-бөлім | Раздел 3 | Section 3

Информатика | Информатика | Computer Science

IRSTI 20.23.17; 20.23.21; 20.23.25 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.08

Zahoor Ahmad, Madina Mansurova*
Al-Farabi Kazakh national university, Kazakhstan, Almaty
*E-mail: madina.mansurova@kaznu.kz

MACHINE LEARNING APPROACH TO PREDICT SIGNIFICANT WAVE HEIGHT

To estimate significant wave height of ocean wave, a machine learning framework is developed. Significant wave height and period can be used by supervised training of machine learning to predict ocean conditions. In this paper we proposed a method to predict significant wave height using Support vector regression (SVR). Buoy dataset taken from the Queensland government open data portal the input from which were aggregated into supervised learning test and training data sets, which were supplied to machine learning models. The SVR model replicated significant wave height with a root-mean-squared-error of 0.044 and performed on the test data with 95% accuracy. Comparing to forecasting with the physics-based model the Machine learning SVR model requires only a fraction ($< 1/1200^{\text{th}}$) of the computation time, to predict Significant wave height.

Key words: Machine learning, significant wave height, Support vector regression.

Захур Ахмад, Мадина Мансурова*
Әл-Фараби атындағы Қазақ ұлттық университеті, Қазақстан, Алматы қ.
*E-mail: madina.mansurova@kaznu.kz

Толқынның елеулі биіктігін болжауға арналған машиналық оқыту негізіндегі тәсіл

Мұхит толқынының елеулі биіктігін бағалауға арналған машиналық оқыту жүйесі құрылды. Толқынның елеулі биіктігі мен толқын периоды мұхит жағдайларын болжау үшін бақыланатын машиналық оқыту барысында пайдаланылуы мүмкін. Бұл жұмыста тірек векторы әдісі негізіндегі регрессия көмегімен (Support vector regression – SVR) толқынның елеулі биіктігін болжау әдісі ұсынылды. Буй деректер жиыны Квинсленд үкіметінің ашық деректер порталынан алынды, кіріс деректері бақыланатын оқыту мен тестілеу үшін деректер жиынтығына біріктірілді. SVR моделі толқынның елеулі биіктігін 0,044 орташа квадраттық қателікпен көрсетті және тестілеу деректерінде 95% дәлдікпен бойынша орындалды. Толқынның елеулі биіктігін физикалық модель негізінде болжаумен салыстырғанда, машиналық оқыту негізіндегі SVR моделі айтарлықтай аз есептеу уақытын ($< 1/1200$) қажет етеді.

Түйін сөздер: Машиналық оқыту, толқынның елеулі биіктігі, тірек векторы әдісі негізіндегі регрессия.

Захур Ахмад, Мадина Мансурова*
Казахский национальный университет имени аль-Фараби, Казахстан, г.Алматы
*E-mail: madina.mansurova@kaznu.kz

Подход на основе машинного обучения для прогнозирования значительной высоты волны

Разработана система машинного обучения для оценки значительной высоты океанской волны. Значительная высота и период волны могут быть использованы при контролируемом машинном обучении для прогнозирования состояния океана. В данной работе предложен метод для прогнозирования значительной высоты волны с помощью регрессии на основе метода опорных векторов (Support vector regression – SVR). Набор данных буев взят с портала открытых данных правительства Квинсленда, входные данные с которого были объединены в наборы данных для контролируемого обучения и тестирования.

Модель SVR воспроизводила значительную высоту волны со среднеквадратической ошибкой 0,044 и выполнялась на тестовых данных с точностью 95%. По сравнению с прогнозированием значительной высоты волны на основе физической модели, для модели SVR с машинным обучением требуется значительно меньше (< 1/1200) времени вычислений.
Ключевые слова: Машинное обучение, значимая высота волны, регрессия на основе метода опорных векторов.

1 Introduction

Many people are unaware of a single climate factor that can have a profound effect on the living conditions and health of coastal people. Wave weather is the distribution of wave signals measured at a given time and place, just as atmospheric weather is defined as the "intermediate weather" of a given time and place. About 10% of the world's population lives within 20 kilometers of coastline and less than 20 meters above sea level (Kummu et al. 2016). For these people, hot weather can affect their daily lives like atmospheric weather. Big waves can disrupt harbors and make boats dangerous, keeping fishermen and boats afloat while their businesses suffer.

Surfers aside, there are basic reasons why information on wave conditions over the next few days is important. For example, delivery routes can be made by avoiding rough seas and thus reducing shipping times. Another industry that benefits from wave information is the $ 160 B (2014) [1] marine fishery, which can improve harvesting activities accordingly. Awareness of critical situations is critical to military and navy operations by Navy and Marine Corps teams. Also, predicting energy production from renewable energy sources is important in maintaining a stable electricity grid because more renewable energy sources (e.g. sun, wind, waves, wave, etc.) are in between. In the deep penetration of the renewable energy market, a combination of increasing energy conservation and improved speculation of energy prediction will be required.

Waves can be defined by three distinct elements: wavelength, wave duration, and direction of wave. The higher the tide, the more dangerous the boat conditions and the greater the potential for the wave to form or erode beaches and coastal cliffs. The direction of the wave is the way in which the wave comes to the observer.

In practice, it is difficult to measure these variables because the waves of different wavelengths, heights and directions can mix and produce very confusing wave patterns. Scientists and engineers use sophisticated calculations to solve the parts of the waves and produce three common summarization calculations: critical wavelengths (H_s), wavelength (T_p), and wave direction (θ_m). These three figures are then used to describe the weather of the waves, just as temperature, rain, wind speed and direction can be used to describe the local climate. Commercialization and distribution of wave energy technology will require not only addressing positive and regulatory issues, but also overcome technological challenges, one of which can provide accurate predictions of energy production. The need for any prediction is that the model that is properly represented is developed, measured and validated. In addition, the model must be able to run fast and include the correct prediction details in its predictions. A mechanical framework for this skill is developed here.

Because wave models can be awfully expensive, a new method of machine learning [2, 3, 4] is being developed here. The purpose of this approach is to train machine learning models in the more realistic model of wave-based physics forced by atmospheric and ocean history conditions to accurately represent wave conditions (in particular, significant wavelengths and feat). Computer costs are often a major limitation of real-time forecasting systems [6, 7]. Here, we use machine learning techniques to predict significant wave height by taking the predictor and predict and variable into account from the dataset. While machine learning were used to predict wave conditions [8, 9, 10, 11, 12, 13], it has not been used in the context of a surrogate model which can obtained highest accuracy with lowest root mean squared error as defined below.

2 Wave modeling

2.1 Numerical Model

The Simulating WAves Nearshore (SWAN) code FORTRAN is a standard industrial tool developed at Delft University of Technology that incorporates wave fields in coastal waters forced by wave conditions at natural boundaries, oceans, and winds [14]. SWAN mimics the energy contained in the waves as they travel in the ocean and disperse ashore. Specifically, data on the surface of the ocean contains a wave-variance spectrum, or energy density $E(\sigma, \theta)$, and these wavelengths are still distributed over wavelengths (as seen in the unused frame of the current speed reference) with distribution directions common to rotate the stems of each spectral object.

The bulk of the action is defined as $N = E/\sigma$, which is saved during the distribution along the wave element before the current one. The appearance of $N(x, y, t; \sigma, \theta)$ in space, x, y, and time, t, is governed by the action balance equation [15, 16]:

$$\frac{\partial N}{\partial t} + \left(\frac{\partial c_x N}{\partial x} + \frac{\partial c_y N}{\partial y}\right) + \left(\frac{\partial c_\sigma N}{\partial \sigma} + \frac{\partial c_\theta N}{\partial \theta}\right) = \frac{S_{tot}}{\sigma} \tag{1}$$

The left side represents the kinematic part of the equation. The second term (parent) describes an increase in the wavelength of a wave in the opposite direction of the Cartesian space where the c is wave wave. The third term represents the effect of a change in radian frequency due to differences in water depth and current mean. The fourth term presents a deeper reflection and current practice. Maximum c_σ and c_θ distribution speed in the spectral space (σ, θ). The right-hand side represents the dynamic sources of space and the sinking of all body processes that produce, disperse, or disperse the wave energy (i.e., wave growth through air, offline power transmission through three or four wave interactions, and wave decay due to white extinguishing, collision, and depth).

Haas et al. [5] define wave power consumption as a function of the critical wavelength, Hs and time wavelength, T. This information can be used to calculate wave power. Therefore, the time limit of T and, in particular, Hs because J is proportional to the wavelength, is necessary to predict the intensity of the wavelength.

3 Machine learning

3.1 Proposed method

Supervised machine learning regression models are tested to perform tasks of predicting significant wave height. Support-vector Regression (SVR) constructs a hyperplane or set of hyperplanes in a high- or infinite-dimensional space, which can be used for regression (Hs prediction), or other tasks like outlier's detection. The function used to map a lower dimensional data into a higher dimensional data though kernel. Two parallel lines drawn to the two sides of Support Vector with the error threshold value, (epsilon) are known as the boundary line. These lines create a margin between the data points.

3.2 Background

Python toolkit SciKit-Learn [34] was used to access high-level programming interfaces to machine learning libraries and to cross validate results. Machine learning have shown the greatest potential for pattern recognition in large data sets. Consider that a physics-based model acts as a non-linear function that converts input (wave signals and variable ocean currents and wind speeds) to output (spatially variable Hs). The predictor and predict and from buoy data can be collected in input vector, x, and output vector, y, respectively.

Because the purpose of this effort is to develop a framework of machine learning to effectively predict Hs from buoy data, the nonlinear function mapping inputs to the best representation of outputs, $\underline{\widehat{y}}$, is sought:

$$g(\underline{x}; \underline{\underline{\Theta}}) = \underline{\widehat{y}}. \tag{2}$$

The machine learning sufficiently trained model provides a mapping matrix, $\underline{\underline{\Theta}}$, which is a machine learning data model driven by vector-matrix functions included in (3).

The Python Toolkit SciKit-Learn [21] has been used to access high-level frameworks for cross-validation results and python machine learning libraries. Machine learning SVR model is used considering the root mean squared error, less training data for better prediction and is faster to compute output (more on this later).

3.3 Training dataset

Cairns wave monitoring of Queensland Coastal weather Observation data from Datawell 0.7 m Waverider Buoys were downloaded. Measured and derived wave parameters from data collected by a wave monitoring buoy anchored at Cairns (1 Jan 2020 to May 2020). The dataset has six fields: *Hs, Hmax, Tz, Tp, Di_ TpTrue* and *SST*. These fields are defined in table (2). There were 130 occasions when data from wave monitoring buoy at cairns were missing. Those missing values were deal using feature engineering by replacing them with the average values. These data were compiled into $\underline{\underline{X}}$ vectors. In total, the design matrix $\underline{\underline{X}}$ has 4,369 rows and 7 columns.

Table 1. Dataset fields (Attributes)

Field	**Definition**
Hs	Significant wave height, an average of the highest third of the waves in a record (26.6 minute recording period)
Hmax	The maximum wave height in the record The zero upcrossing wave period
Tz	The zero upcrossing wave period
Tp	The peak energy wave period
Dir_Tp TRUE	Direction (related to true north) from which the peak period waves are coming from
SST	Approximation of sea surface temperature
Date_Time	The Date and time of the record

For SVR algorithms, $\underline{\underline{Y}}$ is composed of the 4,369 model runs (rows), each of which contains 7 attributes (columns) defining the Hs field.

Note that in practice, data on design matrices is pre-processed. Specifically, $\underline{\underline{X}}$ undergoes a generalized global variable (e.g., all existing members are measured so that their total distribution is Gaussian with zero mean and unit variance). Here, no pre-processing of SVR's $\underline{\underline{Y}}$ is required.

Data $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ were randomly divided into two groups to form a training data set consisting of 90% of 4,369 rows of data and test data sets the remaining 10%. The mapping matrix is calculated using training data and then used in the test data set and RMSE between the vector of the test data, $\underline{y}$, and its machine learning representation, $\underline{\widehat{y}}$ is calculated.

The SVR algorithm needs to be supplied only by $\underline{\underline{X}}$ and the vector of the Hs value column compiled as y. Data were further subdivided into two groups with 90% of $\underline{x}$ vector randomly assembled in training dataset and reserved the remaining for testing. The SVR model returns three files; the first describes the normal change applied to $\underline{x}$, the dot product taken with the mapping matrix $\underline{\underline{\Theta}}$ described in the second file, and the third file is used to convert $\underline{\widehat{y}}$ back to the characteristic Hs.

3.4 Support vector regression model

In training data set Xn is a multivariate set of N observations with Yn response value observed. To find the linear function (4) and make sure it is as flat as possible, find $f(x)$ having minimal norm value

$$f(x) = x'\beta + b \tag{3}$$

(β', β). This is constructed as a convex optimization problem to minimize (5) subject to all residuals

$$J(\beta) = \frac{1}{2}\beta\beta' \tag{4}$$

having a value less than ε; or, in equation form (6):

$$\forall n : |y_n - (x_n'\beta + b)| \leq \varepsilon. \tag{5}$$

It is possible that no such function $f(x)$ exists to satisfy the constraints of all points. To deal with impossible obstacles, enter the slink ξn and $\xi * n$ variables for each point. This approach is similar to the concept of "soft margin" in SVM segmentation, because the flexible flexibility allows regression errors to exist until the ξn and $\xi * n$ values, but still satisfy the required conditions.

The inclusion of slack variables leads to the primal formula, also known as the objective function [25]:

$$J(\beta) = \frac{1}{2}\beta\beta' + C\sum_{n=1}^{N}(\xi_n + \xi_n^*) \tag{6}$$

Subject to:

$$\forall n : y_n - (x_n'\beta + b) \leq \varepsilon + \xi_n \tag{7}$$

$$\forall n : (x_n'\beta + b) - y_n \leq \varepsilon + \xi_n^* \tag{8}$$

$$\forall n : \xi_n^* \geq 0 \tag{9}$$

$$\forall n : \xi_n \geq 0 \tag{10}$$

Constant C is the limit of the box, a positive numerical value that controls the penalty placed on the observation which lies outside the epsilon margin (ε) and helps prevent over-fitting. This value determines the trade-off between $f(x)$ fatness and the value until the deviation greater than ε is tolerated.

The linear loss function of ε-insensitivity ignores errors that are in ranges of ε distance of the observed values by considering them as equal to zero. Loss is measured based on the distance between the observed value y and the ε boundary. This is described by

$$L_\varepsilon = \begin{cases} 0 \quad \text{If} \quad |y - f(x)| \leq \varepsilon \\ |y - f(x)| - \varepsilon \quad \text{Otherwise} \end{cases} \tag{11}$$

In most of the linear regression models, the objective is to minimize the sum of squared errors. For example, take Ordinary Least Squares (OLS). For OLS with one predictor (Maximum wave height) the objective function is as follows:

$$MIN\sum_{i=1}^{n}(y_i - w_i x_i)^2 \tag{12}$$

Where y_i is the target, w_i is the coefficient, and x? is the predictor (Maximum wave Height).

Ridge, Lasso and ElasticNet are all extensions of this simple equation, with an additional penalty parameter, Correlation-based Feature Selection (CFS), that aims to minimize complexity and reduce the number of features used in the final model. The aim is to reduce the error of the test set.

In contrast to OLS, the SVR's objective function is to reduce coefficients – in particular, the $l2$-norm of the vector coefficient – not the squared error. The term error is rather handled in constraints, where we set the absolute error below or equal to the specified margin, called the maximum error, ϵ (epsilon). We can tune the epsilon to get for our model the desired accuracy. The new objective function and constraints for our model are as follows:

$$\begin{aligned} &\text{Minimize } \frac{1}{2}\|w\|^2, \\ &\text{Subject to } |y_i - \langle w, x_i \rangle - b| \leq \varepsilon \end{aligned} \tag{13}$$

Where x_i is a training sample with target value y_i.

The inner product plus intercept $\langle w, x_i \rangle - b$ is the prediction for that sample, and ε is a free parameter that serves as a threshold. The Kernel applied here is RBF(Radial basis function) due to non-linearity in the data set.

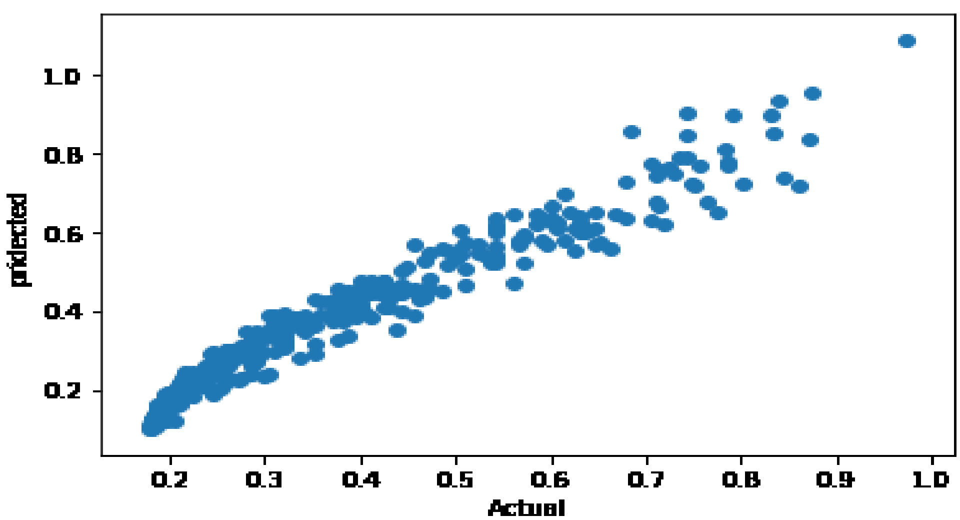


Figure 1: Scatter plot showing the actual and predicted values for Hs. The Swan model values horizontally on X-axis and SVR predicted values are represented by Y-axis vertically.

The SVR model effectiveness was evaluated according to the accuracy percentage in predicting Hs value. In the SVR results no bias was observed and 95.7% of the time correctly predicted the characteristic Hs in the test data set. The scattered plot in Figure 2 visualize the characteristic Hs from SWAN and the SVR representation which revel that there is no outlier found with the final model.

The problem of regression is to find a function that approximates mapping from an input domain to real numbers based on a training sample. To analyze the performance of SVR, the model was trained on 90% of the dataset and the remaining 10% was allocated as test data. The accuracy of SVR was 95% on test dataset with a root mean squared error of 0.044.

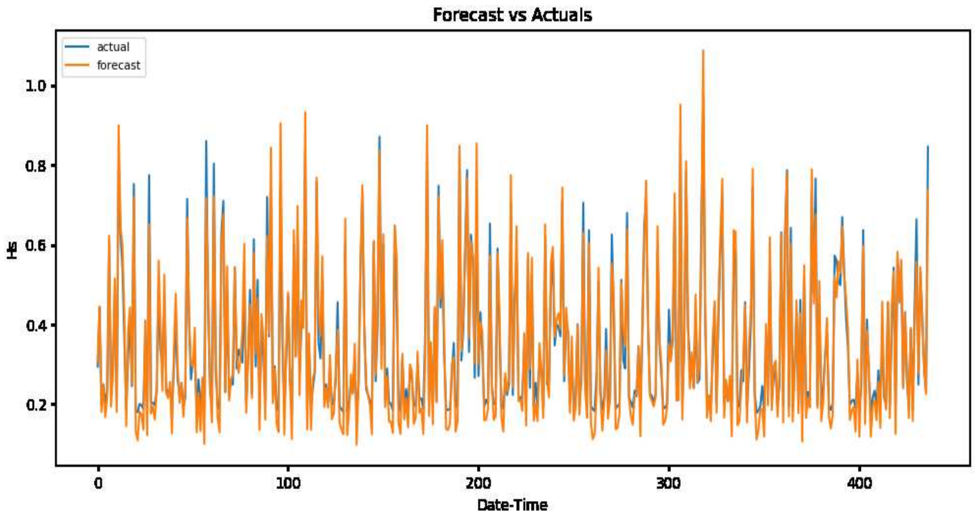


Figure 2: Plot visualizing the actual and forecasted values for Hs against time series. Blue indicate the actual Swan model Hs while the orange represents SVR predicted Hs.

4 Discussion

Advanced machine learning models have been developed here to create improved mapping matrix (or vector) and pre- and post-processor functions, to predict significant wave height. Instead of the historical data used to create an input vector, x, now the weather data can be used. In order to work with the weather mode, the data from buoy are used, both the predictor and predicant from the same data, to train the machine learning algorithm to form Hs field. Such data is part of the Marine Information System which is the state of WAVEWATCH III-predictable waves conditions available for the next 10 days. Also, forecasts for ROMS-simulated ocean-currents and wind forecast are available for the next 48 hours from CeNCOOS and The Weather Company [19], respectively. For the Cairns wave the historical data is available on Queensland Coastal weather Observation [24] for data taken from Datawell 0.7m Waverider Buoys.

The execution of machine learning models quickly produces the Hs field. Computationally, this only need a multiplication of the $L+1$ matrix. In fact, for a 24-hour forecast, on a single-core processor the machine-learningSVR took 0.044 s to calculate the Hs field | well over three orders of magnitude (485,833%) faster than the running the full physics based models. In fact, performance that requires a lot of wall clock time loads metrics files for memory.

The machine learning models presented here are specific to the Queensland coast cairns region and will need to be re-training to apply to other locations. Of course, using a physics-based model on a new site requires the creation of a grid and the integration of all the boundary and conditions of coercion with all the efforts of the server. However, the important thing is that the framework needed to develop this technology is introduced for the first time for wave modeling in 2017. As expected [22], the data-centric modeling machine learning approaches has grown increasingly common in last few years and are expecting to grow in near future rapidly.

5 Conclusion

An improved version of machine learning models has been developed with new approach, as computationally efficient to predict Hs fields. From supervised training of machine learning models determined appropriately trained mapping matrices, give in representations of similarly accurate Hs in the domain of interest. Thus, this approach of machine learning models can contribute to a fast and efficient wave-condition forecast system. The power-generation potential of WECs or surf conditions can be estimated using these forecasted wave conditions. Ultimately, it is envisioned that such improved version of machine learning models which don't required many parameters for accurate prediction could be installed locally on a WEC thereby making their own forecast system. In addition, the buoy itself can collect wave-condition data that can be used to update machine learning models. As machine learning technology advances, they can be adapted to integrate the continuous distribution of real-time data collected locally with predictions available to change and improve the parameters of the machine learning model. In fact, such methods have already been widely used "online learning" [23].

The approach previously proposed by author to predict characters Hs using MLP requires a large amount of data and more calculation to form mapping matrix due to lack of an important parameter maximum wave height, which is considered in this work. This parameter when used as a predictor gives better forecasting with low calculation cost and high model efficiency. Additional efforts are currently underway using ensemble machine learning approaches to predict Hs and Wave period T. The results are expected to further improve the process of wave characteristics prediction and take into account how bathymetry effects wave heights.

References

[1] FAO, The State of the World Fisheries and Aquaculture 2016. Con tributing to the food security and nutrition for all *Technical Report, Food and Agriculture Organization of the United Nations* (2016).

[2] I. Goodfellow, Y. Bengio, A. Courville, "Deep Learning", *MIT Press,* (2016).

[3] Y. LeCun, Y. Bengio, G. Hinton, "Deep learning", *Nature* 521 (2015): 436-444.

[4] J. Schmidhuber, "Deep learning in neural networks: An overview", *Neural Networks* 61 (2015): 85-117.

[5] K. Haas, S. Ahn, V. S. Neary, S. Bredin, "Development of a wave energy resource classification system, in: Waterpower Week", *METS, Washington, DC,* 1-5.

[6] P. M. DeVries, T. B. Thompson, B. J. Meade, "Enabling large-scale viscoelastic calculations via neural network acceleration", *Geophysical Research Letters* 44 (2017): 2662-2669.

[7] V. Mallet, G. Stoltz, B. Mauricette, "Ozone ensemble forecast with machine learning algorithms", *Journal of Geophysical Research: Atmospheres* 114 (2009).

[8] D. Peres, C. Iuppa, L. Cavallaro, A. Cancelliere, E. Foti, "Significant wave height record extension by neural networks and reanalysis wind data", *Ocean Modelling* 94 (2015): 128-140.

[9] O. Makarynskyy, "Improving wave predictions with artificial neural networks", *Ocean Engineering* 31 (2004): 709-724.

[10] A. Etemad-Shahidi, J. Mahjoobi, "Comparison between M5' model tree and neural networks for prediction of significant wave height in lake superior", *Ocean Engineering* 36 (2009): 1175-1181.

[11] J. Mahjoobi, A. Etemad-Shahidi, "An alternative approach for the prediction of significant wave heights based on classification and regression trees", *Applied Ocean Research* 30 (2008): 172-177.

[12] M. Browne, D. Strauss, B. Castelle, M. Blumenstein, R. Tomlinson, C. Lane, "Empirical estimation of nearshore waves from a global deep-water wave model", *IEEE Geoscience and Remote Sensing Letters* 3 (2006): 462-466.

[13] M. Browne, B. Castelle, D. Strauss, R. Tomlinson, M. Blumenstein, C. Lane, "Near-shore swell estimation from a global wind-wave model: Spectral process, linear, and artificial neural network models", *Coastal Engineering* 54 (2007): 445-460.

[14] The SWAN Team, SWAN Scientific and Technical Documentation *Technical Report SWAN Cycle III version 40.51, Delft University of Technology* (2006).

[15] G. J. Komen, L. Cavaleri, M. Donelan, "Dynamics and Modelling of Ocean Waves", *Cambridge University Press* (1996).

[16] C. C. Mei, M. Stiassnie, D. K.-P. Yue, "Theory and Applications of Ocean Surface Waves: Part 1: Linear Aspects. Part 2: Nonlinear Aspects", *World Scientific* (1989).

[17] Y. Song, D. Haidvogel, "A semi-implicit ocean circulation model using a generalized topography-following coordinate system", *Journal of Computational Physics* 115 (1994): 228-244.

[18] J. Patterson, J. Thomas, L. Rosenfeld, J. Newton, L. Hazard, J. Scianna, R. Kudela, E. Mayorga, C. Cohen, M. Cook, et al., "Addressing ocean and coastal issues at the west coast scale through regional ocean observing system collaboration, in: Oceans'12", *IEEE* 1-8.

[19] The Weather Company *The Weather Company* (2017).

[20] G. Chang, K. Ruehl, C. Jones, C. C. Roberts, J. D.and Chartrand, "Numerical modeling of the effects of wave energy converter characteristics on nearshore wave conditions", *Renewable Energy* 89 (2016): 636-648.

[21] F. Pedregosa, G. Varoquaux, A. Gramfort, V. Michel, B. Thirion, O. Grisel, M. Blondel, P. Prettenhofer, R. Weiss, V. Dubourg, et al., "SciKit-Learn: Machine learning in Python", *Journal of Machine Learning Research* 12 (2011): 2825-2830.

[22] Scott C James, Yushan Zhang, Fearghal O'Donncha, "See discussions, A Machine Learning Framework to Forecast Wave Conditions", *Coastal Engineering* (2017): 46556-5637.

[23] N. Cesa-Bianchi, A. Conconi, C. Gentile, "On the generalization ability of on-line learning algorithms", *IEEE Transactions on Information Theory* 50 (2004): 2050-2057.

[24] Queensland Government Coastal Data System - Near real time wave data, open data portal https://www.data.qld.gov.au/dataset/coastal-data-system-near-real-time-wave-data

[25] Vapnik, V. *The Nature of Statistical Learning Theory* (Springer, New York, 1995).

IRSTI 28.23.17; 28.23.29; 20.53.19; 20.53.17 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.09

R.K.Uskenbayeva, A.K.Bolshibayeva*, S.B.Rakhmetulayeva
International Information technology university, Kazakhstan, Almaty
*e-mail: kakim-aigerim@mail.ru

Integration of information systems in the design of an integrated logistics platform

Each logistics information system has its own storage. This is due to the fact that the companies that form the supply chain are independent and may have different legal statuses, legal documents, etc. But in order to ensure the smooth and adequate operation of the platform when making decisions, it is necessary that the data and applications of one system be recognizable in another system. Therefore, integration is needed at several levels. The main goal of the study is to show possible ways of creating integrations starting from the system design stage. Integration theory is a complex task, so it needs to be comprehensively considered. All integration processes are reduced to data integration and software integration. The analysis of integration methods is carried out, as well as factors that negatively affect the integration possibilities are considered. Solutions of integration problems, to ensure the optimization of the design process (reduction of the design time) and design of the system itself (minimization of the content of the system), and during the operation of systems, optimization of its functioning (behavior) of the system. As a result of the study, the need to create data integration and application integration that allows you to structure data is shown. It should be noted that data can be located at all levels of the system architecture. And the system software can be: software modules, applications and systems. The basis of all types of system software, i.e. applications and an integral system are made up of software modules, design and development of software (i.e. based on programming systems) is based on service-oriented technologies, where the basis of systems is software services. The research carried out can be used in the development of information systems.

Keywords: information system, platform, integration, services, data.

Р.К.Ускенбаева, А.К. Болшибаева*, С.Б.Рахметулаева
Халықаралық Ақпараттық Технологиялар Университеті, Қазақстан, Алматы қ.
*e-mail: kakim-aigerim@mail.ru

Кешенді логистикалық платформаны жобалау кезінде ақпараттық жүйелерді біріктіру

Әрбір логистикалық ақпараттық жүйеніңөзіндік қоймасы бар. Бұл логикалық тізбетіқұрайтын компаниялардың тәуелсіз болуына және де түрлізаңды мәртебелері, заңды құжаттары және т.б. барына байланысты болып келеді. Шешімдерқабылдаған кезде платформаның бірқалыпты және адекватты жұмысын қамтамасыз ету үшін, бір жүйенің мәліметтері мен қосымшалары екінші жүйеде танымды болуы қажет. Сондықтан интеграцияны бірнеше деңгейде жасау қажет. Зерттеудіңнегізгі мақсаты - жүйені жобалау кезеңінен бастап интеграцияны құрудың барлық мүмкін жолдарын көрсету. Интеграция теориясы күрделііс, сондықтан оны жан-жақтықарастыруқажет. Барлық интеграциялық процестер мәліметтердің және бағдарламалардыңинтергациясынаәкеліп соқтырады. Интеграциялау әдістерін талдау жүргізіледі, сонымен қатар интеграциялық мүмкіндіктерге теріс әсерететін факторлар қарастырылады. Жобалау процесін (жобалау уақытынқысқарту) және жүйеніңөзін жобалауды (жүйеніңмазмұнын минимизациялау) оңтайландыруды қамтамасызетету интеграциялық мәселелердің шешімдері болып табылады, ал жүйелер жұмыс істеген кезде оның жұмысын (мінез-құлқын) оңтайландыру болып табылады. Зерттеу нәтижесінде, деректердіқұрылымдауға мүмкіндік беретін мәліметтер интеграциясын және қосымшалардың интеграциясын құру қажеттілігі көрсетілген. Деректер жүйеніңархитектурасының барлық деңгейлерінде орналасуы мүмкін екенін ескеру қажет.

Ал жүйенің бағдарламалық жасақтамасы ретінде: бағдарламалық модульдер, қосымшалар және жүйелер болуы мүмкін. Жүйенің бағдарламалық жасақтаманың барлық түрлерініңнегізі, яғни. қосымшалар мен ажырамас жүйе бағдарламалық модульдерінен тұрады, бағдарламалық жасақтаманы жобалау жәнеәзірлеу (яғни бағдарламалау жүйелеріне негізделген) қызмет көрсетуге бағытталған технологияларғанегізделген, мұнда жүйелердіңнегізі бағдарламалыққызмет көрсетуге негізделген. Жүргізілген зерттеулерді ақпараттық жүйелерді дамытуда қолдануға болады.

Кілтік сөздер: ақпараттық жүйе, платформа, интеграция, қызметтер, деректер.

Р.К.Ускенбаева, А.К. Болшибаева*, С.Б.Рахметулаева
Международный университет информационных технологий, Казахстан, г. Алматы
*e-mail: kakim-aigerim@mail.ru

Интеграция информационных систем при проектировании интегрированной платформы логистики

Каждая информационная система логистики имеет свое хранилище. Это связано с тем, что компании, которые образуют логистическую цепочку являются независимыми и могут иметь разные юридические статусы, правовые документы и т.д. Но для того чтобы обеспечить бесперебойную и адекватную работу платформы при принятии решений, необходимо чтобы данные и приложения одной системы были узнаваемы в другой системе. Поэтому необходима интеграция на нескольких уровнях. Основная цель исследования - показать возможные пути создания интеграций начиная с этапа проектирования систем. Теория интеграции является сложной задачей, поэтому ее надо всесторонне рассмотреть. Все интеграционные процессы сводится к интеграции данных и интеграции программ. Осуществлен анализ методов интеграции, а также рассмотрены факторы, негативно влияющие на интеграционные возможности. Решения задач интеграции, обеспечить оптимизацию процесса проектирования (сокращения продолжительность времени проектирования) и проектирование самой системы (минимизация содержание системы), а при эксплуатации систем оптимизация ее функционирования (поведения) системы. В результате проведенного исследования показана необходимость в создании интеграции данных и интеграции приложений, позволяющие структурировать данные. Следует отметить, что данные могут располагаться на всех уровнях архитектуры системы. А программным обеспечением системы может выступить: программные модули, приложения и системы. Основу всех видов ПО системы, т.е. приложений и целостной системы составляют программные модули, проектирование и разработка программного обеспечения (т.е. на основе программирования систем) ведется на основе сервис-ориентированной технологий, где основу систем составляют программные сервисы. Проведенные исследования могут быть использованы при разработке информационных систем.
Ключевые слова: Информационная система, платформа, интеграция, сервисы, данные.

1 Introduction

Companies continue to accumulate disparate integration solutions, while, according to one of the recent studies, it is the presence of many tools that negatively affects the implementation of integration projects. The lack of system in the approach to integration, the use of many tools, often with overlapping functionality, the lack of a clear understanding, leads to the fact that the next integration problem is solved for a long time and with excessive costs, and the maintenance of the integration results turns out to be too complicated and expensive.

Companies are showing increasing interest in methods and tools for managing business processes (business process management, BPM), combining disparate operations in various departments in an end-to-end successful business process - more than the basis for a strategic approach to the integration of corporate applications. On the other hand, the development of BPM technologies, in particular the BPMN and BPEL languages for describing and executing business processes, has resulted in the emergence of a large number of proposals for business processes on the market. Forrester analysts believe that the best candidates for an integrated integration system for modern enterprises are integration-oriented business process management systems [1].

But all of these methods and technologies share common limitations - they all do not take into account the semantics of data and applications when integrating.

Based on their created packages of programs and systems that provide data integration of applications and systems. Integration of systems in most cases is a forced measure aimed at improving the efficiency of business processes in which information systems are used. The article discusses the main approaches to the integration of information systems, the proposed methods for solving various problems.

In automated business logistics, a problem arises for the following reasons:

First, the systems business processes are based on web services technologies, therefore, integration between services is necessary; With service-oriented systems design technology, there is a need to integrate software and data from software modules, software applications, software systems, or ecosystems.

Secondly, the supply chain can be implemented by various economic entities with different legal statuses and organizational structure, management management systems. Therefore, for a system of work between them, integration is required.

Thirdly, during the execution of logistic chains or logistic processes, which are fragments of the logistic supply chain of goods, the installation of data and procedures integration is required.

2 Service-oriented integration

SOA is by far the most sophisticated approach to application integration. The goal of this paradigm is to break down the logical functionality of a software system into smaller logical units, also called services.

The central concepts of SOA integration are "service"and "process". A service is a function that is well-defined, self-contained, and independent of the context or state of other services. Services have the following characteristics:

- Reusable custom encapsulation of a recurring business task that hides implementation details from the service interface.

- They are the building blocks for business processes.

- Can be linked with other services to encapsulate more important business functions, working in the context of specific business needs.

A process (business process) is defined as the logic of their interaction, independent of the implementation of services.

Allocating services based on application functionality only makes sense if they can be used repeatedly and in different contexts. It is customary to distinguish a supplier, a service consumer, and a component that ensures interaction between the supplier and the consumer (the so-called broker). Business processes are encapsulated from the service virtualization layer through direct interaction with application functions. Business processes are connected by connecting services with varying granularity. Together they represent an end-to-end implementation. The flexibility of business processes is a consequence, since one service can be changed to a more suitable implementation, without any impact on the consumer of business processes [3].

In the SOA approach, the information system is divided into several nodes - services that can interact with each other. This is an SOA integration scenario, i.e. coordination of information and data transmission formats.

To solve such a problem, it is necessary first of all to determine how the services are connected. You can connect to each other directly, but then you get a large number of links that will be difficult to use when expanding the platform, and the conversion of the data format when moving from one service to another must be taken into account. Therefore, with such a problem, a BPEL server can be used, which acts as a service bus, it unifies the data format, as well as controls the exchange of data between services [4].

2.1 The basics of building software for a business process automation system based on BPEL

As part of the work, the model of service of goods is considered, which describes the main stages of receipt and shipment of goods. The block diagrams describing the algorithm are shown in Figure 1-2.

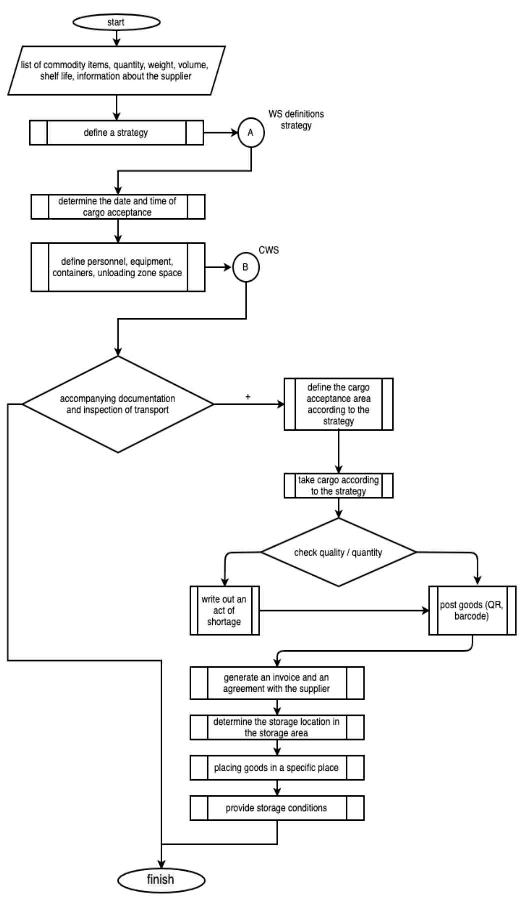


Figure 1: Algorithm for executing the cargo service model (goods receipt)

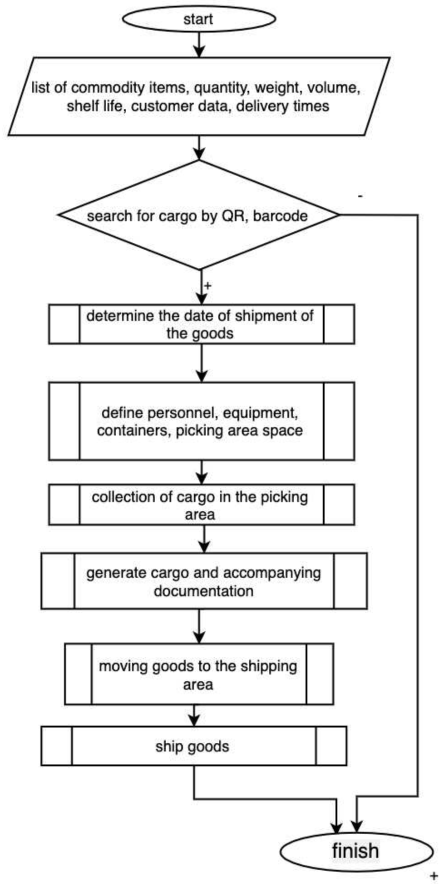


Figure 2: Algorithm for executing the cargo service model (packaging and shipment of goods)

Each task in the presented flowcharts is performed using the BPEL language:

- waiting for the arrival of a new application or the previous process (i.e. waiting for the result of the previous task). To accomplish this goal, BPEL must use Receive semantics;

- initialization of the passed variables to prepare for calling the web service. In this case, data type conversion is possible. BPEL supports Assign semantics to accomplish this goal;

- calling a web service, which is available as a WSDL, to perform a specific task, for example, "determine the date of receipt of the goods."BPEL supports Invoke semantics to accomplish this goal;

- getting the execution result of the called web service to prepare for returning or transferring the result to the next process. To accomplish this, you also need to use the Assign semantics in BPEL;

- return of the received data when the current task is performed next. To accomplish this, you need to use the Reply semantics in BPEL.

In addition to the semantics discussed above, the BPEL language supports a whole set of other semantics, the use of which allows you to build an information model of system management.

Let's model the workflow using BPEL. Here, a typical model of cargo acceptance is considered, in which there are main stages from the moment the application is submitted to the placement of the cargo at the storage address. Figure 3 shows the process for filing a storage claim.

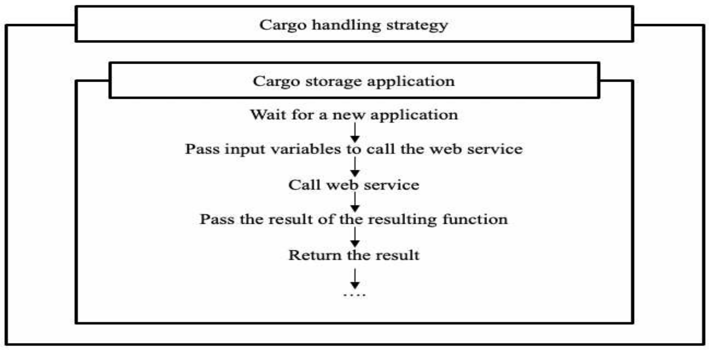


Figure 3: BPEL description of the cargo storage claim process)

Now we will give a description of the same BPEL process in code form:

<process name="Zayavka≫

The process element is followed by partnerLinks, which define other services or processes with which this process interacts. In this case, the only affiliate link automatically generated by the wizard is the link for the client interface of that BPEL process

<partnerLinks> //
<partnerLink name = "client"partnerLinkType = "tns: Client-Admin"
myRole = "Admin"/>
<partnerLink name = "reservation"partnerLinkType = "tns: Admin-reservation"
partnerRole = "zayavkaIssuer"/>
</partnerLinks>

PartnerLinks is followed by global variable definitions that are available throughout the BPEL process. The types of these variables are defined in the WSDL for the process itself.

<variables>
<variable name="purchaseRequest"messageType="tns:purchaseRequest"/>
<variable name="cost"type="xsd:double"/>
<variable name="items"type="tns:ItemSet"/>
<variable name="cancelRequest"messageType="tns:cancelRequest"/>
<variable name="cancelResponse"messageType="tns:cancelResponse"/>
<variable name="detailRequest"messageType="tns:detailRequest"/>
<variable name="detailResponse"messageType="tns:detailResponse"/>
<variable name="dateReached"type="xsd:boolean"/>
</variables>

1. The administrator receives data from the client using a web application;

2. The administrator sends the data to the service. As a result, a new request sends the entered data to the execution server of BPEL processes (which is visible from the outside as a set of web services) and thus generates a new instance of the "request for cargo acceptance"process;

3. The BPEL process calls an external web service that performs the "create a new ticket"task. The function of this service takes input from the request in step 2) and passes the result to the next process;

4. formation of a queue, which includes information about the cargo, the date of receipt, special processes of certain strategies (selection of mechanisms, personnel, etc.)

<bpel:sequence name="strategy≫

<bpel:sequence name="date≫

<bpel:receive name-="new request "partnerLink="client"/>

<bpel:assign name "variables≫

</bpel:assign>

<bpel:invoke name="Calling an external function with parameters≫

</bpel:invoke> <bpel:assign validate="no"name="Getting the result of a function ≫

</bpel:assign>

<bpel:reply name="Returning the result"/>

</bpel:sequence>

</bpel:sequence>

</bpel:process>

The process must also match requests to each other. For example, we can assign a unique ID to each quote and final proposal. BPEL documents these identifiers using the <correlationSet> tag:

A key part of the BPEL document defines the steps required to process a request. The <sequence> tag performs actions sequentially; the <flow> tag runs them in parallel; and the <receive>, <reply>, and <invoke> tags define the basic steps required to interact with web services using WSDL.

The sequence begins after receiving the buyer's request. The <flow> tag takes a parallel set of steps to contact each supplier for a quote. Each action refers to a specific WSDL operation and uses the available variables for input and output. After receiving responses from the supplier, the purchasing agent composes a message to respond to the buyer. An administrator can use the BPEL <assign> tag and W3C XPath to target portions of an XML document to take vendor containers and finalize a proposal back to the customer [5].

The last step is to manage the exceptions in the script. For example, if an error occurs while contacting a supplier, the agent might want to send a message to the customer. BPEL includes error handlers for these error conditions.

3 Material and methods

In this section, the relevance of the distribution of the execution of the operation on the cargo and the web service performing the operation for the subsequent automation of important decision-making processes will be considered.

In the developed platform, the method of decision-making under uncertainty was applied and an attempt was made to model a decision-making system based on common sense.

The problem of combining general consideration and logical reasoning was resolved with the advent of the theory of fuzzy sets, proposed by L. Zadeh, a professor at the University of Berkeley (California, USA) [6-7], in the 60s of the last century. The theory of fuzzy sets made it possible to operate with a mathematically fuzzy representation of concepts that have qualitative and subjective characteristics.

With the help of a fuzzy model, the problem of cargo distribution among web service operations was solved. Obviously, the decision-maker must distribute the load taking into account such characteristics of operations as the speed of the operation, reliability, flexibility of the operation, etc.

Given:

X=$(x_1, x_2, x_3, ..., x_n)$ is a set of loads arriving at a certain moment of time;

Y=$(y_1, y_2, y_3, ..., y_p)$ a set of features that characterize the operation;

Z=$(z_1, z_2, z_3, ..., z_m)$ a set of web services performing the operation.

It is required to distribute the load among web services in an optimal way, taking into account the properties of the operation for this particular load.

Of course, for each group of cargoes, it is advisable to select its own set of features, and here it is obvious that each cargo will have a feature to some extent.

In the work, the main features of the operation are considered, 4 features have been identified that are important for any type of cargo:

- quick response to the order;
- the reliability of the operation;
- the speed of the operation;
- flexibility of the operation;
- execution of the operation requiring special conditions.

All signs of the operation were assessed by experts of DRAGON SYSTEM LLP engaged in international logistics and experts of BK Logistics LLP engaged in warehousing and storage of non-food products. An expert assessment is necessary, since in many cases the decision maker does not have the full amount of data and acts in conditions of inaccurate information. The experts were asked to evaluate the cargo according to these characteristics, i.e. if, for example, we take the sign "speed of operation it is easy to determine which of the goods is perishable and requires quick execution of the operation, and which cargo can be processed in a longer time frame. This allows experts to give an expert assessment of the value of the membership function of a particular cargo to a set of perishable goods: if the cargo is obviously perishable, then the value of the membership function will be close to 1, if, on the contrary, the value of the membership function will be close to 0.

According to the criterion "speed of response to an order an order with high danger (toxicity) and food orders must be processed quickly, that is, if there is a cargo with such characteristics, the expert assigns the order a maximum value of 1;

According to the criterion "reliability of the operation the goods having the criterion of fragile, dangerous, toxic and other characteristics - the maximum value of the membership function 1;

According to the criterion "speed of operation"for perishable goods, the maximum value is 1;

According to the criterion "flexibility of the operation"if the operation when working with the cargo can be rebuilt, stopped without loss, etc. then the value is 0;

According to the criterion "execution of an operation requiring special conditions"goods with a large dimension, or large weight, or special conditions of transportation.

Thus, at the first stage, experts evaluated all 5 features and a formalized condition of the problem was obtained.

We have two sets R and S.

Let r: $X \times Y \rightarrow [0,1]$ be the membership function of the odd binary relation R, which is set by the experts. The physical meaning of this function is to what extent the attribute yj. corresponds to the load x_i.

As a result, we get a representation of the ratio R in matrix form:

$$R = \begin{array}{c} x_1 \\ x_2 \\ \dots \\ x_n \end{array} \begin{bmatrix} r\,(x_1,y_1) & r\,(x_1,y_2) & \dots & r\,(x_1,y_p) \\ r\,(x_2,y_1) & r\,(x_2,y_2) & \dots & r\,(x_2,y_p) \\ \dots & \dots & \dots & \dots \\ r\,(x_n,y_1) & r\,(x_n,y_2) & \dots & r\,(x_n,y_p) \end{bmatrix}$$

Table 1: Fuzzy ratio R

	quick response to the order	the reliability of the operation	the speed of the operation	flexibility of the operation	execution of the operation requiring special conditions
Item 1: solid, light weight, small size, household, not hazardous. No special conditions	0,3	0,1	0,3	0,1	0,1
Item 2: solid, heavy weight, oversized, manufacturing, non-hazardous, special conditions	0,1	0,5	0,1	0,9	0,9
Commodity 3: solid, light weight, small size, food, not dangerous, No special conditions	0,9	0,1	0,9	0,5	0,9
Commodity 4: solid, regular, small size, manufacturing, hazardous, special conditions	0,5	0,9	0,5	0,4	0,8
Item 5: Liquid, Regular, Dimension, Household, Hazardous, Special Conditions	0,7	0,9	0,7	0,5	0,8
Item 6: liquid, regular, small size, food, not hazardous, No special conditions	0,9	0,3	0,9	0,5	0,3
Item 7: Liquid, Heavyweight, Oversized, Household, Hazardous, Special Conditions	0,7	0,5	0,7	0,8	0,8
Item 8: gas, light weight, large size, household, hazardous, special conditions	0,7	0,9	0,7	0,7	0,8
Item 9: gas, light weight, dimension, household, not dangerous, No special conditions	0,5	0,7	0,5	0,4	0,5
Item 10: Gas, Regular, Dimension, Household, Hazardous, Special Conditions	0,7	0,9	0,7	0,4	0,8

We carry out the same procedure with S fuzzy binary relation.

Let s: $Y \times \ Z \rightarrow [0,1]$ be the membership function of a fuzzy binary relation S. Is equal to the degree of importance from the attribute y_i for the web service z_j. In matrix form, this relationship is:

$$S = \begin{array}{c} y_1 \\ y_2 \\ \cdots \\ y_n \end{array} \begin{bmatrix} s\left(y_1, z_1\right) & s\left(y_1, z_2\right) & \cdots & s\left(y_1, z_m\right) \\ s\left(y_2, z_1\right) & s\left(y_2, z_2\right) & \cdots & s\left(y_2, z_m\right) \\ \cdots & \cdots & \cdots & \cdots \\ s\left(y_p, z_1\right) & s\left(y_p, z_2\right) & \cdots & s\left(y_p, z_m\right) \end{bmatrix}$$

Table 2: Fuzzy ratio S

	WS1	WS2	WS3	WS4	WS5	WS6	WS7	WS8	WS9	WS10
quick response to the order	0,9	0,9	0,9	0,9	0,7	0,5	0,7	0,3	0,5	0,3
the reliability of the operation	0,9	0,7	0,5	0,3	0,9	0,9	0,9	0,7	0,3	0,5
the speed of the operation	0,7	0,9	0,3	0,5	0,9	0,7	0,5	0,9	0,9	0,3
flexibility of the operation	0,5	0,3	0,9	0,7	0,5	0,9	0,3	0,9	0,7	0,9
execution of the operation requiring special conditions	0,3	0,5	0,7	0,9	0,3	0,3	0,9	0,5	0,9	0,9

The matrix of fuzzy relations R:

$$R = \begin{bmatrix} 0,3 & 0,1 & 0,3 & 0,1 & 0,1 \\ 0,1 & 0,5 & 0,1 & 0,9 & 0,9 \\ 0,9 & 0,1 & 0,9 & 0,5 & 0,9 \\ 0,5 & 0,9 & 0,5 & 0,4 & 0,8 \\ 0,7 & 0,9 & 0,7 & 0,5 & 0,8 \\ 0,9 & 0,3 & 0,9 & 0,5 & 0,3 \\ 0,7 & 0,5 & 0,7 & 0,8 & 0,8 \\ 0,7 & 0,9 & 0,7 & 0,7 & 0,8 \\ 0,5 & 0,7 & 0,5 & 0,4 & 0,5 \\ 0,7 & 0,9 & 0,7 & 0,4 & 0,8 \end{bmatrix}$$

The matrix of fuzzy relations S:

$$S = \begin{bmatrix} 0,9 & 0,9 & 0,9 & 0,9 & 0,7 & 0,5 & 0,7 & 0,3 & 0,5 & 0,3 \\ 0,9 & 0,7 & 0,5 & 0,3 & 0,9 & 0,9 & 0,9 & 0,7 & 0,3 & 0,5 \\ 0,7 & 0,9 & 0,3 & 0,5 & 0,9 & 0,7 & 0,5 & 0,9 & 0,9 & 0,3 \\ 0,5 & 0,3 & 0,9 & 0,7 & 0,5 & 0,9 & 0,3 & 0,9 & 0,7 & 0,9 \\ 0,3 & 0,5 & 0,7 & 0,9 & 0,3 & 0,3 & 0,9 & 0,5 & 0,9 & 0,9 \end{bmatrix}$$

Further, from the matrices R and S, we obtain the matrix T

$$T = \begin{bmatrix} t(x_1, z_1) & t(x_1, z_2) & \dots & t(x_1, z_m) \\ t(x_2, z_1) & t(x_2, z_2) & \dots & t(x_2, z_m) \\ \dots & \dots & \dots & \dots \\ t(x_n, z_1) & t(x_n, z_2) & \dots & t(x_n, z_m) \end{bmatrix}$$

where each element of the matrix is calculated by the formula:

$$t(x, z_i) = \frac{\sum_y r(x, y) \bullet s(y, z_i)}{\sum_y r(x, y)}$$

thus the matrix T has the form:

$$T = \begin{bmatrix} 0,72 & 0,77 & 0,63 & 0,68 & 0,72 & 0,63 & 0,63 & 0,63 & 0,68 & 0,46 \\ 0,53 & 0,50 & 0,72 & 0,69 & 0,53 & 0,66 & 0,66 & 0,69 & 0,69 & 0,77 \\ 0,62 & 0,69 & 0,67 & 0,74 & 0,62 & 0,57 & 0,65 & 0,62 & 0,74 & 0,56 \\ 0,66 & 0,66 & 0,64 & 0,64 & 0,66 & 0,65 & 0,73 & 0,64 & 0,64 & 0,59 \\ 0,67 & 0,68 & 0,64 & 0,64 & 0,67 & 0,65 & 0,70 & 0,64 & 0,64 & 0,57 \\ 0,71 & 0,73 & 0,65 & 0,68 & 0,71 & 0,65 & 0,61 & 0,65 & 0,68 & 0,49 \\ 0,63 & 0,64 & 0,68 & 0,69 & 0,63 & 0,64 & 0,64 & 0,66 & 0,69 & 0,60 \\ 0,66 & 0,66 & 0,65 & 0,65 & 0,66 & 0,66 & 0,68 & 0,66 & 0,65 & 0,58 \\ 0,68 & 0,68 & 0,64 & 0,63 & 0,68 & 0,67 & 0,69 & 0,65 & 0,63 & 0,56 \\ 0,68 & 0,69 & 0,63 & 0,64 & 0,68 & 0,64 & 0,71 & 0,64 & 0,64 & 0,56 \end{bmatrix}$$

Next, you need to set a threshold number in order to determine the set of products, the operation of which can be performed using the web service, according to the formula:

$$l = \min_{(i,j)} \underset{x}{max} \min\left(t(x, z_i), t(x, z_j)\right)$$

The first step was to compose a matrix of pairwise minima:

$$Matrixofpairwiseminima = \begin{matrix} 0,722 & 0,633 & 0,633 & 0,678 & 0,633 & 0,633 & 0,633 & 0,633 & 0,456 \\ 0,500 & 0,500 & 0,692 & 0,532 & 0,532 & 0,660 & 0,660 & 0,692 & 0,692 \\ 0,621 & 0,670 & 0,670 & 0,621 & 0,573 & 0,573 & 0,621 & 0,621 & 0,561 \\ 0,661 & 0,635 & 0,635 & 0,635 & 0,648 & 0,648 & 0,642 & 0,635 & 0,590 \\ 0,672 & 0,639 & 0,639 & 0,644 & 0,650 & 0,650 & 0,644 & 0,644 & 0,567 \\ 0,707 & 0,652 & 0,652 & 0,679 & 0,652 & 0,610 & 0,610 & 0,652 & 0,486 \\ 0,631 & 0,639 & 0,677 & 0,631 & 0,631 & 0,643 & 0,643 & 0,660 & 0,603 \\ 0,658 & 0,653 & 0,647 & 0,647 & 0,663 & 0,663 & 0,658 & 0,647 & 0,584 \\ 0,677 & 0,638 & 0,631 & 0,631 & 0,669 & 0,669 & 0,654 & 0,631 & 0,562 \\ 0,677 & 0,631 & 0,631 & 0,643 & 0,643 & 0,643 & 0,637 & 0,637 & 0,557 \end{matrix}$$

We calculate the maximum element of each column: 0,722 0,670 0,692 0,679 0,669 0,669 0,660 0,692 0,692

Among the maximum elements, we find the minimum, equal to 0.66. In the matrix T we find a number slightly less than 0.66, it is equal to 0.658, which is a threshold number.

Next, we modify the matrix, and if the element $t\left(x_i,\ z_j\right)$ is greater than or equal to the threshold number, then the product xi is included in the set M_j.

As a result, we get a modified matrix T'.

$$T' = \begin{bmatrix} 0,722 & 0,767 & 0 & 0,678 & 0,722 & 0 & 0 & 0 & 0,678 & 0 \\ 0 & 0 & 0,724 & 0,692 & 0 & 0,66 & 0,66 & 0,692 & 0,692 & 0,772 \\ 0 & 0,694 & 0,67 & 0,742 & 0 & 0 & 0 & 0 & 0,742 & 0 \\ 0,661 & 0,661 & 0 & 0 & 0,661 & 0 & 0,726 & 0 & 0 & 0 \\ 0,672 & 0,678 & 0 & 0 & 0,672 & 0 & 0,70 & 0 & 0 & 0 \\ 0,707 & 0,734 & 0 & 0,679 & 0,707 & 0 & 0 & 0 & 0,679 & 0 \\ 0 & 0, & 0,677 & 0,689 & 0 & 0 & 0 & 0,66 & 0,689 & 0 \\ 0,663 & 0,658 & 0 & 0 & 0,663 & 0,663 & 0,679 & 0,658 & 0 & 0 \\ 0,685 & 0,677 & 0 & 0 & 0,685 & 0,669 & 0,692 & 0 & 0 & 0 \\ 0,677 & 0,689 & 0 & 0 & 0,677 & 0 & 0,711 & 0 & 0 & 0 \end{bmatrix}$$

4 Results and discussion

We represent matrix T' in the form of a table 3.

Table 3: Modified fuzzy ratio T'

Thus, the distribution of the operation execution for different categories of goods by web services was considered, which have the characteristics of the operations that they can perform.

As can be seen from matrix T', the distribution of product categories by web services was obtained, taking into account the priority of the product.

If a service is busy performing an operation of one product, another product with the same characteristics can implement another service. When evaluating the services, the "preferences"of the web service were taken into account. Therefore, each web service works only with the product that is present in its set of preferred products.

4.1 Integration solution for logistics system applications

Apache Camel, which implements Enterprise Integration Patterns, is used as a basis for integrating applications within the system. The advantages of using this framework are as follows:

1. Open source framework
2. Implemented EIPs
3. Ability to use different transport (TCP, UDP, HTTP, etc.) and API
4. Lightweight routing description language, based on Java DSL, and it is also possible to use Spring XML, Scala, PHP and other languages.

If you need to integrate with web services, the SOA architecture is used in conjunction with the Apache CXF framework, which also simplifies application development. At the moment, it has a huge number, 328, plug-in components for interacting with third-party systems

For a more detailed application, it is necessary to understand the architecture of the framework itself. The following figure shows the main components of the framework:

	WS1	WS2	WS3	WS4	WS5	WS6	WS7	WS8	WS9	WS10
Item 1: solid, light weight, small size, household, not hazardous. No special conditions	0,722	0,767	-	0,678	0,722	-	-	-	0,678	-
Item 2: solid, heavy weight, oversized, manufacturing, non-hazardous, special conditions	-	-	0,724	0,692	-	0,660	0,660	0,692	0,692	0,772
Commodity 3: solid, light weight, small size, food, not dangerous, No special conditions	-	0,694	0,670	0,742	-	-	-	-	0,742	-
Commodity 4: solid, regular, small size, manufacturing, hazardous, special conditions	0,661	0,661	-	-	0,661	-	0,726	-	-	-
Item 5: Liquid, Regular, Dimension, Household, Hazardous, Special Conditions	0,672	0,678	-	-	0,672	-	0,700	-	-	-
Item 6: liquid, regular, small size, food, not hazardous, No special conditions	0,707	0,734	-	0,679	0,707	-	-	-	0,679	-
Item 7: Liquid, Heavyweight, Oversized, Household, Hazardous, Special Conditions	-	-	0,677	0,689	-	-	-	0,660	0,689	-
Item 8: gas, light weight, large size, household, hazardous, special conditions	0,663	0,658	-	-	0,663	0,663	0,679	0,658	-	-
Item 9: gas, light weight, dimension, household, not dangerous, No special conditions	0,685	0,677	-	-	0,685	0,669	0,692	-	-	-
Item 10: Gas, Regular, Dimension, Household, Hazardous, Special Conditions	0,677	0,689	-	-	0,677	-	0,711	-	-	-
Lots of products to be distributed across web services	T1, T6, T9, T10, T5, T8, T4	T1, T6, T3, T10, T5, T9, T4, T8	T2, T7, T3	T3, T2, T7, T6, T1	T1, T6, T9, T10, T5, T8, T4	T9, T8, T2	T4, T10, T5, T9, T8, T2	T2, T7, T8	T3, T2, T7, T6, T2	T2

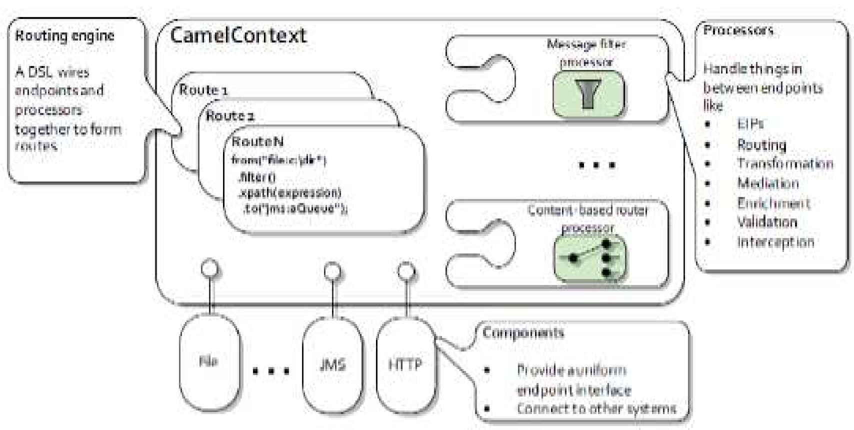


Figure 4: Components of the framework

1. CamelContext - context of code execution
2. Routing engine - message routing engine
3. Processors - message handlers

4. Components - components of interaction with third-party systems

The exchange of messages shown in Figure 5 is carried out between systems using "neutral"endpoints, described using a message endpoint (URI).

An example of an endpoint description (Figure 6):

1. Schema - determines the type of transport (in this case, the file will be read)
2. Context path - context address (data / inbox directory)
3. Options - additional option (delay of 5000 ms, update interval is determined)

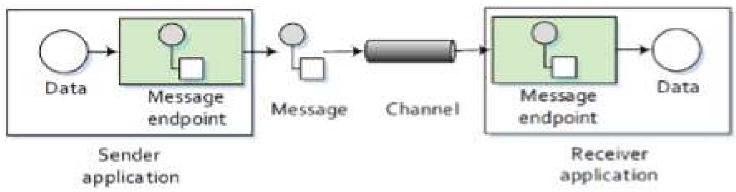


Figure 5: Messaging between applications

Figure 6: Endpoint description

5 Conclusion

To solve a common task - the task of automating logistics processes, it is required to perform various integration tasks. The need for integration arises both at the level of data and software modules (within models and intermodular relations) and at the level of the logistics system as a whole, i.e. to harmonize the robots of the logistics system with other systems, for example, ERP systems, CPM systems, enterprise PLM systems, etc. This factor requires that the system be integration-oriented or integration-adapted, i.e. adapted to integration processes.

Thus, the system should be designed on the basis of integration-oriented design technologies, and also, which is no less important, the operation of the system should be carried out on the basis of technology oriented to the integration process, while integrating data, software modules and applications [8-9].

Integration of data and software modules during design will occur in different cases and in different parts of the system. In other words, it can be argued that the integration process occurs when the logistics processes are performed in various places / tasks and subtasks or operations. To complete the integration process, it is required to build a system based on certain requirements, i.e. the system must have a number of conditions.

1. the system should be built on the basis of POSIX principles.

2. the system must be interoperable, i.e. interoperability, interfaces must be fully open, interact and function with other products or systems without any restrictions on access and implementation.

Various integration processes are carried out in places of origin in the procedural composition of the corresponding system, or by collecting a set of separate procedures that locally solve only one integration process as part of a single system, which we will call a system or a subsystem, or an integration bus [10].

References

[1] Dergachev A.A., Safronov A.G. Integratsiya veb-servisov na osnove ontologii predmetnoI oblasti // Sbornik trudov molodykh uchenykh i sotrudnikov kafedry VT, Vypusk 4. / Pod red. d.t.n., prof. T.I. Alieva. SPb: SPb NIU ITMO, 2013. S. 14-17

[2] Morozova O.A. Integratsiya korporativnykh informatsionnykh sistem: M80 uchebnoe posobie. M.: FinansovyI universitet, 2014. 140 s

[3] Khop, G Shablony integratsii korporativnykh prilozheniI / Gregor Khop, Bobbi Vul'f; per. s angl. M.: Izdatel'skiI dom "Vil'yams 2007. 672 s.

[4] Gabriel Brown Service Chaining in Carrier Networks, https://www.qosmos.com/wp-content/uploads/Service-Chaining-in-Carrier-NetworksWPHeavy-ReadingQosmosFeb2015.pdf

[5] https://www.osp.ru/os/2009/10/11171290

[6] https://cyberleninka.ru/article/n/primenenie-nechetkoy-kompozitsii-dlya-modelirovaniya-logisticheskih-potokov

[7] Dergachev A.A., Chernyak EH.A. Vybor kompozitsii veb-servisov na osnove otsenki parametrov kachestva obsluzhivaniya// Sbornik tezisov dokladov kongressa molodykh uchenykh, Vypusk 1. SPb: NIU ITMO, 2014. S. 57

[8] Akho A. Struktury dannykh i algoritmy. / A. Akho, D. Khopkroft, D.Ul'man; per. s angl. M.: Izdatel'skiI dom "Vil'yams 2010. 400 s

[9] Baboshin, A.A. Podkhod k organizatsii vzaimodeIstviya veb-servisov na osnove modeli potoka rabot / A.A. Baboshin, A.M. Kashevnik // Trudy SPIIRAN, vyp. 5 SPb.: Nauka, 2007. ss. 247-254

[10] Dergachev, A.M. Problemy ehffektivnogo ispol'zovaniya setevykh servisov // Nauchno-tekhnicheskiI vestnik SPBGU ITMO. 2011. No 1 (71). S. 83-87

Литература

[1] Дергачев А.А., Сафронов А.Г. Интеграция веб-сервисов на основе онтологии предметной области // Сборник трудов молодых ученых и сотрудников кафедры ВТ, Выпуск 4. / Под ред. д.т.н., проф. Т.И. Алиева. - СПб: СПб НИУ ИТМО, 2013. - С. 14-17.

[2] Морозова О.А. Интеграция корпоративных информационных систем: М80 учебное пособие. - М.: Финансовый университет, 2014. - 140 с.

[3] Хоп, Г. Шаблоны интеграции корпоративных приложений / Грегор Хоп, Бобби Вульф; пер. с англ. - М.: Издательский дом "Вильямс 2007. - 672 с.

[4] Gabriel Brown Service Chaining in Carrier Networks, https://www.qosmos.com/wp-content/uploads/Service-Chaining-in-Carrier-NetworksWPHeavy-ReadingQosmosFeb2015.pdf

[5] https://www.osp.ru/os/2009/10/11171290

[6] https://cyberleninka.ru/article/n/primenenie-nechetkoy-kompozitsii-dlya-modelirovaniya-logisticheskih-potokov

[7] Дергачев А.А., Черняк Э.А. РВыбор композиции веб-сервисов на основе оценки параметров качества обслуживания /Сборник тезисов докладов конгресса молодых ученых, Выпуск 1. - СПб: НИУ ИТМО, 2014. - С. 57.

[8] Ахо А. Структуры данных и алгоритмы/ А. Ахо, Д. Хопкрофт, Д.Ульман; пер. с англ. - М.: Издательский дом "Вильямс 2010. - 400 с.

[9] Бабошин, А.А. Подход к организации взаимодействия веб-сервисов на основе модели потока работ/ А.А. Бабошин, А.М. Кашевник // Труды СПИИРАН, вып. 5 - СПб.: Наука, 2007. сс. 247-254.

[10] Дергачев, А.М. Проблемы эффективного использования сетевых сервисов // Научно-техническиий вестник СПбГУ ИТМО. 2011. No 1 (71). С. 83-87

4-бөлім **Раздел 4** **Section 4**

Қолданбалы математика **Прикладная математика** **Applied Mathematics**

IRSTI 27.41.19 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.10

N.B. Alimbekova[1,2*], N.M. Oskorbin[3]
[1]Abai Kazakh National Pedagogical University, Kazakhstan, Almaty
[2]S. Amanzholov East Kazakhstan University, Kazakhstan, Ust-Kamenogorsk
[3]Altai State University, Russia, Barnaul
*e-mail: nurlana1101@gmail.com

STUDY OF THE INITIAL BOUNDARY VALUE PROBLEM FOR A TWO-DIMENSIONAL CONVECTION-DIFFUSION EQUATION WITH A FRACTIONAL TIME DERIVATIVE IN THE SENSE OF CAPUTO-FABRIZIO

In this paper, we study an initial boundary value problem for a differential equation with a fractional order derivative in time in the Caputo-Fabrizio sense. This equation is of great practical importance in modeling the processes of fluid motion in porous media and anomalous dispersion. The uniqueness and continuous dependence of the solution of the problem on the input data in differential form is proved. A computationally efficient implicit difference scheme with weights is proposed. A priori estimates are obtained for the solution of the problem under the assumption that the solution exists in the class of sufficiently smooth functions. The uniqueness of the solution and the stability of the difference scheme with respect to the initial data and the right-hand side of the equation follows from the obtained estimates. The convergence of the difference problem solution to the differential problem solution with the second order in time and space variables is proved. The results of computational experiments confirming the reliability of the theoretical analysis are presented.

Key words: Fractional differential equation, fractional derivative in the sense of Caputo-Fabrizio, finite difference method, energy inequality method, stability, convergence, a priori estimate.

Н.Б. Алимбекова[1,2*], Н.М. Оскорбин[3]
[1]Абай атындағы Қазақ ұлттық педагогикалық университеті, Қазақстан, Алматы қ.
[2]С. Аманжолов атындағы Шығыс Қазақстан университеті, Қазақстан, Өскемен қ.
[3]Алтай мемлекеттік университеті, Ресей, Барнаул қ.
*e-mail: nurlana1101@gmail.com

Уақыт бойынша Капуто-Фабрицио мағынасындағы бөлшек туындысы бар екі өлшемді конвекция-диффузия теңдеуі үшін қойылған бастапқы шекаралық есепті зерттеу

Бұл жұмыста Капуто-Фабрицио мағынасындағы уақыт бойынша бөлшек ретті туындысы бар дифференциалдық теңдеу үшін қойылған бастапқы шекаралық есеп зерттелді. Бұл теңдеу фильтрация үрдістерін және аномалды дисперсияны модельдеуде үлкен қолданбалы мәнге ие. Есеп шешімінің жалғыздығы мен бастапқы берілген мәндерден тәуелділігі дифференциалдық формада дәлелденді. Есептеуге тиімді салмағы бар айқын емес айырымдық сұлба ұсынылды. Жеткілікті тегіс функциялар класында шешімі бар деген болжаммен есептің шешімі үшін априорлық бағалаулар алынды. Осы бағалаулардан шешімнің жалғыздығы және бастапқы берілген мәндер мен теңдеудің оң жағы бойынша айырымдық сұлбаның орнықтылығы шығады. Айырымдық есептің шешімінің дифференциалдық есептің шешіміне уақыт және кеңістіктік айнымалылары бойынша екінші ретпен жинақталуы дәлелденді. Теориялық талдаудың дұрыстығын растайтын есептеу тәжірибелерінің нәтижелері ұсынылды.

Түйін сөздер: Бөлшек ретті дифференциалдық теңдеу, Капуто-Фабрицио мағынасындағы бөлшек туынды, ақырлы айырымдар әдісі, энергиялық теңсіздіктер әдісі, орнықтылық, жинақтылық, априорлы бағалау.

Н.Б. Алимбекова[1,2*], Н.М. Оскорбин[3]
[1]Казахский Национальный педагогический университет им. Абая, Казахстан, г. Алматы
[2]Восточно-Казахстанский университет им. С. Аманжолова, Казахстан, г. Усть-Каменогорск
[3]Алтайский государственный университет, Россия, г. Барнаул
*e-mail: nurlana1101@gmail.com

Исследование начально-краевой задачи для двумерного уравнения конвекции-диффузии с дробной производной по времени в смысле Капуто-Фабрицио

В настоящей работе исследуется начально-краевая задача для дифференциального уравнения с производной дробного порядка по времени в смысле Капуто-Фабрицио. Данное уравнение имеет большую прикладную значимость при моделировании процессов фильтрации и аномальной дисперсии. Доказаны единственность и непрерывная зависимость решения задачи от входных данных в дифференциальной форме. Предложена вычислительно эффективная неявная разностная схема с весами. Получены априорные оценки для решения задачи в предположении существования решения в классе достаточно гладких функций. Из этих оценок следуют единственность решения и устойчивость разностной схемы по начальным данным и правой части уравнения. Доказана сходимость решения разностной задачи к решению дифференциальной задачи со вторым порядком по временной и пространственной переменным. Представлены результаты вычислительных экспериментов, подтверждающие достоверность теоретического анализа.

Ключевые слова: Дифференциальное уравнение дробного порядка, дробная производная в смысле Капуто-Фабрицио, метод конечных разностей, метод энергетических неравенств, устойчивость, сходимость, априорная оценка.

1 Introduction

Differential equations containing fractional derivatives have become popular because they are more suitable for modeling specific real-world problems than ordinary differential equations. Therefore, the development of analytical and numerical methods for the theory of fractional differential equations is an urgent and important problem. One of the important examples of applying this type of equations is the equations describing flows of a multiphase fluid in highly porous fractured formations with fractal well geometry.

In this paper, we obtain a priori estimates in differential form for this problem, which implies the uniqueness of the solution and its continuity from the input data. An implicit finite difference scheme of the second order of approximation in time and in a spatial variable is proposed. The stability of the proposed scheme as well as convergence with a speed equal to the approximation order is proved. The results obtained are confirmed by numerical calculations performed for two test problems.

2 Literature review

In recent years, the use of fractional order derivatives to construct mathematical models of various physical processes involving electrical circuits [1], thermal and diffusion processes [2,3], medicine [4,5], and other processes [6,7], as well as the development of numerical or analytical solutions for these fractional mathematical models are very relevant. Among them, the problems of fluid flows in porous media are of great interest, where their dynamics are significantly affected by memory effects, which are described by the theory of fractional-order integro-differentiation [8,9]. In the fluid flow problems, whose state and observation processes are controlled by the time-varying Brownian motion or the Levy process, the Riemann-Liouville fractional derivative was used for the Zakai equation [10]. In [9], several models were proposed to describe fluid flow processes in complex fractured porous media containing fractional Riemann–Liouville derivatives in time and space. For single-phase fluid flow, a nonlinear pressure equation containing fractional Riemann-Liouville derivatives with respect to time is obtained, a fractional differential modification of Darcy's law is proposed, and a fractional differential equation for anisotropic fluid flow is obtained. A fractional differential modification of the Barenblatt-Gilman model for nonequilibrium two-phase countercurrent capillary impregnation is also proposed, taking into account the effects of power memory when the system relaxes to a local equilibrium state. For the two-phase flow of an incompressible and immiscible fluid in porous media, a memory formalism using the fractional Caputo derivative was introduced and a two-level discrete time method was developed that uses a large time step for pressure and a small time step for saturation [11]. In [12], a nonlinear two-dimensional orthotropic fluid flow equation with a fractional Riemann–Liouville derivative in time is considered. In [13], a fractional model was presented for two immissible fluids flowing through a porous medium with an average capillary pressure, and the solution was obtained using the Mittag-Leffler function, the Sumudu transform, the sinusoidal Fourier transform and their inversions after obtaining the corresponding formulas for fractional integrals and derivatives. In [14], the laminar flow of a fluid in an axisymmetric porous cylindrical channel exposed to a magnetic field was investigated. The governing equations consisted of fractional partial differential equations based on Caputo-Fabrizio fractional derivatives in time.

As we can see, many papers have been devoted to the theoretical development and application of fractional derivatives in various branches of science, but in this paper we want to use the recently introduced fractional derivative in the sense of Caputo-Fabrizio without a singular core [15]. The properties of the Caputo-Fabrizio fractional derivative are studied in [16], and various boundary value problems for the fractional heat equation involving this fractional derivative are studied in [17].

The use of the Caputo-Fabrizio fractional derivative has been studied in many papers. For example, in [18], the equation of groundwater flow within an unlimited aquifer is modified using the concept of the Caputo-Fabrizio fractional derivative without the singular core. In [19], the model of groundwater motion through a geological formation was extended using the Caputo-Fabrizio fractional order derivative and the equation was solved analytically using some integral transformations.

The main contribution of [20] is the construction and analysis of stable schemes based on the third-order finite difference method in time and spectral methods in space for the effective solution of the two-dimensional diffusion equation containing a fractional Caputo-Fabrizio time derivative. In [21], the Caputo-Fabrizio fractional derivative is used to introduce two new types of high-order derivatives and the existence of solutions for two such fractional high-order integro-differential equations is studied. The article [22] presents a parallel algorithm for solving a two-dimensional fractional differential equation. For this algorithm, a distribution model and a data layout with a virtual boundary are developed. In addition, in [23] application to a nonlinear Fischer-type reaction-diffusion equation was investigated, in [24] application to a stationary heat flow, in [25] application to a groundwater flow, and in [26] application to the study of chaos on the Wallis model for El Nino, in [27] the fractional Nagumo equation with nonlinear diffusion and convection was studied using the Caputo-Fabrizio fractional derivative.

3 Material and methods

3.1 Formulation of the problem

Let $\Omega = (0,1) \times (0,1)$ and $Q_T = \Omega \times (0,T)$ for $T > 0$. We consider the following initial boundary value problem: find $u \in \bar{Q}_T$ such that

$$\partial_{0t}^{\alpha} u = Ku + Du + f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$u(x,0) = \rho(x), \quad x \in \bar{\Omega}, \tag{2}$$

$$u(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \tag{3}$$

where $0 < \alpha < 1$, $x = (x_1, x_2)$; $K = K_1 + K_2$, $D = D_1 + D_2$,

$$K_m u = q_m(x,t) \frac{\partial u}{\partial x_m}, \quad D_m u = \frac{\partial}{\partial x_m} \left(k_m(x,t) \frac{\partial u}{\partial x_m} \right), \quad m = 1,2.$$

The fractional derivative is defined in the sense of the Caputo-Fabrizio definition:

$$\partial_{0t}^{\alpha} u(x,t) = \frac{1}{1-\alpha} \int_0^t \exp(-\gamma(t-\tau)) \frac{\partial u}{\partial \tau}(x,\tau)\, d\tau, \quad \gamma = \frac{\alpha}{1-\alpha}. \tag{4}$$

Assume that the following conditions hold for the coefficients and the right-hand side of (1):

$$k_m(x,t) \in C^{1,0}\left(\bar{Q}_T\right), \; q_m(x,t), \, f(x,t) \in C\left(\bar{Q}_T\right), \tag{5}$$

$$0 < c_1 \le k_m(x,t) \le c_2, \quad |q_m(x,t)| \le c_2, \quad 2c_1 > c_2^2. \tag{6}$$

Assume that there exists a solution to the problem (1)-(3) in a class of sufficiently smooth functions.

3.2 Uniqueness of the solution and its continuous dependence on input data

Introduce the following scalar products and norms:

$$\|u\|_{0,\bar{Q}_T}^2 = \int_0^T \int_{\Omega} u^2 dx\, dt, \quad \|u\|_{0,\Omega}^2 = \int_{\Omega} u^2 dx,$$

$$\|\nabla u\|_{0,\bar{Q}_T}^2 = \left\| \frac{\partial u}{\partial x_1} \right\|_{0,\bar{Q}_T}^2 + \left\| \frac{\partial u}{\partial x_2} \right\|_{0,\bar{Q}_T}^2, \quad (u,v) = \int_{\Omega} uv\, dx,$$

$$\|u\|_*^2 = \int_0^T \int_{\Omega} D_{0t}^{-\alpha} \|u\|_{0,\Omega}^2 \, dx\, dt,$$

where u and v are functions defined in $\bar{Q}_T$; $D_{0t}^{-\alpha}u$ is the Caputo-Fabrizio fractional integration operator [28]:

$$D_{0t}^{-\alpha}u = \frac{2\,(1-\alpha)}{2-\alpha}u\,(t) + \frac{2\alpha}{2-\alpha}\int_0^t u\,(\tau)\,d\tau, \quad t \geq 0.$$

The following two lemmas are proved similarly to [29].

Lemma 1 *For any absolutely continuous function on* $[0,T]$, $y\,(t)$, *the following inequality holds:*

$$y\partial_{0t}^{\alpha}y \geq \frac{1}{2}\partial_{0t}^{\alpha}y^2, \quad 0 < \alpha < 1.$$

Lemma 2 *Let* $y\,(t)$ *be a non-negative absolutely continuous function satisfying the inequality*

$$\partial_{0t}^{\alpha}y \leq \gamma_1 y\,(t) + \gamma_2\,(t)\,,\ 0 \leq \alpha \leq 1$$

for almost every $t \in [0,T]$, *where* $\gamma_1 > 0$, $\gamma_2\,(t)$ *are nonnegative summable functions on* $[0,T]$. *Then*

$$y\,(t) \leq y\,(0)\,E_{\alpha}\,(\gamma_1 t^{\alpha}) + \Gamma\,(\alpha)\,E_{\alpha,\alpha}\,(\gamma_1 t^{\alpha})\,D_{0t}^{-\alpha}\gamma_2\,(t)\,,$$

where $E_{\alpha}\,(z)$, $E_{\alpha,\mu}\,(z)$ *are Mittag-Leffler functions:*

$$E_{\alpha}\,(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma\,(\alpha n+1)}, \quad E_{\alpha,\mu}\,(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma\,(\alpha n+\mu)}.$$

Boundedness of the functions $E_{\alpha}\,(t^{\alpha})$ and $E_{\alpha,\alpha}\,(t^{\alpha})$ for $0 \leq t \leq T$ yields the following inequality for a non-negative absolutely continuous function $y\,(x,t)$ under the conditions of Lemma 2:

$$\|y\|_{0,\bar{Q}_T}^2 \leq M_1\,\|y\,(x,0)\|_{0,\bar{Q}_T}^2 + M_2\,\|\gamma_2\|_*^2\,. \tag{7}$$

Theorem 1 *If* $u\,(x,t) \in C^{2,0}\,(Q_T) \cap C^{1,0}\left(\bar{Q}_T\right)$, $\partial_{0t}^{\alpha}u\,(x,t) \in C\left(\bar{Q}_T\right)$, *then under the conditions (5)-(6) the following inequality holds for the solution of the problem (1)-(3):*

$$\|u\,(x,t)\|_{0,\bar{Q}_T}^2 \leq M_1\,\|u\,(x,0)\|_{0,\bar{Q}_T}^2 + M_2\,\|f\,(x,t)\|_*^2\,, \quad M_1, M_2 > 0,$$

which yields the uniqueness and continuous dependence of the solution on input data.

Proof. Using (1), we get

$$\int_0^T\int_{\Omega} u\partial_{0t}^{\alpha}u\,dx\,dt = \int_0^T\int_{\Omega} Ku\cdot u\,dx\,dt + \int_0^T\int_{\Omega} Du\cdot u\,dx\,dt + \int_0^T\int_{\Omega} f\,(x,t)\,u\,dx\,dt. \tag{8}$$

Estimate the integral on the left-hand side of (8) using Lemma 1:

$$\int_0^T\int_{\Omega} u\partial_{0t}^{\alpha}u\,dx\,dt \geq \frac{1}{2}\int_0^T \partial_{0t}^{\alpha}\,\|u\|_{0,\Omega}^2\,dt. \tag{9}$$

The integrals on the right-hand side of (8) are estimated as follows:

$$\int_0^T \int\limits_\Omega Ku \cdot u\, dx\, dt \leq 2c_2\varepsilon_1 \|u\|^2_{0,\bar{Q}_T} + \frac{c_2}{4\varepsilon_1} \|\nabla u\|^2_{0,\bar{Q}_T}, \tag{10}$$

$$-\int_0^T \int\limits_\Omega Du \cdot u\, dx\, dt \geq c_1 \|\nabla u\|^2_{0,\bar{Q}_T}, \tag{11}$$

$$\int_0^T \int\limits_\Omega fu\, dx\, dt \leq \frac{\varepsilon_2}{2} \|f\|^2_{0,\bar{Q}_T} + \frac{1}{2\varepsilon_2} \|u\|^2_{0,\bar{Q}_T}. \tag{12}$$

Choosing $\varepsilon_1 = (2c_2)^{-1}$, $\varepsilon_2 = \left(2c_1 - c_2^2\right)^{-1}$, it follows from (8) that

$$\int_0^T \partial^\alpha_{0t} \|u\|^2_{0,\Omega}\, dt + \left(c_1 - \frac{c_2^2}{2}\right) \|\nabla u\|^2_{0,\bar{Q}_T} \leq \|u\|^2_{0,\bar{Q}_T} + \frac{1}{2c_1 - c_2^2} \|f\|^2_{0,\bar{Q}_T}.$$

Using Lemma 2 implies

$$\int_0^T \|u(x,t)\|^2_{0,\Omega}\, dt \leq \int_0^T \|u(x,0)\|^2_{0,\Omega} E_\alpha(t^\alpha)\, dt +$$

$$+\frac{\Gamma(\alpha)}{2c_1 - c_2^2} \int_0^T E_{\alpha,\alpha}(t^\alpha)\, D^{-\alpha}_{0t} \|f(x,t)\|^2_{0,\Omega}\, dt. \tag{13}$$

Using the inequality (7), we obtain the statement of the theorem from (13).

3.3 Construction of the numerical method

For the numerical solution of the problem (1)-(3) we apply the finite difference method. In $\bar{Q}_T$, we introduce a uniform finite difference grid $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$, where

$$\overline{\omega}_h = \{x_{ij} = (ih,\, jh): \quad i = 0,1,...,N, \;\; j = 0,1,...,N, \;\; Nh = 1\},$$

$$\overline{\omega}_\tau = \{t_n = n\tau, \;\; n = 0,1,...,M; \;\; T = \tau M\}.$$

First let us derive a discrete analog of the fractional derivative in the sense of Caputo-Fabrizio. For this purpose we use the technique applied in [30] for the derivation of the discrete analog of the fractional derivate in the sense of Caputo. In the following lemma we assume $u(t) = u(\cdot, t)$.

Lemma 3 *Let $u(t) \in C^3[0,T]$. The discrete analog of the derivative (4) with the approximation order $O(\tau^{3-\alpha})$ is given by*

$$\partial^\alpha_{0t_{n+\sigma}} u(t) \approx \Delta^\alpha_{0t_{n+\sigma}} u \equiv \frac{1}{\alpha\tau} \sum_{s=0}^{n} g_{n-s} \left(u(t_{s+1}) - u(t_s)\right), \tag{14}$$

where

$$g_s^{\alpha,\sigma} = \begin{cases} A_0^{\alpha,\sigma}, & s = 0,\ n = 0, \\ A_0^{\alpha,\sigma} + B_1^{\alpha,\sigma}, & s = 0,\ n > 0, \\ A_s^{\alpha,\sigma} + B_{s+1}^{\alpha,\sigma} - B_s^{\alpha,\sigma}, & 1 \le s \le n-1,\ n > 0, \\ A_n^{\alpha,\sigma} - B_n^{\alpha,\sigma}, & s = n,\ n > 0, \end{cases}$$

$$A_0^{\alpha,\sigma} = \frac{e^{\gamma\tau\sigma} - 1}{e^{\gamma\tau\sigma}}; \quad A_s^{\alpha,\sigma} = \frac{e^{\gamma\tau} - 1}{e^{\gamma\tau(\sigma+s)}}, \quad B_s^{\alpha,\sigma} = \frac{e^{\gamma\tau}(\gamma\tau - 2) + \gamma\tau + 2}{2\gamma e^{\gamma\tau(\sigma+s)}}, \quad s \ge 1. \tag{15}$$

Proof. Following [30], let $\sigma = 1 - \alpha/2$. Using the definition (4), construct the following approximation for the fractional derivative of the function $u(t) \in C^3[0,T]$ of order α, $0 < \alpha < 1$, in the sense of Caputo-Fabrizio at a fixed point $t_{n+\sigma}$, $n \in \{0, 1, ..., M-1\}$:

$$\partial_{0t_{n+\sigma}}^{\alpha} u(t) \approx \Delta_{0t_{n+\sigma}}^{\alpha} u = \frac{1}{1-\alpha} \sum_{s=1}^{n} \int_{t_{s-1}}^{t_s} \exp\left(-\gamma\left(t_{n+\sigma} - \eta\right)\right) \tilde{u}_s'(\eta)\, d\eta +$$

$$+ \frac{1}{1-\alpha} \int_{t_n}^{t_{n+\sigma}} \exp\left(-\gamma\left(t_{n+\sigma} - \eta\right)\right) \tilde{u}_s'(\eta)\, d\eta, \tag{16}$$

where $\tilde{u}_s(\eta)$ is the approximation of $u(\eta)$ on $[t_{s-1}, t_s]$, $s \in \{1, 2, ..., n\}$. Various approaches to approximate $\tilde{u}_s(\eta)$ result in different computational schemes which differ by the approximation error, the complexity of the calculations. Among them, approaches based on applying the trapezoidal rule, interpolation and predictor-corrector methods are known. In this paper, we utilize the quadratic interpolation polynomial of u using three nodes t_{s-1}, t_s and t_{s+1}:

$$\tilde{u}_s(t) = \frac{(t - t_s)(t - t_{s+1})}{2\tau^2} u(t_{s-1}) - \frac{(t - t_{s-1})(t - t_{s+1})}{\tau^2} u(t_s) + \frac{(t - t_{s-1})(t - t_s)}{2\tau^2} u(t_{s+1}), \tag{17}$$

for which

$$u(t) - \tilde{u}_s(t) = \frac{u'''\left(\overline{\xi}_s\right)}{6} (t - t_{s-1})(t - t_s)(t - t_{s+1}) \tag{18}$$

holds, where $t \in [t_{s-1}, t_{s+1}]$, $\overline{\xi}_s \in (t_{s-1}, t_{s+1})$. Using (17) in (16), we obtain

$$\Delta_{0t_{n+\sigma}}^{\alpha} u = \frac{1}{1-\alpha} \sum_{s=1}^{n} \int_{t_{s-1}}^{t_s} \frac{u_{t,s-1} + u_{\bar{t}t,s}\left(\eta - t_{s-\frac{1}{2}}\right)}{\exp\left(\gamma\left(t_{n+\sigma} - \eta\right)\right)} d\eta + \frac{u_{t,n}}{1-\alpha} \int_{t_n}^{t_{n+\sigma}} \frac{d\eta}{\exp\left(\gamma\left(t_{n+\sigma} - \eta\right)\right)},$$

where $u_{t,s} = (u^{s+1} - u^s)\tau^{-1}$, $u_{\bar{t},s} = (u^s - u^{s-1})\tau^{-1}$. Taking into account the equality

$$\int_{t_{s-1}}^{t_s} \frac{\eta - t_{s-\frac{1}{2}}}{\exp\left(\gamma\left(t_{n+\sigma} - \eta\right)\right)} d\eta = \frac{\exp(\gamma\tau)(\gamma\tau - 2) + \gamma\tau + 2}{2\gamma^2 \exp\left(\gamma\tau\left(n + \sigma - s + 1\right)\right)}, \quad 1 \le s \le n,$$

we arrive at

$$\Delta_{0t_{n+\sigma}}^{\alpha} u = \frac{1}{\alpha} \left(\sum_{s=1}^{n} \frac{\exp(\gamma\tau) - 1}{\exp\left(\gamma\tau\left(n + \sigma - s + 1\right)\right)} u_{t,s-1} + \right.$$

$$+\sum_{s=1}^{n}\frac{\exp(\gamma\tau)(\gamma\tau-2)+\gamma\tau+2}{2\gamma\exp(\gamma\tau(n+\sigma-s+1))}(u_{t,s}-u_{t,s-1})+\frac{\exp(\gamma\tau\sigma)-1}{\exp(\gamma\tau\sigma)}u_{t,n}\Bigg).$$

Finally, using the notations (15), we arrive at the assertion of the lemma.

In $\overline{\omega}_{h\tau}$ we introduce the following difference scheme with weights of the approximation order $O\left(h^2+\tau^2\right)$:

$$\Delta^{\alpha}_{0t_{n+\sigma}}y_{ij}=\Theta y_{ij}+\Psi y_{ij}+\varphi^n_{ij}, \tag{19}$$

$$y^0_{ij}=\rho_{ij}, \tag{20}$$

$$y^{(\sigma)}\big|_{\gamma_h}=0,\ \ n>0, \tag{21}$$

where $\Theta=\Theta_1+\Theta_2$, $\Psi=\Psi_1+\Psi_2$,

$$\Theta_m y_{ij}=0.5\left(\eta_{m,ij}-|\eta_{m,ij}|\right)\xi_{m,ij}y^{(\sigma)}_{\bar{x}_m,ij}+0.5\left(\eta^{+1_m}_{m,ij}+\left|\eta^{+1_m}_{m,ij}\right|\right)\xi^{+1_m}_{m,ij}y^{(\sigma)}_{x_m,ij},$$

$$\Psi_m y_{ij}=\left(\xi_m y^{(\sigma)}_{\bar{x}_m}\right)_{x_m,ij},\quad \varphi^n_{ij}=f\left(x_{ij},t_{n+\sigma}\right),$$

$$\xi^n_{1,ij}=k\left(x_{i-\frac{1}{2},j},t_{n+\sigma}\right),\quad \xi^n_{2,ij}=k\left(x_{i,j-\frac{1}{2}},t_{n+\sigma}\right),\quad \eta^n_{m,ij}=\frac{q_m\left(x_{ij},t_{n+\sigma}\right)}{k_m\left(x_{ij},t_{n+\sigma}\right)},$$

$$y^{(\sigma)}=\sigma y^{n+1}+(1-\sigma)\,y^n, \tag{22}$$

γ_h is the set of boundary nodes of $\overline{\omega}_h$. Here we use standard notations from the theory of difference schemes.

3.4 Stability and convergence of the difference scheme

We introduce scalar products and norms:

$$(u,v)=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}u_{ij}v_{ij}h^2,\quad (u,v]=\sum_{i=1}^{N}\sum_{j=1}^{N}u_{ij}v_{ij}h^2,$$

$$\|u\|^2=\sum_{i=0}^{N}\sum_{j=0}^{N}u_{ij}v_{ij}h^2,\quad \|\nabla u\|^2=\|u_{\bar{x}_1}\|^2+\|u_{\bar{x}_2}\|^2,$$

$$\|u_{\bar{x}_1}\|^2=\sum_{i=1}^{N}\sum_{j=0}^{N}u_{\bar{x}_1,ij}h^2,\quad \|u_{\bar{x}_2}\|^2=\sum_{i=0}^{N}\sum_{j=1}^{N}u_{\bar{x}_2,ij}h^2.$$

We prove several auxiliary lemmas.

Lemma 4 *For any function $y(t)$ defined on the grid $\overline{\omega}_{h\tau}$, the following inequality holds:*

$$y^{(\sigma)}\Delta^{\alpha}_{0t_{n+\sigma}}y\geq\frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}}y^2.$$

This Lemma is proved similarly to Lemma 1 from [29]. Below, the letters μ with indices denote positive numbers that do not depend on h and τ.

Lemma 5 *Under the conditions (6), the following inequality holds for the solution of the difference problem (19)-(21):*

$$\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 + c_1 \left\|\nabla y^{(\sigma)}\right\|^2 \leq \mu_1 \left\|y^{(\sigma)}\right\|^2 + \|\varphi\|^2 .$$

Proof. Multiply the equation (19) scalarly by $y^{(\sigma)}$:

$$\left(\Delta^{\alpha}_{0t_{n+\sigma}} y,\, y^{(\sigma)}\right) = \left(\Theta y,\, y^{(\sigma)}\right) + \left(\Psi y,\, y^{(\sigma)}\right) + \left(\varphi^n,\, y^{(\sigma)}\right). \tag{23}$$

Estimate the scalar products on the left-hand side of (23) using Lemma 4:

$$\left(\Delta^{\alpha}_{0t_{n+\sigma}} y, y^{(\sigma)}\right) \geq \frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 . \tag{24}$$

Estimate the terms on the right-hand side of (23) as follows:

$$\left(\Theta y,\, y^{(\sigma)}\right) = \sum_{m=1}^{2} \left(\Theta_m y,\, y^{(\sigma)}\right) \leq \frac{c_2^2}{4\varepsilon c_1} \left\|\nabla y^{(\sigma)}\right\|^2 + \frac{\varepsilon c_2^2}{2c_1} \left\|y^{(\sigma)}\right\|^2 , \tag{25}$$

$$-\left(\Psi y,\, y^{(\sigma)}\right) \geq c_1 \sum_{m=1}^{2} \left(1,\, \left(y^{(\sigma)}_{\bar{x}_m}\right)^2\right] = c_1 \left\|\nabla y^{(\sigma)}\right\|^2 , \tag{26}$$

$$\left(\varphi, y^{(\sigma)}\right) \leq \frac{1}{2}\left(\|\varphi\|^2 + \left\|y^{(\sigma)}\right\|^2\right). \tag{27}$$

Taking into account (24)-(27) and choosing $\varepsilon = \dfrac{c_2^2}{2c_1^2}$, we obtain from (23):

$$\frac{1}{2}\Delta^{\alpha}_{0t_{n+\sigma}} \|y\|^2 + \frac{c_1}{2} \left\|\nabla y^{(\sigma)}\right\|^2 \leq \frac{2c_1^3 + c_2^4}{4c_1^3} \left\|y^{(\sigma)}\right\|^2 + \frac{1}{2} \|\varphi\|^2 . \tag{28}$$

Using the definition $y^{(\sigma)}$, from (28) we arrive at the statement of the lemma.

Lemma 6 *[30] Let the non-negative sequences y^n and φ^n, $n = 0, 1, 2,\ldots$ satisfy the inequality*

$$\Delta^{\alpha}_{0t_{n+\sigma}} y \leq \lambda_1 y^{n+1} + \lambda_2 y^n + \varphi^n, \quad n \geq 1,$$

where $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ are some constants. Then there exists t_0 such that for $\tau \leq t_0$, the inequality holds

$$y^{n+1} \leq 2\left(y^0 + \frac{(t_n)^{\alpha}}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \varphi^m\right) E_{\alpha}\left(2\lambda\,(t_n)^{\alpha}\right),$$

where $\lambda = \lambda_1 + \dfrac{\lambda_2}{2 + 2^{1-\alpha}}$.

Based on Lemma 5 and Lemma 6, the following theorem is proved.

Theorem 2 *Under the conditions (6), there exists t_0 such that for $\tau \leq t_0$, the following inequality holds for the solution of the difference problem (19)-(21):*

$$\left\|y^{n+1}\right\|^2 \leq \mu_2 \left(\left\|y^0\right\|^2 + \frac{(t_n)^\alpha}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \|\varphi^m\|^2\right),$$

which implies the uniqueness of the solution and stability of the difference scheme (19)-(21) with respect to the initial data and the right-hand side.

Theorem 3 *Under the the conditions of Theorem 2, the solution of the difference problem (19)-(21) converges to the solution of the differential problem (1)-(3) and the following inequality holds:*

$$\left\|y^{n+1} - u(\cdot, t_{n+1})\right\| \leq \mu_3 \left(h^2 + \tau^2\right).$$

Proof. Consider the problem for the difference $z = y - u$:

$$\Delta^\alpha_{0t_{n+\sigma}} z_{ij} = \Theta z_{ij} + \Psi z_{ij} + \psi^n_{ij}, \quad (x,t) \in \omega_{h\tau}, \tag{29}$$

$$z^0_{ij} = 0, \tag{30}$$

$$z^{(\sigma)}\Big|_{\gamma_h} = 0, \quad n > 0, \tag{31}$$

where $\psi^n_{ij} = \varphi^n_{ij} - \Delta^\alpha_{0t_{n+\sigma}} u^n_i + \Theta u^{(\sigma)}_{ij} + \Psi u^{(\sigma)}_{ij}$. The following inequality holds for the solution of the difference problem (29)-(31):

$$\left\|z^{n+1}\right\|^2 \leq \frac{\mu_4 t^\alpha_n}{\Gamma(1+\alpha)} \max_{0\leq m\leq n} \|\psi^m\|^2, \tag{32}$$

where $\|\psi^m\| = O\left(h^2 + \tau^2\right)$. (32) yields the convergence of the solution of the difference problem (19)-(21) to the solution of the differential problem (1)-(3).

3.5 Implementation of the difference scheme

To solve the problem (19)-(21), we use the alternating directions method, which consists of two stages [31]:

$$\frac{g^{\alpha,\sigma}_0}{\tau\alpha}\left(y^{n+\frac{1}{2}}_{i,j} - y^n_{i,j}\right) + \frac{1}{\tau\alpha}\sum_{s=0}^{n-1} g^{\alpha,\sigma}_{n-s}\left(y^{s+1}_{i,j} - y^s_{i,j}\right) =$$

$$= \sigma(\Theta_1 + \Psi_1)\, y^{n+\frac{1}{2}}_{i,j} + (1-\sigma)(\Theta_2 + \Psi_2)\, y^n_{i,j}, \tag{33}$$

$$\frac{g^{\alpha,\sigma}_0}{\tau\alpha}\left(y^{n+1}_{i,j} - y^{n+\frac{1}{2}}_{i,j}\right) + \frac{1}{\tau\alpha}\sum_{s=0}^{n-1} g^{\alpha,\sigma}_{n-s}\left(y^{s+1}_{i,j} - y^s_{i,j}\right) =$$

$$= \sigma(\Theta_1 + \Psi_1)\, y^{n+\frac{1}{2}}_{i,j} + (1-\sigma)(\Theta_2 + \Psi_2)\, y^{n+1}_{i,j}. \tag{34}$$

On each time layer, the solution of the problem (19)-(21) is reduced to a sequential solution of tridiagonal systems of equations, which are solved by the Thomas algorithm. By checking directly, one can make sure that the stability condition of the Thomas algorithm holds. To check the accuracy of the difference scheme (19)-(21), a number of computational experiments were performed on the example of two test problems.

4 Results and discussion

Problem 1. Consider the equation

$$\partial_{0t}^{\alpha} u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2, t) \tag{35}$$

with the right-hand side

$$f(x_1, x_2, t) = -\frac{3}{\alpha^3} \sin^2(\pi x_1) \sin^2(\pi x_2) \left(2\alpha^2 \exp\left(\frac{t\alpha}{\alpha - 1}\right) - 4\alpha \exp\left(\frac{t\alpha}{\alpha - 1}\right) + \right.$$
$$\left. +2\alpha^2 \exp\left(\frac{t\alpha}{\alpha - 1}\right) - t^2\alpha^2 - 2t\alpha^2 - 2\alpha^2 + 2ta + 4\alpha - 2 \right) +$$
$$+4\pi^2 t^3 \sin^2(\pi x_1) \sin^2(\pi x_2) - 2\pi t^3 \cos(\pi x_1) \sin(\pi x_1) \sin^2(\pi x_2) -$$
$$-2\pi^2 t^3 \cos^2(\pi x_1) \sin^2(\pi x_2) - 2\pi t^3 \sin^2(\pi x_1) \cos(\pi x_2) \sin(\pi x_2) -$$
$$-2\pi^2 t^3 \sin^2(\pi x_1) \cos^2(\pi x_2)$$

and homogeneous initial and boundary conditions.

The exact solution to this problem is as follows:

$$u(x_1, x_2, t) = t^3 \sin^2(\pi x_1) \sin^2(\pi x_2).$$

When analyzing the dependence of the error order on the spatial step, the value of the time step is selected as $\tau = 10^{-5}$. The step value for the spatial variable h varied between $h = 10^{-2}$ and $h = 10^{-5}$.

The error value was determined by the formula

$$E = \max_{0 \le n \le M} \max_{0 \le i \le N} \max_{0 \le j \le N} \left| y_{ij}^n - u(ih, jh, t_n) \right|.$$

When analyzing the dependence of the error order on the time step, the value of the spatial step is selected as $h = 10^{-4}$. The value of the time step varied between $\tau = 10^{-5}$ and $\tau = 10^{-8}$. The order of the fractional derivative is set to $\alpha = 0.3$, $\alpha = 0.45$ and $\alpha = 0.85$.

Tables 1 and 2 show error values for various values of the parameters σ, h and τ.

Table 1: Error analysis for Problem 1

	$\sigma = 0.85$ ($\alpha = 0.3$)	$\sigma = 0.775$ ($\alpha = 0.45$)	$\sigma = 0.575$ ($\alpha = 0.85$)
$h = 1/100$	$1.582643 \cdot 10^{-7}$	$1.535625 \cdot 10^{-7}$	$5.953420 \cdot 10^{-8}$
$h = 1/500$	$4.543282 \cdot 10^{-9}$	$6.625594 \cdot 10^{-9}$	$2.655208 \cdot 10^{-9}$
$h = 1/1000$	$1.683145 \cdot 10^{-9}$	$1.659836 \cdot 10^{-9}$	$8.956221 \cdot 10^{-10}$
$h = 1/2000$	$5.325643 \cdot 10^{-10}$	$4.859264 \cdot 10^{-10}$	$6.958645 \cdot 10^{-10}$
$h = 1/5000$	$2.546234 \cdot 10^{-10}$	$2.654822 \cdot 10^{-10}$	$5.659750 \cdot 10^{-10}$
$h = 1/10000$	$8.203144 \cdot 10^{-11}$	$2.659820 \cdot 10^{-10}$	$3.659504 \cdot 10^{-10}$

Table 2: Error analysis for Problem 1

	$\sigma = 0.85$ $(\alpha = 0.3)$	$\sigma = 0.775$ $(\alpha = 0.45)$	$\sigma = 0.575$ $(\alpha = 0.85)$
$\tau = 10^{-5}$	$1.625822 \cdot 10^{-9}$	$1.956430 \cdot 10^{-9}$	$8.659832 \cdot 10^{-10}$
$\tau = 10^{-6}$	$1.659832 \cdot 10^{-11}$	$8.956268 \cdot 10^{-12}$	$2.956354 \cdot 10^{-12}$
$\tau = 10^{-7}$	$8.659825 \cdot 10^{-14}$	$4.986372 \cdot 10^{-14}$	$4.356320 \cdot 10^{-15}$
$\tau = 10^{-8}$	$5.953167 \cdot 10^{-16}$	$2.956363 \cdot 10^{-16}$	$7.923544 \cdot 10^{-18}$

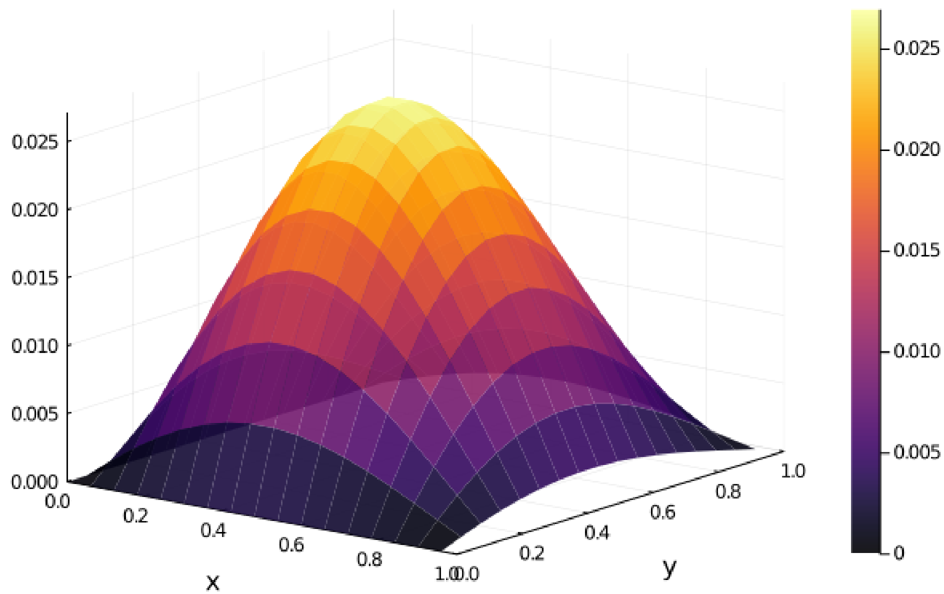


Figure 1: Solution of Problem 1, $\alpha = 0.85$, $n = 1000$

Figure 1 shows a graph of the approximate solution of the problem on time layer $n = 1000$ at $\alpha = 0.85$.

Problem 2. Consider the equation (35) with the right-hand side

$$f(x_1, x_2, t) = -\frac{2}{\alpha}\pi \sin(2\pi x_1)\sin(2\pi x_2)\left(\exp\left(\frac{t\alpha}{\alpha-1}\right)-1\right)+$$

$$+16\pi^3 t \sin(2\pi x_1)\sin(2\pi x_2) - 4\pi^2 t\cos(2\pi x_1)\sin(2\pi x_2) - 4\pi^2 t\sin(2\pi x_1)\cos(2\pi x_2)$$

and homogeneous initial and boundary conditions.

The exact solution to this problem is as follows:

$$u(x_1, x_2, t) = 2\pi t \sin(2\pi x_1)\sin(2\pi x_2).$$

Tables 3 and 4 show error values for various values of the parameters σ, h and τ. In Figure 2 the graph of the approximate solution of the problem on the layer $n = 1000$ at $\alpha = 0.85$ is given.

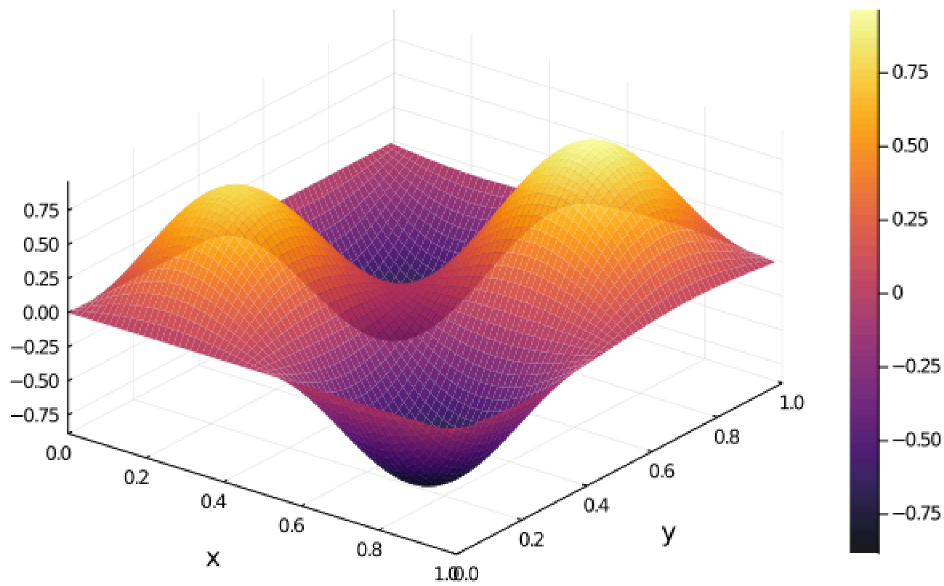


Figure 2: Solution of Problem 2, $\alpha = 0.85$, $n = 1000$

Table 3: Error analysis for Problem 2

	$\sigma = 0.85$ $(\alpha = 0.3)$	$\sigma = 0.775$ $(\alpha = 0.45)$	$\sigma = 0.575$ $(\alpha = 0.85)$
$h = 1/100$	$8.568827 \cdot 10^{-7}$	$6.884689 \cdot 10^{-7}$	$3.954957 \cdot 10^{-7}$
$h = 1/500$	$3.448455 \cdot 10^{-8}$	$2.817526 \cdot 10^{-8}$	$1.665078 \cdot 10^{-8}$
$h = 1/1000$	$8.754912 \cdot 10^{-9}$	$7.523579 \cdot 10^{-9}$	$4.796715 \cdot 10^{-9}$
$h = 1/2000$	$2.324212 \cdot 10^{-9}$	$2.378951 \cdot 10^{-9}$	$1.838244 \cdot 10^{-9}$
$h = 1/5000$	$5.345784 \cdot 10^{-10}$	$9.680494 \cdot 10^{-10}$	$1.015005 \cdot 10^{-9}$
$h = 1/10000$	$2.876746 \cdot 10^{-10}$	$7.735310 \cdot 10^{-10}$	$8.981407 \cdot 10^{-10}$

Table 4: Error analysis for Problem 2

	$\sigma = 0.85$ $(\alpha = 0.3)$	$\sigma = 0.775$ $(\alpha = 0.45)$	$\sigma = 0.575$ $(\alpha = 0.85)$
$\tau = 10^{-5}$	$2.523844 \cdot 10^{-9}$	$1.635420 \cdot 10^{-9}$	$6.623524 \cdot 10^{-10}$
$\tau = 10^{-6}$	$3.520531 \cdot 10^{-11}$	$7.435820 \cdot 10^{-12}$	$1.023465 \cdot 10^{-12}$
$\tau = 10^{-7}$	$7.025432 \cdot 10^{-14}$	$3.023564 \cdot 10^{-14}$	$3.342564 \cdot 10^{-15}$
$\tau = 10^{-8}$	$3.623501 \cdot 10^{-16}$	$3.623524 \cdot 10^{-16}$	$6.526534 \cdot 10^{-18}$

It follows from the results shown in Tables 1 and 3 that the error is a value of magnitude $O\left(h^2\right)$. Similarly, the results in Tables 2 and 4 show that the error is of magnitude $O\left(\tau^2\right)$.

Thus, computational experiments have confirmed that the difference scheme converges with the second order in both spatial and temporal variables.

5 Conclusion

Thus, an implicit finite difference scheme is constructed for a fractional differential equation with variable coefficients containing a fractional time derivative in the sense of Caputo-Fabrizio. The stability and error estimates of the difference scheme are established. The empirical convergence agrees well with the theoretical estimates.

The results obtained in this work are the basis for the construction of finite element methods for solving fluid flow problems in fractured porous media. In particular, the constructed discrete analogue of the fractional derivative in the sense of Caputo-Fabrizio will be used in subsequent works. Also, a comparison of solutions obtained using finite element and finite difference methods will be carried out. Moreover, the results obtained can be applied to the numerical solution of other equations containing a fractional time derivative.

6 Acknowledgement

The work was supported by grant funding of scientic and technical programs and projects of the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. AP08053189, 2020-2022).

References

[1] Latawiec K.J., Stanislawski R., Lukaniszyn M., Czuczwara W., and Rydel M., "Fractional-order modeling of electric circuits: Modern empiricism vs. classical science" , *Proceedings of Progress in Applied Electrical Engineering* (2017).

[2] Oprzedkiewicz K., Mitkowski W., "A Memory-Efficient Noninteger-Order Discrete-Time State-Space Model of a Heat Transfer Process" , *Int. J. Appl. Math. Comput. Sci.,* 28 (2018): 649–659.

[3] Wang K.L., Liu S.Y., "He's fractional derivative and its application for fractional Fornberg-Whitham equation" , *Therm. Sci.,* 21 (2017): 2049-2055.

[4] Altaf M., Atangana A., "Dynamics of Ebola Disease in the Framework of Different Fractional Derivatives" , *Entropy,* 21 (2019): 303.

[5] Lichae B.H., Biazar J., Ayati Z., "The Fractional Differential Model of HIV-1 Infection of CD+T-Cells with Description of the Effect of Antiviral Drug Treatment" , *Comput. Math. Methods Med.,* 2019:4059549 (2019).

[6] Caputo M., "Models of flux in porous media with memory" , *Water Resources Research,* 36 (2000): 693–705.

[7] Agarwal R., Yadav M. P., Baleanu D., Purohit S. D., "Existence and uniqueness of miscible flow equation through porous media with a non singular fractional derivative" , *AIMS Mathematics,* 5:2 (2020): 1062–1073.

[8] Di Giuseppe E., Moroni M., Caputo M., "Flux in porous media with memory: models and experiments" , *Transport in Porous Media,* 83:3 (2010): 479–500.

[9] Gazizov R. K., Lukaschuk S. Yu., "Drobno-differencialny podhod k modelirovaniyu processov filtratsii v slozhnyh neodnorodnyh poristyh sredah [Fractional-differential approach to modeling filtration processes in complex inhomogeneous porous media]" , *Vestnik UGATU,* 21:4 (2017): 104–112.

[10] Umarov S., Daum F., Nelson K., "Fractional nonlinear filtering problems and their associated fractional Zakai equations" , *ArXiV,* 1305.2658 (2013): 1-15.

[11] Zhong W., Li C., and Kou J.,. "Numerical Fractional-Calculus Model for Two-Phase Flow in Fractured Media" , *Advances in Mathematical Physics,* 2013:429835 (2013): 1-7.

[12] Lukaschuk V. O., Lukaschuk S. Yu, "Gruppovaya klassifikaciya, invariantnye reshenija i zakony sohraneniya nelineinogo dvumernogo ortotropnogo uravneniya filtratsii s drobnoi proizvodnoi Rimana–Liuvillya po vremeni [Group classification, invariant solutions, and conservation laws of a nonlinear two-dimensional orthotropic filtration equation with a Riemann–Liouville fractional derivative in time]", *Vestnik SGTU. Seriya fiziko-matematicheskie nauki,* 24:2 (2020): 226–248.

[13] Choudharya A., Kumarb D., Singh J., "A fractional model of fluid flow through porousmedia with mean capillary pressure", *Journal of the Association of Arab Universities for Basic and Applied Sciences,* 21 (2016): 59–63.

[14] Uddin1 S., Mohamad M., "Caputo-Fabrizio Time Fractional Derivative Applied to Visco Elastic MHD Fluid Flow in the Porous Medium,"*International Journal of Engineering & Technology,* 7 (2018): 533-537.

[15] Caputo M., Fabrizio M., "A New Definition of Fractional Derivative without Singular Kernel", *Progress in Fractional Differentiation and Applications,* 2 (2015): 73–85.

[16] Losada J., Nieto J. J., "Properties of a New Fractional Derivative without Singular Kernel,"*Progress in Fractional Differentiation and Applications,* 1:2 (2015): 87–92.

[17] AlSalti N., Karimov E., Kerbal S., "Boundary value problems for fractional heat equation involving Caputo-Fabrizio derivative", *New Trends in Mathematical Sciences,* 4:4 (2016): 79–89.

[18] Feulefack P. A., Djida J. D., Atangana A., "A new model of groundwater flow within an unconfined aquifer: Application of Caputo-Fabrizio fractional derivative", *American Institute of Mathematical Sciences,* 24:7 (2019): 3227–3247.

[19] Atangana A., Baleanu D., "Caputo-Fabrizio Derivative Applied to Groundwater Flow within Confined Aquifer", *Journal of Engineering Mechanics,* 143:5 (2016): D4016005.

[20] Yu F., Chen M., "Finite difference/spectral approximations for the twodimensional time Caputo-Fabrizio fractional diffusion equation", *arXiv,* 1906.00328 (2019): 1-18.

[21] Aydogan M., Baleanu D., Mousalou A., Rezapour S., "On high order fractional integro differential equations including the Caputo–Fabrizio derivative", *Boundary Value Problems,* 2018:90 (2018): 1–15.

[22] Gong C., Bao W., Tang G. et al., "A Parallel Algorithm for the Two-Dimensional Time Fractional Diffusion Equation with Implicit Difference Method", *The Scientific World Journal,* 219580 (2014): 1–8.

[23] Atangana A., "On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation", *Applied Mathematics and Computation,* 273 (2016): 948–956.

[24] Yang X. J., Srivastava H. M., "Machado Tenreiro J. A. A new fractional derivative without singular kernel: Application to the modelling of the steady heat flow", *Thermal Science,* 20:2 (2016): 753–756.

[25] Atangana A., Alkahtani B., "New model of groundwater flowing within a confine aquifer: application of Caputo-Fabrizio derivative", *Arabian Journal of Geosciences volume,* 9:8 (2016): 1–6.

[26] Alkahtani B., Atangana A., "Chaos on the Vallis Model for El Nino with Fractional Operators", *Entropy,* 18:100 (2016): 1–17.

[27] Alqahtani R. T., "Fixed point theorem for Caputo–Fabrizio fractional Nagumo equation with nonlinear diffusion and convection", *Journal of Nonlinear Sciences and Application,* 9:5 (2016): 1991–1999.

[28] Atangana A., *Fractional Operators with Constant and Variable Order with Application to GeoHydrology* (Elsevier, 2018).

[29] Alikhanov A. A., "A priori estimates for solutions of boundary value problems for fractional order equations", *Differential Equations,* 46 (2010): 660–666.

[30] Alikhanov A. A., "A new difference scheme for the time fractional diffusion equation", *Journal of Computational Physics,* 280 (2015): 424–438.

[31] Zhumagulov B. T., Temirbekov N. M., Baigereyev D. R. "Efficient difference schemes for the three-phase non-isothermal flow problem", *AIP Conference Proceedings,* 1880:060001 (2017): 1-10.

IRSTI 27.31.15; 27.43.17 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.11

K.M. Bazikova*, **G.A. Abdenova**, **G.E. Sagyndykova**
L.N. Gumilyov Eurasian National University, Kazakhstan, Nur-Sultan
*e-mail: kmbazikova@mail.ru

LINEAR STOCHASTIC DISTRIBUTED MODEL OF MONEY ACCUMULATION IN THE FORM OF A STATE SPACE

The article deals with the problem of the passive parametric identification of systems for modeling the evolution of money savings income and expenses of one household using a linear stochastic distributed model in the form of a state space taking into account white noises model of the investigated object dynamics' and white noises of the linear model measuring system of a distributed type. The use of the finite difference method allowed reducing the solution of partial differential equations to the solution of linear finite difference system with private derivatives to be reduced to the solution of a system of linear finite-difference and algebraic equations represented by models in the form of state space. It was proposed the use of a Kalman filtering algorithm for reliable evaluation of object behavior. The statement of the problem of estimating the coefficients of the equation of evolution of money savings income and expenses of one household is given. The structure of household income and expenses is described, taking into account additional additive white noise meters. An algorithm for numerical approbation of method for solving the problem of estimating the coefficients of an equation in the form of the state space for the evolution of money savings income and expenses of one household is considered. Calculations were carried out using the Matlab mathematical system based on statistical data for five years, taken from the site "Agency for Strategic planning and reforms of the Republic of Kazakhstan Bureau of National statistics". The proposed method for solving the problem of coefficients assessment's passive identification using the equations of money savings for one household in the form of a state space is sufficiently universal.

Key words: linear finite-difference equation, model in the form of a state space, evolution of one household money savings, passive identification, Kalman filter, prediction estimates, filtering estimates.

К.М. Базикова*, Г.А. Абденова, Г.Е. Сагындыкова
Л.Н. Гумилев атындағы Еуразия ұлтық университеті, Қазақстан, Нұр-Сұлтан қ.
*e-mail: kmbazikova@mail.ru

Ақшалай жинақталуының күй кеңістігі түріндегі сызықтық стохастикалық үлестірілген моделі

Мақалада зерттелетін объектінің динамика моделінің ақ шуларын және таратылған типтегі өлшеу жүйесінің сызықтық моделінің ақ шуларын есекере отырып, күй кеңістігі түріндегі сызықтық стохастикалық үлестірілген модельдің көмегімен, жеке үй шаруашылығының кірістері мен шығыстарының ақшалай жинақталуының эволюциясын модельдеу үшін жүйелердің пассивті параметрлік идентификациясының есебі қарастырылады. Ақырғы айырымдар әдісін қолдану дербес туындылы теңдеулер шешімін күй кеңістігі түріндегі модельдермен ұсынылған сызықтық ақырғы-айырымдық және алгебралық теңдеулер жүйесінің шешіміне келтіруге мүмкіндік береді. Объектінің әрекетін дұрыс бағалау үшін Калман сүзгісінің алгоритмін қолдану ұсынылды. Бір үй шаруашылығының кірістері мен шығыстарының ақшалай жинақталу эволюциясы теңдеуінің коэффициенттерін бағалау мәселесінің тұжырымдамасы келтірілген. Есептегіштердің қосымша ақ шуын ескере отырып, үй шаруашылығының кірістері мен шығыстардың құрылымы сипатталған. Бір үй шаруашылығының кірістері мен шығыстардың ақшалай жинақтары эволюциясы күйінің кеңістігі түріндегі теңдеу коэффициенттерін бағалау мәселесін шешудің әдістемесін сандық апробациялау алгоритмі қарастырылған. Есептеулер Matlab математикалық жүйесін қолдана отырып, "Қазақстан Республикасы Стратегиялық жоспарлау және реформалар агенттігі ұлттық статистика бюросы"сайтынан алынған бес жылдағы бақылау деректері негізінде жүргізілді. Күй кеңістігі түрінде бір үй шаруашылығы үшін ақша жинақтау теңдеулерінің коэффициенттерін бағалауды пассивті идентификациялау мәселесін шешудің ұсынылған әдісі жеткілікті түрде әмбебап болып табылады.

Түйін сөздер: сызықтық ақырғы-айырымдық теңдеу, күй кеңістігіндегі модель, бір үй шаруашылығының ақша жинақтарының эволюциясы, пассивті идентификация, Калман сүзгісі, болжауды бағалау, сүзуді бағалау.

К.М. Базикова*, Г.А. Абденова, Г.Е. Сагындыкова
Евразийский национальный университет имени Л.Н. Гумилева, Казахстан, г. Нур-Султан
*e-mail: kmbazikova@mail.ru

Линейная стохастическая распределенная модель денежных накоплений в форме пространства состояний

В статье рассматривается задача пассивной параметрической идентификации систем для моделирования эволюции денежных накоплений доходов и расходов одного домохозяйства, с помощью линейной стохастической распределенной модели в форме пространства состояний с учетом белых шумов модели динамики исследуемого объекта и белых шумов линейной модели измерительной системы распределенного типа. Использование метода конечных разностей позволило свести решение уравнений с частными производными к решению системы линейных конечно-разностных и алгебраических уравнений, представленных моделями в форме пространства состояний. Для достоверного оценивания поведения объекта было предложено использование алгоритма калмановской фильтрации. Приведена постановка задачи оценивания коэффициентов уравнения эволюции денежных накоплений доходов и расходов одного домохозяйства. Описана структура доходов и расходов домохозяйства с учетом дополнительных аддитивных белых шумов измерителей. Рассмотрен алгоритм численной апробации методики по решению задачи оценивания коэффициентов уравнения в форме пространства состояний эволюции денежных накоплений доходов и расходов одного домохозяйства. Осуществлены расчеты с помощью математической системы Matlab на основе данных наблюдений за пять лет, взятых с сайта "Бюро национальной статистики Агентства по стратегическому планированию и реформам Республики Казахстан". Предложенная методика решения задачи пассивной идентификации оценивания коэффициентов уравнений денежных накоплений для одного домохозяйства в форме пространства состояний в достаточной степени универсальна.

Ключевые слова: линейное конечно-разностное уравнение, модель в пространстве состояний, эволюция денежных накоплений одного домохозяйства, пассивная идентификация, фильтр Калмана, оценки предсказания, оценки фильтрации.

1 Introduction

The identification of dynamic objects is one of the main directions of modern control theory. In this area, there are many works devoted mainly to the identification of linear dynamic objects [1]- [8]. Moreover, the well-known works cover a variety of situations that arise during identification: the presence of additive noise at the input and output of the object [9, 10], or the impossibility of submitting test signals to the input [6], discrete or continuous form of signals [7], correlation or uncorrelatedness of signals and interference [11, 12], etc. Naturally, these methods generally give good results when analyzing objects in the vicinity of "standard"modes. In all other cases, objects are presented as essentially nonlinear, and at present there are few or no general identification methods to describe them. But recently, partial differential equations are often used to describe the dynamics of the object under study.

Thus, in [13], a number of partial differential equations (PDE) are studied. They are based on models developed to study some of the most important economic issues. At the same time, they are very interesting for mathematicians, because their structure is often quite complex. This paper shows a number of examples of such PDEs, discusses what is known about their properties, and lists some open questions for future research. The paper [14,15] introduces and discusses a nonlinear market equation of the Boltzmann type, which describes the influence of knowledge on the evolution of wealth in a system of agents who interact through binary transactions. The article [16] presents a semigroup approach to the mathematical analysis of problems with inverse parameters when identifying unknown parameters in a linear parabolic equation with mixed boundary conditions. In [17], an estimate of the parameters of stochastic differential equations of the return to mean type caused by Brownian motion is shown. At the same time, when identifying dynamic objects and systems, models in the state space are used with the use of a modified Kalman filter. For example [18], a new bilinear model is introduced in the form of a state space. The development of this model is linear-bilinear with respect to the state of the system. The classical Kalman filter is not applicable to this model, and therefore a new Kalman filter is introduced. The identification of systems described by partial differential equations is considered in papers [19]- [24].

In that research, for modeling of one household money savings dynamics we are going to use a linear stochastic distributed model in the form of a state space (SS) that describes the dynamics of money savings of income and expenses in the form of linear differential equations with partial derivatives, but the model of measuring system in the form of linear distributed algebraic equations with additive white noises in both the dynamics model and the model of the measuring system. Then, we are going to present an economic interpretation of the values included in the proposed model in the form of SS [25, 26].

2 Materials and Methods

In reality, the household money savings have a discrete character: the household receives a salary and household savings in a form that increases spasmodically and does not change further until the nearest waste of money. With expenses (we will take into account the total expenses by the end of the month), household savings are abruptly decreased, that is savings are determined by a piecewise constant function of time.

As time passes, the point moves through the space of savings with rate $\frac{dx}{dt} = \dot{x}$. Suppose that rate can be calculated in another way, using additional terms, for example, as additive white noise dynamics of the savings income or expenses. The possibility of calculations in another way arises in a detailed study in the process of earnings and costs in the household. Let rate is expressed as a function like $F(x, t)$ - the function of two variables x, t and the additive white noise of money savings dynamics $w(t)$. As a result, we get the following relation:

$$\frac{dx(t)}{dt} = F(x(t), t) + w(t). \tag{1}$$

Equation (1) is a stochastic ordinary differential equation that describes the dynamics of household income or expenses; $x(t)$ is an unknown function; $F(x, t)$ - given function. If at the initial moment of time $t = 0$ savings of fixed household are known, then we have the initial condition:

$$x(t)\,|_{t=0} = \bar{x}_0 \;, \tag{2}$$

where $x(t)\,|_{t=0}$ is the white Gaussian value with mathematical expectation $\bar{x}_0$ and unknown variance P_0.

Relations (1) and (2) allow us to formulate the Cauchy problem for a stochastic ordinary differential equation. The type $F(x,t)$ of function in (1) depends on the particular household and on its additive white noises of the economic activity. The more accurately we write its analytical function $F(x,t)$ (based on statistical data), with the most reliable characteristics of additive noise, the more accurate will be the mathematical model. The function F can be represented globally in the form like $F = D - R$, $(D \geq 0, R \geq 0)$, where $D(x,t)$ is the function that describes the household income and the function $R(x,t)$ is the expenses of the household. In [25, 26], examples of defining the functions D and R are given.

1. The income structure of the household $D(x(t),t)$ taking into account the dynamics of the additive white noise of the investigated object: $D(x(t),t) = D_0(x(t),t) + D_1(x(t),t) + w_1(x(t),t)$, where

a) $D_0(x(t),t)$ - household wages. Suppose that additive noise $w_1(x(t),t)$ is some white distributed Gaussian noise with zero expectation and unknown variance Q.

b) $D_1(x(t),t)$ - solid income from investments in money savings in the bank.

$$D_1(x(t),t) = \alpha \cdot x(t) \cdot \theta(x(t), x_0). \tag{3}$$

Suppose a household invests all available money $x(t)$ in a bank on $p\frac{\%}{month}$.

Function $\theta(x(t), x_0)$ - threshold function (θ-function):

$$\theta(x(t), x_0) = \left\{ \begin{array}{ll} 0, & at\ x < x_0 \\ 1, & at\ x > x_0 \end{array} \right\}, \tag{4}$$

where x_0 is the minimum amount of savings that allows you to make an investment in the bank.

As a result, we get the household income function:

$$D(x(t),t) \approx D_0(x(t),t) + \alpha \cdot x(t) \cdot \theta\left(x\left(t\right), x_0\right) + w_1(x(t),t). \tag{5}$$

2. The structure of the household expenses $R(x(t),t)$, taking into account the additional additive white noise meters $w_2(t)$, can be written in the form: $R(x(t),t) \approx R_0(x(t),t) + R_1(x(t),t) + R_2(x(t),t) + w_2(x(t),t)$, where

a) $R_0(x(t),t)$ - average daily expenses to ensure the existence of the household. This part of the costs includes utility bills, average food costs, expenses for necessary clothes, transportation costs.

b) $R_1(x(t),t)$ - daily expenses that ensure the well-being of the household. This category of expenses is connected with the fact that if a household has surplus money, then it increases the cost of improving the quality of life.

c) $R_2(x(t),t)$ - expenses of elite goods. With sufficiently large savings, a household can allow the purchase of goods which are not the essential goods.

Thus, equation (1) has the form

$$\frac{dx}{dt} \approx D_0\left(x(t),t\right) + \alpha \cdot x(t) \cdot \tilde{\theta}\left(x,x_0\right) - R_0(x(t),t) - C_1\frac{x(t)}{x(t)+y_1}\tilde{\theta}\left(x,y_0\right) -$$

$$-C_2\frac{x(t)-z_1}{(x(t)-z_1)+(z_2-z_1)}\tilde{\theta}\left(x,z_1\right) + w(x(t),t), \tag{6}$$

where $w(x(t),\ t) = w_1(x(t),\ t) + w_2(x(t),\ t)$ - the additive generalized white noise.

The given mathematical description, taking into account white Gaussian income and expenses, based on statistical data, can be refined with a more detailed study of the economic activity of the household.

We write equation (6) in the differentials:

$$dx(t) = F(x(t),t)dt + w(x(t),t)dt. \tag{7}$$

In reality, a household, in addition to guaranteed income and expenses, may have random income and expenses. For a mathematical description of such a phenomenon, we introduce a random variable $X(t)$ that means the total amount of money where the household will save from random sources by that time t. The value $X(t+dt)$ is the total random savings of the household at the time $t+dt$, where dt is an infinitely small time interval.

$$dX = X(t+dt) - X(t), \tag{8}$$

which means random household income for an elementary period dt of time at $dX > 0$ and random expenditure $dX < 0$. We are going to call this value the stochastic differential of the random process $X(t)$. If we add the quantity (7) to equation (8), we obtain

$$dx = F\left(x,t\right)dt + dX + wdt. \tag{9}$$

Equation (9) is called a *stochastic differential equation*, where dX is not a differential in the usual sense. In a more general case, taking into account the additive noise of household money savings, relation (9) can be written as

$$dx = F\left(x,t\right)dt + G\left(x,t\right)dX + wdt, \quad G \geq 0. \tag{10}$$

Further, we suppose, that $F\left(x,t\right)$, $G\left(x,t\right)$ - nonrandom functions, but X - Markov's stochastic process.

We divide the time interval into elementary time intervals Δt_i. We denote it as $t_{i+1} = t_i + \Delta t_i$, $t_0 = 0$, $x_i = x\left(t_i\right)$, $X\left(t_i\right) = z_i$, then

$$x_{i+1} = F\left(x_i,t_i\right)\Delta t_i + G\ \left(x_i,t_i\right)\ z_{i+1} + w(x_i,t_i), \tag{11}$$

where random variables z_{i+1} are determined by probability density function and can be implemented numerically.

Households at each moment of time are distributed unevenly along the axis Ox. In [25], based on the application of the principle of continuous media and the introduction of a function, a linear equation with partial derivatives of a distributed type is obtained, which is satisfied by the density of households in the space of savings $u(x,t)$.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F \cdot u) = d \cdot f + p_1 \cdot w, \tag{12}$$

Equation (12) is called the linear equation of monetary accumulations of the ensemble of households, consisting of m classes taking into account the white additive noise of the dynamics of the object under study, the initial conditions (2) regarding the process of monetary accumulation, as well as the equations of the measuring system taking into account the white additive noise (12), in a complex is called a stochastic distributed equation in the form of a SS for each i-class, where $i = 1, 2, ..., m$.

Using the initial condition (2) and the difference scheme (11), we determine the approximate values $x(t)$ of the quantity at time instants $t = t_i$, then we determine one of the possible trajectories of a random variable $x(t)$.

To calculate a discrete analog of any equations in ordinary or partial derivatives is constantly used. Therefore, the task is to estimate the coefficients of equation (12) from observations of discrete input $z(x_k, t_s)$ and a discrete output $y(x_k,\ t_s)$ of a measuring system of appropriately distributed types, which can be written as

$$y(x_k, t_s) = h \cdot x(x_k, t_s) + p_2 \cdot \varepsilon(x_k, t_s), \quad k = \overline{1, n}; \quad s = \overline{0, m}, \tag{13}$$

where $y(x_k, t_s)$ is the output of the measuring system in which the indices k and s mean that the spatio-temporal state function $x(x,t)$ can be measured only at discrete spatial points x_k and at discrete time instants t_s, i.e $\left\{x(x,t) \approx x(x_k, t_s) = x_{k,s}, \quad k = \overline{1,\ n}, \quad s = \overline{1,\ m}\right\}$, h - a given weight coefficient to the measuring system; $\left\{y(x_k, t_s) = h \cdot x(x_k, t_s) = y_{k,s}, \quad k = \overline{1,\ n}, \quad s = \overline{1,\ m}\right\}$ - output of the measuring system; $\left\{\varepsilon(x_k, t_s) = \varepsilon_{k,s}, \quad k = \overline{1,\ n}, \quad s = \overline{1,\ m}\right\}$ - white Gaussian noise of a distributed type measuring system with zero mathematical expectation and unknown variance $p_2 = Q_2(x_k, t_s)$.

Under these conditions, the task is to estimate the parameters F, d, p_1, p_2 based on a distributed discrete input signal $\left\{u(x_k, t_s) = u_{k,s}, \quad k = \overline{1,\ n}, \quad s = \overline{1,\ m}\right\}$, initial conditions (2), as well as a distributed discrete output of the measuring system $\left\{y(x_k, t_s), \quad k = \overline{1,\ n}, \quad s = \overline{1,\ m}\right\}$.

3 Results and Discussion

In order to make the most reliable calculations for researching one household money savings based on the coefficients of the equations of one household money income and expenses, and subsequently, to get the most reliable estimates of the prediction and filtering behavior to the researched object, the scheme of the Kalman filter algorithm is used. We are going to look at the statistical data accumulated over the five years of 2014-2018 [27]. We give the brief calculation data for this example, which were carried out using the Matlab mathematical system based on the following algorithm:

1. The total number of research dates in months for five years - $n = 60$. At the first step of the algorithm, we conduct a linear regression for calculating the scatter based on research data and linear regression [28, 29]. The regression equation is constantly supplemented by a close coupling index, which allows making the most reliable variance calculations for dynamic models and measuring system. For $n = 60$, the coefficients of the regression model will be $a = -0.0089$; $b = 2474000$. The regression line on the graph is chosen so that the sum of the squares of the vertical distances between the points of the regression line and the observational data is minimal. Data on regular household solid income D_0: y_1 - observational data accumulated over five years and calculated data on household income z (based on $n = 60$ points) with calculated coefficients of the regression model made it possible to solve the problem of constructing a linear regression with minimal variance of residuals (see Fig. 1).

y_1=[56330 56419 59929 60913 61887 63025 64126 62873 61956 63107 63329 73362 61913 61824 61770 66499 66384 66320 68193 68106 68053 72985 72913 72895 71652 71638 71549 76263 76162 76084 76291 76200 76097 82343 82285 82339 79187 79111 79013 82485 82393 82307 83346 83286 83161 90188 90049 89992 86508 86425 86299 91191 91025 90960 94894 94789 94706 100374 100262 100184];

z=[56857 57526 58196 58865 59534 60204 60873 61542 62212 62881 63550 64220 64889 65558 66228 66897 67566 68236 68905 69574 70244 70913 71582 72252 72921 73590 74260 74929 75598 76268 76937 77606 78276 78945 79615 80284 80953 81623 82292 82961 83631 84300 84969 85639 86308 86977 87647 88316 88985 89655 90324 90993 91663 92332 93001 93671 94340 95009 95679 96348].

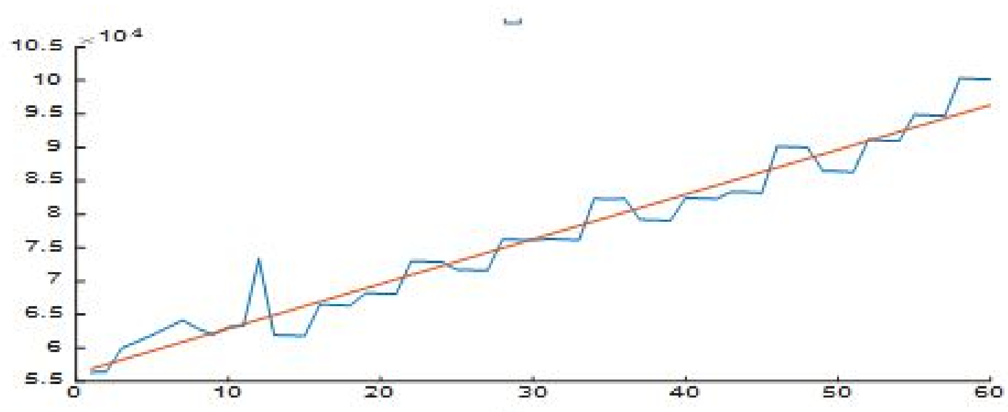


Figure 1: Regression line with the minimal residue variance

2. Then, we are going to reflect on a series of data, where each member of the series makes up the difference between the calculated values of the regression equation on the abscissa axis and the research data about $y = 0$ axes. The maximum line deviation of the regression equation from the research data is $x(12); max = 9142.3$ (see Fig. 2).

3. Further, the observational data is divided into two data sets $n_1 = 48$ and $n_2 = 12$. 48 observations were used to calculate the coefficients of the differential equation ($a = 24687$;

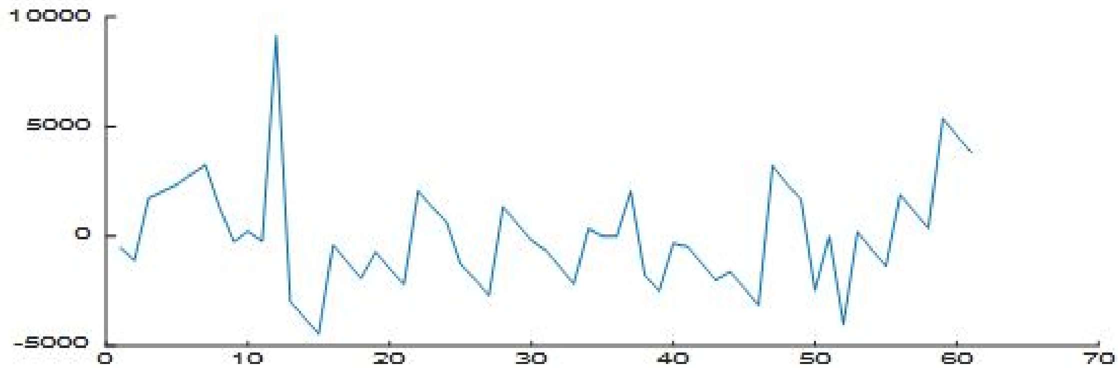


Figure 2: Calculation graph $x(12)$; $max = 9142.3$

b = 0.652;), as well as to calculate the variances of the model in the form of SS ($p_1 = Q_1$ = 25171 - variance for the dynamics model; variance for the model of the measuring system $p_2 = Q_2$= 50999; variance for the initial moment of time $P_0 = Q_1$ in the Kalman filter algorithm) [30,31]. The remaining 12 observations are used to calculate prediction estimates and adjusted filtering estimates based on equations from the Kalman filter algorithm under initial conditions with respect to the state $\mathrm{x}_0 = \bar{x}_0$ and variance at the initial time $P_0 = Q_1$ [31,32].

The calculation data of prediction and filtering assessment according to the Kalman scheme (see Fig. 3):

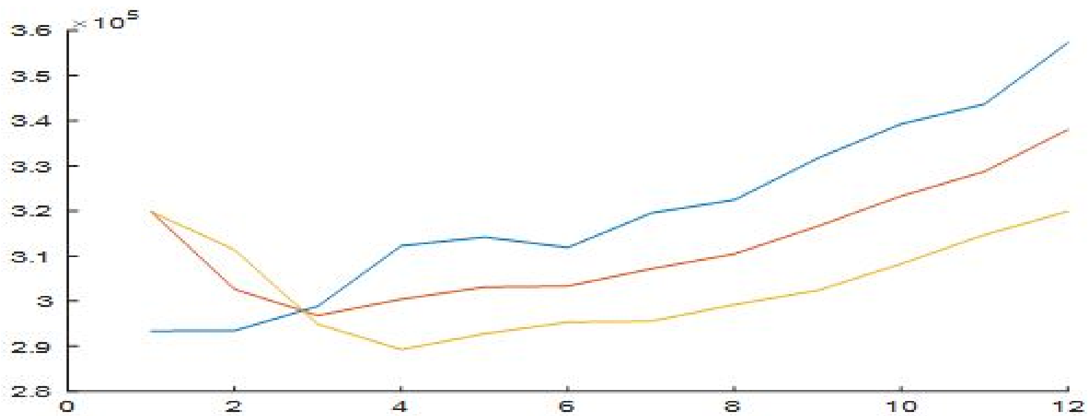


Figure 3: Graphs of prediction (x_p), filtration (x_f) and observational estimates (y_s)

y_s=[79187 86425 86299 91191 91025 90960 94894 94789 94706 100374 100262 100184]: observational data;

x_f=[79187 80491 80828 82895 83635 83899 85571 86184 86392 88733 89606 89917]: estimates of filtration;

x_p=[79187 76317 77167 77387 78735 79217 79389 80479 80879 81015 82541 83110]: prediction estimates.

4 Conclusion

The proposed techniques for solving the problem of coefficient assessment's passive identification using the equations of money savings for one household in the form of SS is sufficiently simple and universal. By analogy with the research techniques that was used for the data from the section «Regular solid household income D_0», other researches can be carried out in other sections: «Total solid expenses R», «Household living expenses R_0», «Household well-being expenses R_1», «Expenses of elite goods R_2». Regarding the last section, we note that in section R_2, it is constantly important to conduct a regular calculation to the total amount of savings at the time of purchase of the elite item and if this total amount exceeds the cost of the elite item R_2, then we must make the purchase of this item.

To clarify the values of the coefficients included in the regression equation, as well as in linear differential equations, there are many other possibilities that refine these coefficients. In particular, to such leverages, which can increase the accuracy coefficient assessment, according to the Kalman scheme, the following items can be attributed:

1. Increase in sample size relative to research data due to the increase in sample size by the growth of the data amount in months;

2. For interior points, more accurate approximation formulas can be used

$$\left(\frac{\partial u}{\partial x}\right)_k \approx \frac{u_{k+1} - u_{k-1}}{2 \cdot h}$$

3. The accumulated information about the estimates of the desired parameters, which ultimately would allow using the ideas of Bayesian parameter estimation [32, 33].

References

[1] Schoukens J., Pintelon R., *Identification of Linear Systems: A Practical Guideline to Accurate Modeling,* (Pergamon, 1991), 354.

[2] Van Den Hof P., Wahlberg B., Weiland S., *System Identification, 1st Edition*, (Elsevier Science, 2004), 2088.

[3] Bensoussan A. et al., "Identification of linear dynamical systems and machine learning", *Optimization and Control* (2020): 1-26, https://www.semanticscholar.org/paper/Identification-of-linear-dynamical-systems-and-Bensoussan-Gelir/141e7250c785c093fec97b4cd51245c307c9e122.

[4] Ramirez R.I., Montejo L.A., "On the identification of damping from non-stationary free decay signals using modern signal processing techniques", *Int J Adv Struct Eng* 7 (2015): 321–328, https://doi.org/10.1007/s40091-015-0096-3.

[5] Wagenmaker A., Jamieson K., "Active Learning for Identification of Linear Dynamical Systems", *Proceedings of Machine Learning Research* vol. 125 (2020): 1-96, https://arxiv.org/abs/2002.00495.

[6] Diblik J., Dzhalladova I., Michalkova M. et al., "Modeling of applied problems by stochastic systems and their analysis using the moment equations", *Adv Differ Equ* 152 (2013), https://doi.org/10.1186/1687-1847-2013-152.

[7] Xu L., "Machine learning and causal analyses for modeling financial and economic data", *Appl Inform* 5, article number: 11 (2018). https://doi.org/10.1186/s40535-018-0058-5.

[8] Wang D., Gao Y., "Recursive maximum likelihood identification method for a multivariable controlled autoregressive moving average system", *IMA Journal of Mathematical Control and Information* vol. 33 (2016): 1015-1031, https://doi.org/10.1093/imamci/dnv021.

[9] Sircar P., Dutta M.K., Mukhopadhyay S., "Signal parameter estimation of complex exponentials using fourth order statistics: additive Gaussian noise environment", *SpringerPlus* 4, article number: 360 (2015), https://doi.org/10.1186/s40064-015-1131-3.

[10] Cui J., Li Y., Yan L., "Temporal variation for fractional heat equations with additive white noise", *Bound Value Probl* 123 (2016), https://doi.org/10.1186/s13661-016-0630-7.

[11] Zhang E., Pintelon R., "Identification of dynamic errors-in-variables systems with guasi-stationary input and colored noise", *Automatica* 123 (2020):1-10.

[12] Sharabati W., Xi B., "A neighborhood regression approach for removing multiple types of noises", *J Image Video Proc.* 19 (2018), https://doi.org/10.1186/s13640-018-0259-9.

[13] Achdou Y., Buera F.J., Lasry J-M., Lions P-L., Moll B., "Partial differential equation models in macroeconomics", *Phil. Trans. R. Soc.* A372:20130397 (2014), http://dx.doi.org/10.1098/rsta.2013.0397.

[14] Pareschi L., Toscani G., "Wealth distribution and collective knowledge: a Boltzmann approach", *Phil. Trans. R. Soc.* A 372:20130396 (2014), http://dx.doi.org/10.1098/rsta.2013.0396.

[15] Albi G., Pareschi L., Zanella M., "Boltzmann-type control of opinion consensus through leaders", *Phil. Trans. R. Soc.* A372:20140138 (2014), http://dx.doi.org/10.1098/rsta.2014.0138.

[16] Ozbilge E., Demir A., "Identification of the unknown diffusion coefficient in a linear parabolic equation via semigroup approach", *Adv Differ Equ* 47 (2014), https://doi.org/10.1186/1687-1847-2014-47.

[17] Li J., Wu JL., "On drift parameter estimation for mean-reversion type stochastic differential equations with discrete observations", *Adv Differ Equ* 90 (2016), https://doi.org/10.1186/s13662-016-0819-1.

[18] Al-Mazrooei A., Al-Mutawa J., El-Gebeily M. et al., "Filtering and identification of a state space model with linear and bilinear interactions between the states", *Adv Differ Equ* 176 (2012), https://doi.org/10.1186/1687-1847-2012-176.

[19] Boikov I.V., Krivulin N.P., "Identifikatsiia sistem, opisyvaemykh differentsial'nymi uravneniiami v chastnykh proizvodnykh [Identification of systems described by partial differential equations]", *Mater. VII mezhdunar. nauch. konf. «Matematicheskoe i komp'iuternoe modelirovanie estestvennonauchnykh i sotsial'nykh problem»,* Penza (2012): 23–27.

[20] Krivulin N.P., "Opredelenie parametrov fizicheskikh protsessov, opisyvaemykh differentsial'nymi uravneniiami v chastnykh proizvodnykh s peremennymi koeffitsientami [Determination of the parameters of physical processes described by partial differential equations with variable coefficients]", *Mater. VIII mezhdunar. nauch. konf. «Matematicheskoe i komp'iuternoe modelirovanie estestvennonauchnykh i sotsial'nykh problem»,* Penza (2014): 172–178.

[21] Boikov I.V., Krivulin N.P., "Parametricheskaia identifikatsiia sistem, matematicheskie modeli kotorykh opisyvaiutsia differentsial'nymi uravneniiami s proizvodnymi drobnykh poriadkov[Parametric identification of systems whose mathematical models are described by differential equations with derivatives of fractional orders]", *Metrologiia,* No 9 (2013): 3–17.

[22] Boikov I.V., Krivulin N.P., "Parametricheskaia identifikatsiia ereditarnykh sistem s raspredelennymi parametrami [Parametric identification of distributed parameter hereditary systems]", *Izvestiia vysshikh uchebnykh zavedenii. Tekhnicheskie nauki,* No 2(26) (2013): 120–129.

[23] Ivanov D.V., "Identifikatsiia diskretnykh lineinykh dinamicheskikh sistem drobnogo poriadka s oshibkoi na vykhode [Identification discrete fractional order linear dynamic systems with output error]", *Mater. mezhdunar. sibirskoi konf. po upravleniiu i sviazi (SIBCON),* Krasnoiarsk (2013): 1-3, doi: 10.1109 / SIBCON.2013.6693623.

[24] Vasil'ev V.A., Konev V.V., "Posledovatel'naia identifikatsiia lineinykh dinamicheskikh sistem v nepreryvnom vremeni po shumnym nabliudeniiam [Sequential identification of linear dynamical systems in continuous time from noisy observations]", *Problemy upravleniia i teorii informatsii,* No 16(2) (1987): 101-112.

[25] Yerofeenko V.T., Kozlovskaia I.S., *Uravneniia s chastnymi proizvodnymi i matematicheskie modeli v ekonomike* [Equations with private derivatives and mathematical models in economics] (Minsk: Izd-vo Bel. gos. un-ta, 2004), 245.

[26] Chernavskii D.S., Popkov Iu.S., Rakhimov A.Kh., "Matematicheskie modeli tipologii semeinykh nakoplenii [Mathematical models of family savings typology]", *Ekonomika i matematicheskie metody,* vol.30, is. 2 (1994): 98-106.

[27] Statistika urovnia zhizni [Statistics of the standard of living], accessed December 8, 2020, https://stat.gov.kz/official/industry/64/statistic/7.

[28] Kopchenova N.V., Maron I.A., *Vychislitelnaia matematika v primerakh i zadachakh* [Computational mathematics in examples and tasks] (M.: Nauka, 2009), 370.

[29] Eliseeva I.I. (Eds), *Ekonometrika: Uchebnik* [Econometrics: Textbook] (M.: Finansy i statistika. 2001), 344.

[30] Abdenova G.A., Voevoda A.A., "Otsenivanie parametrov i kharakteristik shumov nestatsionarnykh protsessov v stokhasticheskikh sistemakh, opisyvaemykh v prostranstve sostoianii [Estimation of parameters and characteristics of noises of non-serial processes in stochastic systems described in the space of states]", *Sb. nauch. trudov NGTU,* No 3 (61) (2010): 11-18.

[31] Abdenova G.A., "Prognozirovanie znachenii urovnia vremennogo riada na osnove uravnenii filtra Kalmana [Prediction of time series level values based on Kalman filter equations]", *Polzunovskii vestnik,* No 2 (2010): 4-6.

[32] Abdenov A., Abdenova G., Kulbaev D., "Estimation of equation coefficients of free and forced string vibrations in continuous medium with consideration of dynamic noise and measurement", *J. Materials Today: Procedings* vol. 16 (2019): 336-342.

[33] Gorskii V.G., Adler Iu.P., Talalai A.M., *Planirovanie promyshlennykh eksperimentov. Modeli dinamiki* [Planning of industrial experiments. Models of dynamics] (M.: Metallurgiia, 1978), 112.

IRSTI 20.23.17 DOI: https://doi.org/10.26577/JMMCS.2021.v110.i2.12

A.T.Rakhymova[1*], **M.B.Gabbassov**[2], **K.M.Shapen**[1]
[1]L.N. Gumilyov Eurasian National University, Kazakhstan, Nur-Sultan
[2]System research company «Factor», Kazakhstan, Nur-Sultan
*E-mail: aigerim_rakhimova@mail.ru

FUNCTIONS IN ONE SPACE OF FOUR-DIMENSIONAL NUMBERS

For the first time, the theory of functions of four-dimensional numbers with commutative product was described in works of Abenov M.M., in which the mathematical apparatus was defined, algebraic operations and their properties were determined, functions of four-dimensional numbers, their limits, continuity and differentiability were found. The continuation was the joint work of Abenov M.M. and Gabbasov M.B., where similar anisotropic four-dimensional spaces (with the notation M2-M7) were defined, which are also commutative with zero divisors. This work is devoted to the study of functions of a four-dimensional variable, definitions and analysis of four-dimensional functions, their properties, as well as the regularity of functions. The purpose of this work is to analyze the definition of functions of four-dimensional variables of the space M5, as well as theorems on the continuity and existence of differentiability of functions of four-dimensional variables. This work is descriptive for comparing the spaces of four-dimensional numbers M5 and M3. In the article, theorems on the continuity and differentiability of functions of four-dimensional variables and their properties are proved, and the Cauchy-Riemann conditions are found. The form of trigonometric, exponential, logarithmic, exponential and power functions of four-dimensional variables is determined and the regularity of functions of M5 space is proved.
Key words: four-dimensional function, continuity, differentiability, regular function, Cauchy-Riemann condition.

А.Т.Рахымова[1*], М.Б.Габбасов[2], К.М.Шапен[1]
[1]Л.Н. Гумилев атындағы Еуразия ұлттық университеті, Қазақстан, Нұр-Сұлтан қ.
[2] «Factor» жүйелік зерттеулер компаниясы, Қазақстан, Нұр-Сұлтан қ.
*E-mail: aigerim_rakhimova@mail.ru

ТӨРТ ӨЛШЕМДІ САНДАР КЕҢІСТІГІНДЕГІ ФУНКЦИЯЛАР

Төрт өлшемді сандардың функциялар теориясы алғаш рет М.М.Абеновтың еңбектерінде сипатталған, онда математикалық аппарат анықталған, алгебралық амалдар және олардың ауыстырымдылық көбейтіндісі, қасиеттері анықталған, сонымен қатар төрт өлшемді сандардың функциялары, олардың шектері, үзіліссіздігі және дифференциалдануы зерттелді. Бұл жұмыстың жалғасы М.М. Әбенов пен М.Б. Ғаббасовтың бірлескен зетрреу мақаласы болды, ол жерде ұқсас нөлдік бөлгіштері бар коммутативті болатын анизотропты төртөлшемді кеңістіктер (М2-М7 белгісімен белгіленген) анықталған. Бұл жұмыс төртөлшемді айнымалылы функцияларын, сол функциялардың анықтамалары мен талдауларын, олардың қасиеттерін, сонымен қатар олардың регулярлығын зерттеуге арналған. Бұл жұмыстың мақсаты М5 кеңістігінің төрт өлшемді айнымалыларының функцияларының анықтамасын, сондай-ақ төрт өлшемді айнымалылар функцияларының дифференциалдануы мен үздіксіздігі туралы теоремаларды талдау болып табылады. Бұл жұмыс М5 және М3 төрт өлшемді сандарының кеңістіктерін салыстыру арқылы жүзеге асырылған. Мақалада төрт өлшемді айнымалылар функцияларының үздіксіздігі мен дифференциалдылығы, олардың қасиеттері туралы теоремалар дәлелденген, Коши-Риман шарттары анықталған. Төрт өлшемді айнымалылардың тригонометриялық, экспоненциалдық, логарифмдік, көрсеткіштік және қуаттық функциясының түрлері анықталған, М5 кеңістігінің төртөлшемді айнымалыларының функцияларының заңдылығы дәлелденген.
Түйін сөздер: төртөлшемді функция, үзіліссіздік, дифференциалдылық, регуляр функция, Коши-Риман шарты.

А.Т.Рахымова[1*], М.Б.Габбасов[2], К.М.Шапен[1]
[1]Евразийский национальный университет имени Л.Н. Гумилева, Казахстан, г. Нур-Султан
[2]Компания системных исследований «Factor», Казахстан, г. Нур-Султан
[*]E-mail: aigerim_rakhimova@mail.ru

ФУНКЦИИ В ОДНОМ ПРОСТРАНСТВЕ ЧЕТЫРЕХМЕРНЫХ ЧИСЕЛ

Впервые теория функций четырехмерных чисел с коммутативным произведением были описаны в работах Абенова М.М., в которых был определен математический аппарат, определены алгебраические операции и их свойства, были найдены функции четырехмерных чисел, их пределы, непрерывность и дифференцируемость. Продолжением была совместная работа Абенова М.М. и Габбасова М.Б., где были определены подобные анизотропные четырехмерные пространства (с обозначениями M2-M7), которые также являются коммутативными с делителями нуля. Данная работа посвящена изучению функций четырехмерного переменного, определений и анализа четырехмерных функций, их свойств, а также регулярности функций. Целью данной работы являются анализ определения функций четырехмерных переменных пространства M5, а также теоремы о непрерывности и существования дифференцируемости функций четырехмерных переменных. Данная работа имеет описательный характер для сравнения пространств четырехмерных чисел M5 и M3. В статье доказаны теоремы о непрерывности и дифференцируемости функций четырехмерных переменных, их свойства, а также найдены условия Коши-Римана. Определен вид тригонометрических, экспоненциальной, логарифмической, показательной и степенной функций четырехмерных переменных и доказана регулярность функций четырехмерных переменных пространства M5.

Ключевые слова: четырехмерная функция, непрерывность, дифференцируемость, регулярная функция, условие Коши-Римана.

1 Introduction

The existence of the theory of functions of four-dimensional numbers originates from the investigations of M.M. Abenov, where four-dimensional numbers, functions of four-dimensional numbers, their limit, continuity and differentiability were found [1]. In work [2], Abenov M.M. and Gabbasov M.B. identified all the existing six (M2, M3, M4, M5, M6, M7) anisotropic four-dimensional spaces, which are also associative and commutative with zero divisors. In paper [3], the space of four-dimensional numbers M5 was investigated, where algebraic operations on four-dimensional numbers were described, the eigenvalues for finding the norm were found, and the metric is defined. In this paper we study the concept of the four-dimensional function in the space M5, their continuity and differentiability, as well as analysis of their properties.

2 Material and methods

It is known from the researches of many authors [4-9] that complex analysis is an extension of real analysis, i.e. all mathematical operations, definitions, functions, their properties, differentiability and continuity are performed by analogy with real analysis. In papers [2-3], complex analysis is generalized by the analysis of functions of four-dimensional variables. Let us define functions, their properties, continuity and differentiability of functions in the four-dimensional space M5.

2.1 Functions given in the space of four-dimensional numbers

Definition 1 *A function of a four-dimensional variable of the space M5 is a mapping F of some four-dimensional number from the set D into a four-dimensional number of the set G.*

If only one value $X \in D$ corresponds to each value of $Y \in G$, then the function is called single-valued, if more than one value of Y corresponds to some X, then the function is called multivalued.

To describe the function, we use the notation $Y = F(X)$ and define functions that map four-dimensional numbers to four-dimensional numbers, that is $F : R^4 \to R^4$.

Definition 2 *Let a function $f : C \to C$ has the following property: if $f(x + yi) = c(x, y) + d(x, y)i$, then $f(x - yi) = c(x, y) - d(x, y)i$, that is, it maps complex conjugate numbers to complex conjugate numbers. Let us call such functions self-adjoint functions.*

Theorem 1 *Let the function $f(x + yi) = c(x, y) + d(x, y)i : C \to C$ be differentiable, $c(x, y) = c(x, -y)$ for $\forall (x, y) \in C$ and there is a point $(x_0, 0) \in C$ such that $d(x_0, 0) = 0$. Then it is self-adjoint.*

Proof. Since the function f is differentiable, then it satisfies the Cauchy-Riemann conditions

$$\frac{\partial c(x,y)}{\partial x} = \frac{\partial d(x,y)}{\partial y}$$

$$\frac{\partial c(x,y)}{\partial y} = -\frac{\partial d(x,y)}{\partial x}$$

and the function $f(x - yi) = c(x, -y) + d(x, -y)i$ satisfies the following conditions

$$\frac{\partial c(x,-y)}{\partial x} = -\frac{\partial d(x,-y)}{\partial y}$$

$$\frac{\partial c(x,-y)}{\partial y} = \frac{\partial d(x,-y)}{\partial x}$$

considering that $c(x, y) = c(x, -y)$ and equating the right sides, we get

$$\frac{\partial d(x,y)}{\partial y} = -\frac{\partial d(x,-y)}{\partial y}$$

$$-\frac{\partial d(x,y)}{\partial x} = \frac{\partial d(x,-y)}{\partial x}$$

that is

$$\begin{array}{c} \dfrac{\partial\left(d(x,y) + \partial d(x,-y)\right)}{\partial y} = 0 \\ \dfrac{\partial\left(d(x,y) + d(x,-y)\right)}{\partial x} = 0 \end{array}.$$

Therefore, $d(x, y) = -d(x, -y) + \text{const}$.
Substituting $x = x_0, y = 0$ we get const $= 0$ which corresponds to the equality

$$f(x, -y) = c(x, y) - d(x, y)i.$$

The theorem is proved.

Remark 1 *There is a differentiable mapping* $f : C \to a + ib =$ const, *which has a generalization to the four-dimensional space M5* $F : R^4 \to (a, b, 0, 0) =$ const.

Consequence 1 *Any complex polynomial with zero intercept is a self-adjoint function.*

Proof. As a point $(x_0, 0) \in C$ it is sufficient to take the point *(0,0)*.

Further, we extend the definition of a self-adjoint function to the four-dimensional space.

Let $S : R^4 \to C^4$ be a bijection assigning to each four-dimensional number (x_1, x_2, x_3, x_4) its spectrum $(\mu_1, \mu_2, \mu_3, \mu_4)$, defined in [4]

$$\mu_1 = x_1 - x_4 + (x_2 + x_3)\, i, \quad \mu_2 = x_1 - x_4 - (x_2 + x_3)\, i,$$
$$\mu_3 = x_1 + x_4 + (x_2 - x_3)\, i, \quad \mu_4 = x_1 + x_4 - (x_2 - x_3)\, i. \tag{1}$$

For each component of the spectrum μ_i, we can apply the function *f(*μ*)* and denote the resulting numbers by $f(\mu_1) = f_1 + f_2 i$, $f(\mu_2) = f_1 - f_2 i$, $f(\mu_3) = f_3 + f_4 i$, $f(\mu_4) = f_3 - f_4 i$. It is easy to understand that this number is the spectrum of some four-dimensional number $Y = (y_1, y_2, y_3, y_4)$, the elements of which are found as follows:

$$y_1 = \frac{f_1 + f_3}{2},\ y_2 = \frac{f_2 + f_4}{2},\ y_3 = \frac{f_2 - f_4}{2},\ y_4 = \frac{-f_1 + f_3}{2}. \tag{2}$$

Thus, we have defined a function $F(X)$ that assigns to each $X = (x_1, x_2, x_3, x_4) \in R^4$ the number $Y = (y_1, y_2, y_3, y_4) \in R^4$. The function defined in this way can be briefly written as

$$F(X) = S^{-1}(f(S(X)), \tag{3}$$

where $f(\mu_1, \mu_2, \mu_3, \mu_4)$ means $(f(\mu_1), f(\mu_2), f(\mu_3), f(\mu_4))$, and S^{-1} is the inverse function to S.

Theorem 2 *The function defined by equality* (3) *in the space M5 is a generalization of the functions of real and complex analysis.*

Proof.

1. Consider the real analysis case. Let $X = (x_1, 0, 0, 0) \in R^1$, then by (1)

$$\mu_1 = \mu_2 = \mu_3 = \mu_4 = x_1.$$

Applying the mapping $f(\mu) : C \to C$ we obtain

If $f(x_1 + 0i) = f_1 + f_2 i$ and $f(x_1 - 0i) = f_1 - f_2 i$, then $f(x_1) = f_1 + f_2 i = f_1 - f_2 i = f_1$ and $f_2 = 0$

$$f(\mu_1) = f(\mu_2) = f(\mu_3) = f(\mu_4) = f(x_1) = f_1.$$

By formula (2), we obtain

$y_1 = f_1,\ y_2 = y_3 = y_4 = 0$ or $Y = (f_1, 0, 0, 0)$.

2. Consider the complex analysis case. Let $X = (x_1, x_2, 0, 0) \in C$, then by (1)

$$\mu_1 = \mu_3 = x_1 + x_2 i, \mu_2 = \mu_4 = x_1 - x_2 i.$$

Applying the mapping $f(\mu) : C \to C$ and, due to the complex conjugacy of the function, we obtain

$$f(\mu_1) = f(\mu_3) = f_1 + f_2 i, f(\mu_2) = f(\mu_4) = f_1 - f_2 i.$$

By formula (3), we obtain
$y_1 = f_1, \quad y_2 = f_2, \quad y_3 = y_4 = 0$ or $Y = (f_1, f_2, 0, 0)$.
The theorem is proved.

This theorem states that if we take $(x_1, 0, 0, 0)$ as the argument of the function f, then their image will be numbers of the form $(f_1, 0, 0, 0)$ and this function coincides with the corresponding one-dimensional function. And if we take $(x_1, x_2, 0, 0)$ as the argument of the function f, then it coincides with the corresponding complex-valued function from which it generated.

Let us define the form of elementary functions in the space M5. Consider a complex exponential function that is a self-adjoint function. Let $X = (x_1, x_2, x_3, x_4)$ be a four-dimensional number. Then the spectrum of this number $S(X) = ((x_1 - x_4) + (x_2 + x_3) i, \ (x_1 - x_4) - (x_2 + x_3) i, \ (x_1 + x_4) + (x_2 - x_3) i, (x_1 + x_4) - - (x_2 - x_3) i)$. Let us apply an exponent to each component.

$$\begin{aligned}
exp\left((x_1 - x_4) + (x_2 + x_3) i\right) &= exp\,(x_1 - x_4) \cos(x_2 + x_3) + exp\,(x_1 - x_4) \sin(x_2 + x_3) i \\
exp\left((x_1 - x_4) - (x_2 + x_3) i\right) &= exp\,(x_1 - x_4) \cos(x_2 + x_3) - exp\,(x_1 - x_4) \sin(x_2 + x_3) i \\
exp\left((x_1 + x_4) + (x_2 - x_3) i\right) &= exp\,(x_1 + x_4) \cos(x_2 - x_3) + exp\,(x_1 + x_4) \sin(x_2 - x_3) i \\
exp\left((x_1 + x_4) - (x_2 - x_3) i\right) &= exp\,(x_1 + x_4) \cos(x_2 - x_3) - exp\,(x_1 + x_4) \sin(x_2 - x_3) i
\end{aligned}$$

consequently

$$\left\{ \begin{array}{l} f_1 = exp\,(x_1 - x_4) \cos(x_2 + x_3) \\ f_2 = exp\,(x_1 - x_4) \sin(x_2 + x_3) \\ f_3 = exp\,(x_1 + x_4) \cos(x_2 - x_3) \\ f_4 = exp\,(x_1 + x_4) \sin(x_2 - x_3) \end{array} \right. .$$

Then, by formula (1), we obtain

$$exp\,(X) = \frac{1}{2} \left(\begin{array}{c} exp\,(x_1 - x_4) \cos(x_2 + x_3) + exp\,(x_1 + x_4) \cos(x_2 - x_3) \\ exp\,(x_1 - x_4) \sin(x_2 + x_3) + exp\,(x_1 + x_4) \sin(x_2 - x_3) \\ exp\,(x_1 - x_4) \sin(x_2 + x_3) - exp\,(x_1 + x_4) \sin(x_2 - x_3) \\ -exp\,(x_1 - x_4) \cos(x_2 + x_3) + exp\,(x_1 + x_4) \cos(x_2 - x_3) \end{array} \right).$$

Define the logarithmic function $U(X) = Ln(X)$ through the transformation $X = exp(U)$. Let us write this formula componentwise

$$\begin{gathered} x_1 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right) \\ x_2 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) sinu_2 - u_3\right) \\ x_3 = \tfrac{1}{2}\left(exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)\right) \\ x_4 = \tfrac{1}{2}\left(-exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right) \end{gathered}$$

Hence, by simple calculations, we obtain

$$x_1 + x_4 = \frac{1}{2}\left(exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right.$$

$$\left.-exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right)$$

$$x_2 + x_3 = \frac{1}{2}\left(exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) sin\left(u - u_3\right)\right.$$

$$\left.+exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)\right)$$

$$x_1 - x_4 = \frac{1}{2}\left(exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right.$$

$$\left.+exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right) - exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right)\right)$$

$$x_2 - x_3 = \frac{1}{2}\left(exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) + exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)\right.$$

$$\left.-exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right) \ + exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)\right)$$

$$x_1 + x_4 = exp\left(u_1 + u_4\right) cos\left(u_2 - u_3\right), x_2 + x_3 = exp\left(u_1 - u_4\right) sin\left(u_2 + u_3\right)$$

$$x_1 - x_4 = exp\left(u_1 - u_4\right) cos\left(u_2 + u_3\right), x_2 - x_3 = exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)$$

$$\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2 = exp\left(2(u_1 - u_4)\right) cos^2\left(u_2 + u_3\right) + exp\left(2(u_1 - u_4)\right) sin^2\left(u_2 + u_3\right)$$

$$= exp\left(2\left(u_1 - u_4\right)\right)\left(cos^2\left(u_2 + u_3\right) + sin^2\left(u_2 + u_3\right)\right) = exp\left(2\left(u_1 - u_4\right)\right)$$

$$\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2 = exp\left(2(u_1 + u_4)\right) cos^2\left(u_2 - u_3\right) + exp\left(2(u_1 + u_4)\right) sin^2\left(u_2 - u_3\right)$$

$$= exp\left(2\left(u_1 + u_4\right)\right)\left(cos^2\left(u_2 - u_3\right) + sin^2\left(u_2 - u_3\right)\right) = exp\left(2\left(u_1 + u_4\right)\right)$$

Further, simplifying the above formulas, we calculate the components of the function $U = (u_1, u_2, u_{3,}u_4)$. From the relations

$$exp\left(u_1 - u_4\right) = \sqrt{\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2}, exp\left(u_1 + u_4\right) = \sqrt{\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2}$$

we obtain

$$u_1 = ln\sqrt[4]{\left[\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2\right]\left[\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2\right]}$$

$$u_4 = \frac{1}{4} ln\frac{\left(x_1 + x_4\right)^2 + \left(x_2 - x_3\right)^2}{\left(x_1 - x_4\right)^2 + \left(x_2 + x_3\right)^2}$$

From the relations

$$x_2 - x_3 = exp\left(u_1 + u_4\right) sin\left(u_2 - u_3\right)$$

$$x_1 + x_4 = exp\,(u_1 + u_4)\,cos\,(u_2 - u_3)$$

obtain

$$\frac{sin\,(u_2 - u_3)}{cos\,(u_2 - u_3)} = tg\,(u_2 - u_3) = \frac{x_2 - x_3}{x_1 + x_4}.$$

And from

$$x_2 + x_3 = exp\,(u_1 - u_4)\,sin\,(u_2 + u_3)$$

$$x_1 - x_4 = exp\,(u_1 - u_4)\,cos\,(u_2 + u_3)$$

obtain

$$\frac{sin\,(u_2 + u_3)}{cos\,(u_2 + u_3)} = tg\,(u_2 + u_3) = \frac{x_2 + x_3}{x_1 - x_4}.$$

Simplifying the expressions and due to the periodicity of the trigonometric functions, we obtain

$$u_2 = \frac{1}{2}(arctg\frac{x_2 + x_3}{x_1 - x_4} + arctg\frac{x_2 - x_3}{x_1 + x_4}) + 2\pi k, k = 0, \pm 1, \pm 2, \ldots$$

$$u_3 = \frac{1}{2}(arctg\frac{x_2 + x_3}{x_1 - x_4} - arctg\frac{x_2 - x_3}{x_1 + x_4}) + 2\pi k, k = 0, \pm 1, \pm 2, \ldots$$

Thus [12]

$$Ln\,(X) = \begin{pmatrix} ln\sqrt[4]{\left[(x_1 + x_4)^2 + (x_2 - x_3)^2\right]\left[(x_1 - x_4)^2 + (x_2 + x_3)^2\right]} \\ \frac{1}{2}\left(arctg\frac{x_2 + x_3}{x_1 - x_4} + arctg\frac{x_2 - x_3}{x_1 + x_4}\right) + 2\pi k \\ \frac{1}{2}\left(arctg\frac{x_2 + x_3}{x_1 - x_4} - arctg\frac{x_2 - x_3}{x_1 + x_4}\right) + 2\pi k \\ \frac{1}{4}ln\frac{(x_1 + x_4)^2 + (x_2 - x_3)^2}{(x_1 - x_4)^2 + (x_2 + x_3)^2} \end{pmatrix}$$

where $k = 0, \pm 1, \pm 2, \ldots$ It is a multivalued function, as in the complex analysis.

In a similar way, the following elementary functions can be defined

$$sin\,(X) = \frac{1}{2}\begin{pmatrix} sin\,(x_1 - x_4)\,ch\,(x_2 + x_3) + sin\,(x_1 + x_4)\,ch\,(x_2 - x_3) \\ cos\,(x_1 - x_4)\,sh\,(x_2 + x_3) + cos\,(x_1 + x_4)\,sh\,(x_2 - x_3) \\ cos\,(x_1 - x_4)\,sh\,(x_2 + x_3) - cos\,(x_1 + x_4)\,sh\,(x_2 - x_3) \\ -sin\,(x_1 - x_4)\,ch\,(x_2 + x_3) + sin\,(x_1 + x_4)\,ch\,(x_2 - x_3) \end{pmatrix},$$

$$cos\,(X) = \frac{1}{2}\begin{pmatrix} cos\,(x_1 - x_4)\,ch\,(x_2 + x_3) + cos\,(x_1 + x_4)\,ch\,(x_2 - x_3) \\ -sin\,(x_1 - x_4)\,sh\,(x_2 + x_3) - sin\,(x_1 + x_4)\,sh\,(x_2 - x_3) \\ -sin\,(x_1 - x_4)\,sh\,(x_2 + x_3) + sin\,(x_1 + x_4)\,sh\,(x_2 - x_3) \\ -cos\,(x_1 - x_4)\,ch\,(x_2 + x_3) + cos\,(x_1 + x_4)\,ch\,(x_2 - x_3) \end{pmatrix},$$

$$sh\,(X) \;=\frac{1}{2}\left(\begin{array}{c} sh\,(x_1-x_4)\,cos\,(x_2{+}x_3)+sh\,(x_1+x_4)\,cos\,(x_2{-}x_3)\\ ch\,(x_1-x_4)\,sin\,(x_2{+}x_3)+ch\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ ch\,(x_1-x_4)\,sin\,(x_2{+}x_3)-ch\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ -sh\,(x_1-x_4)\,cos\,(x_2{+}x_3)+sh\,(x_1+x_4)\,cos\,(x_2{-}x_3)\end{array}\right),$$

$$ch\,(X) \;=\frac{1}{2}\left(\begin{array}{c} ch\,(x_1-x_4)\,cos\,(x_2{+}x_3)+ch\,(x_1+x_4)\,cos\,(x_2{-}x_3)\\ sh\,(x_1-x_4)\,sin\,(x_2{+}x_3)+sh\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ sh\,(x_1-x_4)\,sin\,(x_2{+}x_3)-sh\,(x_1+x_4)\,sin\,(x_2{-}x_3)\\ -ch\,(x_1-x_4)\,cos\,(x_2{+}x_3)+ch\,(x_1+x_4)\,cos\,(x_2{-}x_3)\end{array}\right),$$

$$a^X=\frac{1}{2}\left(\begin{array}{c} a^{x_1-x_4}cos\,((x_2+x_3)\,lna)\;\;+a^{x_1+x_4}cos\,((x_2-x_3)\,lna)\\ a^{x_1-x_4}sin\,((x_2+x_3)\,lna)\;\;+a^{x_1+x_4}sin\,((x_2-x_3)\,lna)\\ a^{x_1-x_4}sin\,((x_2+x_3)\,lna)\;\;-a^{x_1+x_4}sin\,((x_2-x_3)\,lna)\\ -a^{x_1-x_4}cos\,((x_2+x_3)\,lna)\;\;+a^{x_1+x_4}cos\,((x_2-x_3)\,lna)\end{array}\right).$$

2.2 Continuity of four-dimensional functions in the space of four-dimensional numbers

Let $F\,(X) = (f_1, f_2, f_3, f_4)$, $G\,(X) = (g_1, g_2, g_3, g_4)$ be four-dimensional functions of the space M5. $X_0 = (x_{10}, x_{20}, x_{30}, x_{40})$ - specified four-dimensional point.

Definition 3 *A four-dimensional function $F(X)$ is called continuous at the point X_0 if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that, under the condition $0 < \|X - X_0\|_C < \delta$, the following inequality holds*

$$\|F\,(X) - F(X_0)\|_C < \varepsilon,$$

where

$$\|X - X_0\|_C = \frac{1}{2}\sqrt{((x_1 - x_1{}^0) - (x_4 - x_4{}^0))^2 + ((x_2 - x_2{}^0) + (x_3 - x_3{}^0))^2}+$$

$$+\frac{1}{2}\sqrt{((x_1 - x_1{}^0) + (x_4 - x_4{}^0))^2 + ((x_2 - x_2{}^0) - (x_3 - x_3{}^0))^2}$$

is a spectral norm of the four-dimensional space M5 [3].

Definition 4 *The value $F(X_0)$ is called the limit of the function $F(X)$ at the point X_0 if, for any sequence of points $\left\{X^{(n)}\right\} \to X_0$, the corresponding sequence of numbers will be $F(X^{(n)}) \to F(X_0)$.*

Or we can write it as follows

$$\lim_{X\to X_0} F(X) \;= F(X_0).$$

Theorem 3 *The four-dimensional function $F\left(X\right)=\left(f_1,f_2,f_3,f_4\right)$ is continuous at the point $X_0=\left(x_{10},x_{20},x_{30},x_{40}\right)$ if and only if each component of the function $F(X)$ is continuous at the point X_0.*

Otherwise $lim_{X\to X_0}F(X)\ =F\left(X_0\right)=\left(f_1^0,f_2^0,f_3^0,f_4^0\right)$ if and only if

$$\underset{X\to X_0}{lim}f_1\ =f_1^0,\underset{X\to X_0}{lim}f_2\ =f_2^0,$$

$$\underset{X\to X_0}{lim}f_3\ =f_3^0,\underset{X\to X_0}{lim}f_4\ =f_4^0.$$

Proof There is a limit $lim_{X\to X_0}F(X)\ =F(X_0)$ if and only if if for any $\varepsilon>$ *0* there exists $\delta>0$, which under the condition $\left\|X-X_0\right\|_C<\delta$, the inequality $\left\|F\left(X\right)-F(X_0)\right\|_C<\varepsilon$ holds. This estimate is reduced to the following form

$$\left\|(f_1,f_2,f_3,f_4)-\left(f_1^0,f_2^0,f_3^0,f_4^0\right)\right\|_C<\varepsilon$$

$$\frac{1}{2}\sqrt{\left(f_1-f_4-f_1^0+f_4^0\right)^2+\left(f_2+f_3-f_2^0-f_3^0\right)^2}+$$

$$+\frac{1}{2}\sqrt{\left(f_1+f_4-f_1^0-f_4^0\right)^2+\left(f_2-f_3-f_2^0+f_3^0\right)^2}<\varepsilon.$$

$$\sqrt{\left(f_1-f_4-f_1^0+f_4^0\right)^2+\left(f_2+f_3-f_2^0-f_3^0\right)^2}<2\varepsilon$$

$$\sqrt{\left(f_1+f_4-f_1^0-f_4^0\right)^2+\left(f_2-f_3-f_2^0+f_3^0\right)^2}<2\varepsilon.$$

$$\left|f_1-f_1^0-f_4+f_4^0\right|<2\varepsilon,\left|f_2-f_2^0+f_3-f_3^0\right|<2\varepsilon,$$

$$\left|f_1-f_1^0+f_4-f_4^0\right|<2\varepsilon,\left|f_2-f_3-f_2^0+f_3^0\right|<2\varepsilon.$$

After summation and subtraction, we get

$$\left|f_1-f_1^0\right|<4\varepsilon,\left|f_2-f_2^0\right|<4\varepsilon,\left|f_3-f_3^0\right|<4\varepsilon,\left|f_4-f_4^0\right|<4\varepsilon\Longleftrightarrow$$

$$\lim_{X\to X_0}f_1\ =f_1^0,\lim_{X\to X_0}f_2\ =f_2^0,$$

$$\lim_{X\to X_0}f_3\ =f_3^0,\lim_{X\to X_0}f_4\ =f_4^0.$$

The theorem is proved.

Definition 5 A four-dimensional function $F\left(X\right)$ is called continuous in some domain $E\subset R^4$ if it is continuous at every point of this domain.

Theorem 4 Let the four-dimensional functions $F\left(X\right)=\left(f_1,f_2,f_3,f_4\right)$ and $G\left(X\right)=\left(g_1,g_2,g_3,g_4\right)$ are continuous in the domain $E\subset R^4$. Then the functions

1) $cF(X)$, $c=(c_1,c_2,c_3,c_4)$ – four-dimensional constant,

2)$F(X)\pm\ G(X)$,

3)$F(X) \cdot G(X)$,

4) $\frac{F(X)}{G(X)}$ for $G(X_0) \neq 0$
are also continuous in the domain $E \subset R^4$ [4, 8, 9].

Proof Let us prove, for example, 3)

According to Definition 3, for each $X_0 \in \Omega \subset R^4$, functions $F(X_0)$ and $G(X_0)$ are continuous at a given point in some domain $E \subset R^4$.

The components of the function $W(X) = F(X) \cdot G(X) = (f_1, f_2, f_3, f_4)(g_1, g_2, g_3, g_4) = (w_1, w_2, w_3, w_4)$ in the M5 space have the form

$$w_1 = f_1g_1 - f_2g_2 - f_3g_3 + f_4g_4,$$

$$w_2 = f_2g_1 + f_1g_2 - f_4g_3 - f_3g_4,$$

$$w_3 = f_3g_1 - f_4g_2 + f_1g_3 - f_2g_4,$$

$$w_4 = f_4g_1 + f_3g_2 + f_2g_3 + f_1g_4.$$

According to Theorem 2, $\lim_{X \to X_0} F(X) G(X)$ consider limit componentwise

$$\underset{X\to X_0}{lim} w_1 = \underset{X\to X_0}{lim} f_1 \underset{X\to X_0}{lim} g_1 - \underset{X\to X_0}{lim} f_2 \underset{X\to X_0}{lim} g_2 - \underset{X\to X_0}{lim} f_3 \underset{X\to X_0}{lim} g_3 + \underset{X\to X_0}{lim} f_4 \underset{X\to X_0}{lim} g_4 =$$
$$= f_1^0g_1^0 - f_2^0g_2^0 - f_3^0g_3^0 + f_4^0g_4^0 = w_1^0,$$
$$\underset{X\to X_0}{lim} w_2 = \underset{X\to X_0}{lim} f_2 \underset{X\to X_0}{lim} g_1 + \underset{X\to X_0}{lim} f_1 \underset{X\to X_0}{lim} g_2 - \underset{X\to X_0}{lim} f_4 \underset{X\to X_0}{lim} g_3 - \underset{X\to X_0}{lim} f_3 \underset{X\to X_0}{lim} g_4 =$$
$$= f_2^0g_1^0 + f_1^0g_2^0 - f_4^0g_3^0 - f_3^0g_4^0 = w_2^0,$$
$$\underset{X\to X_0}{lim} w_3 = \underset{X\to X_0}{lim} f_3 \underset{X\to X_0}{lim} g_1 - \underset{X\to X_0}{lim} f_4 \underset{X\to X_0}{lim} g_2 + \underset{X\to X_0}{lim} f_1 \underset{X\to X_0}{lim} g_3 - \underset{X\to X_0}{lim} f_2 \underset{X\to X_0}{lim} g_4 =$$
$$= f_3^0g_1^0 - f_4^0g_2^0 + f_1^0g_3^0 - f_2^0g_4^0 = w_3^0,$$
$$\underset{X\to X_0}{lim} w_4 = \underset{X\to X_0}{lim} f_4 \underset{X\to X_0}{lim} g_1 + \underset{X\to X_0}{lim} f_3 \underset{X\to X_0}{lim} g_2 + \underset{X\to X_0}{lim} f_2 \underset{X\to X_0}{lim} g_3 + \underset{X\to X_0}{lim} f_1 \underset{X\to X_0}{lim} g_4 =$$
$$= f_4^0g_1^0 + f_3^0g_2^0 + f_2^0g_3^0 + f_1^0g_4^0 = w_4^0.$$

Then

$$\lim_{X \to X_0} F(X) \cdot G(X) = \left(f_1^0, f_2^0, f_3^0, f_4^0\right)\left(g_1^0, g_2^0, g_3^0, g_4^0\right) = F(X_0) \cdot G(X_0)$$

The theorem is proved.

2.3 Differentiable functions defined in the space of four-dimensional numbers

After we have determined the continuity of four-dimensional functions in the space of four-dimensional numbers M5, we next study the differentiability.

Definition 5 *The derivative of the function* $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ *at the point* $X = (x_1, x_2, x_3, x_4)$ *is called the limit* $\lim_{\Delta X \to 0} \frac{F(X+\Delta X)-F(X)}{\Delta X}$ *, if it exists as* $\Delta X = (\Delta x_1, \Delta x_2, \Delta x_3, \Delta x_4) \to 0$ *tends to zero along any path consisting of non-degenerate points.*

Theorem 4 *Let all components of the four-dimensional function $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4) \in R^4$ have continuous derivatives in a neighborhood of the point X. Then the necessary and sufficient conditions for the differentiability of the function $F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ at point X are the following generalized Cauchy-Riemann conditions:*

$$\left\{ \begin{array}{c} \frac{\partial f_1}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = \frac{\partial f_4}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} = -\frac{\partial f_1}{\partial x_2} = \frac{\partial f_4}{\partial x_3} = -\frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} = \frac{\partial f_4}{\partial x_2} = -\frac{\partial f_1}{\partial x_3} = -\frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} = -\frac{\partial f_3}{\partial x_2} = -\frac{\partial f_2}{\partial x_3} = \frac{\partial f_1}{\partial x_4} \end{array} \right. . \tag{4}$$

Proof. Necessity. Let the derivative exists. Then it does not depend on the way $\triangle X \to 0$ tends to zero, and consider the following ways of tending the increment to zero.

Let $\triangle X = (\triangle x_1, 0, 0, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \lim_{(\triangle x_1, 0,0,0) \to (0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(\triangle x_1, 0, 0, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1 + \triangle x_1, x_2, x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1, 2, 3, 4.$

By virtue of Theorem 2 from [3] $\frac{1}{(\triangle x_1, 0,0,0)} = \left(\frac{1}{\triangle x_1}, 0, 0, 0\right)$.

Consequently $\frac{dF}{dX} = \lim_{\triangle x_1 \to 0} \left(\frac{\triangle f_1}{\triangle x_1}, \frac{\triangle f_2}{\triangle x_1}, \frac{\triangle f_3}{\triangle x_1}, \frac{\triangle f_4}{\triangle x_1}\right) = \left(\frac{\partial f_1}{\partial x_1}, \frac{\partial f_2}{\partial x_1}, \frac{\partial f_3}{\partial x_1}, \frac{\partial f_4}{\partial x_1}\right)$.

Now let us choose another way of $\triangle X$ tending to zero, namely, $\triangle X = (0, \triangle x_2, 0, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \mathop{lim}_{(0,\triangle x_2, 0,0) \to (0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(0, \triangle x_2, 0, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1, x_2 + \triangle x_2, x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1, 2, 3, 4.$

By virtue of Theorem 2 from [3] $\frac{1}{(0,\triangle x_2,0,0)} = \left(0, -\frac{1}{\triangle x_2}, 0, 0\right)$.

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \mathop{lim}_{\triangle x_2 \to 0} \left(\frac{\triangle f_2}{\triangle x_2}, -\frac{\triangle f_1}{\triangle x_2}, \frac{\triangle f_4}{\triangle x_2}, -\frac{\triangle f_3}{\triangle x_2}\right) = \left(\frac{\partial f_2}{\partial x_2}, -\frac{\partial f_1}{\partial x_2}, \frac{\partial f_4}{\partial x_2}, -\frac{\partial f_3}{\partial x_2}\right).$$

Choose the third way of $\triangle X$ tending to zero, $\triangle X = (0, 0, \triangle x_3, 0) \to 0 = (0, 0, 0, 0)$. Then

$$\frac{dF}{dX} = \mathop{lim}_{(0,0,\triangle x_3,0) \to (0,0,0,0)} \frac{(\triangle f_1(x_1,x_2,x_3,x_4), \triangle f_2(x_1,x_2,x_3,x_4), \triangle f_3(x_1,x_2,x_3,x_4), \triangle f_4(x_1,x_2,x_3,x_4))}{(0, 0, \triangle x_3, 0)},$$

where $\triangle f_i(x_1,x_2,x_3,x_4) = f_i(x_1, x_2, x_3 + \triangle x_3, x_4) - f_i(x_1,x_2,x_3,x_4), \ i = 1, 2, 3, 4.$

By virtue of Theorem 2 from [3] $\frac{1}{(0,0,\triangle x_3,0)} = \left(0, 0, \frac{1}{\triangle x_3}, 0\right)$.

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \mathop{lim}_{\triangle x_3 \to 0} \left(\frac{\triangle f_3}{\triangle x_3}, \frac{\triangle f_4}{\triangle x_3}, -\frac{\triangle f_1}{\triangle x_3}, -\frac{\triangle f_2}{\triangle x_3}\right) = \left(\frac{\partial f_3}{\partial x_3}, \frac{\partial f_4}{\partial x_3}, -\frac{\partial f_1}{\partial x_3}, -\frac{\partial f_2}{\partial x_3}\right).$$

Finally, choose the fourth way of $\triangle X$ tending to zero, $\triangle X = (0,0,0,\triangle x_4) \to 0 = (0,0,0,0)$. Then

$$\frac{dF}{dX} = \lim_{(0,0,0,\triangle x_4)\to(0,0,0,0)} \frac{\left(\triangle f_1\left(x_1,x_2,x_3,x_4\right),\triangle f_2\left(x_1,x_2,x_3,x_4\right),\triangle f_3\left(x_1,x_2,x_3,x_4\right),\triangle f_4\left(x_1,x_2,x_3,x_4\right)\right)}{(0,0,0,\triangle x_4)},$$

where $\triangle f_i\left(x_1,x_2,x_3,x_4\right) = f_i\left(x_1,x_2,x_3,x_4+\triangle x_4\right) - f_i\left(x_1,x_2,x_3,x_4\right),\ i=1,2,3,4.$

By virtue of Theorem 2 from [3] $\frac{1}{(0,0,0,\triangle x_4)} = \left(0,0,0,-\frac{1}{\triangle x_4}\right)$.

Therefore, by the rule of multiplication of four-dimensional numbers

$$\frac{dF}{dX} = \lim_{\triangle x_4\to 0}\left(\frac{\triangle f_4}{\triangle x_4},-\frac{\triangle f_3}{\triangle x_4},-\frac{\triangle f_2}{\triangle x_4},\frac{\triangle f_1}{\triangle x_4}\right) = \left(\frac{\partial f_4}{\partial x_4},-\frac{\partial f_3}{\partial x_4},-\frac{\partial f_2}{\partial x_4},\frac{\partial f_1}{\partial x_4}\right).$$

Thus, we got that

$$\frac{dF}{dX} = \left(\frac{\partial f_1}{\partial x_1},\frac{\partial f_2}{\partial x_1},\frac{\partial f_3}{\partial x_1},\frac{\partial f_4}{\partial x_1}\right) = \left(\frac{\partial f_2}{\partial x_2},-\frac{\partial f_1}{\partial x_2},\frac{\partial f_4}{\partial x_2},-\frac{\partial f_3}{\partial x_2}\right) =$$

$$= \left(\frac{\partial f_3}{\partial x_3},\frac{\partial f_4}{\partial x_3},-\frac{\partial f_1}{\partial x_3},-\frac{\partial f_2}{\partial x_3}\right) = \left(\frac{\partial f_4}{\partial x_4},-\frac{\partial f_3}{\partial x_4},-\frac{\partial f_2}{\partial x_4},\frac{\partial f_1}{\partial x_4}\right).$$

Equating the components according to the rule of equality of four-dimensional numbers, we obtain (4).

Sufficiency. Let conditions (4) hold. By definition $F\left(X+\triangle X\right) - F\left(X\right) = \left(f_1\left(x_1+\triangle x_1,x_2+\triangle x_2,x_3+\triangle x_3,x_4+\triangle x_4\right) - f_1\left(x_1,x_2,x_3,x_4\right),\dots\right)$. Since the components f_i are continuously differentiable functions of four variables, for each of them we can write $f_i\left(x_1+\triangle x_1,x_2+\triangle x_2,x_3+\triangle x_3,x_4+\triangle x_4\right) - f_i\left(x_1,x_2,x_3,x_4\right) = \frac{\partial f_i}{\partial x_1}\triangle x_1 + \frac{\partial f_i}{\partial x_2}\triangle x_2 + \frac{\partial f_i}{\partial x_3}\triangle x_3 + \frac{\partial f_i}{\partial x_4}\triangle x_4 + o(\sqrt{\triangle x_1^2+\triangle x_2^2+\triangle x_3^2+\triangle x_4^2})$, i=1,2,3,4. This easily implies the existence of a limit from the definition of the derivative.

The theorem is proved.

Consequence 2 *To calculate the derivative of a four-dimensional function can be used one of the following formulas:*

$$\frac{dF}{dX} = \left(\frac{\partial f_1}{\partial x_1},\frac{\partial f_2}{\partial x_1},\frac{\partial f_3}{\partial x_1},\frac{\partial f_4}{\partial x_1}\right) = \left(\frac{\partial f_2}{\partial x_2},-\frac{\partial f_1}{\partial x_2},\frac{\partial f_4}{\partial x_2},-\frac{\partial f_3}{\partial x_2}\right) =$$

$$= \left(\frac{\partial f_3}{\partial x_3},\frac{\partial f_4}{\partial x_3},-\frac{\partial f_1}{\partial x_3},-\frac{\partial f_2}{\partial x_3}\right) = \left(\frac{\partial f_4}{\partial x_4},-\frac{\partial f_3}{\partial x_4},-\frac{\partial f_2}{\partial x_4},\frac{\partial f_1}{\partial x_4}\right). \tag{5}$$

Example 1 *Find the derivative of the function X^3. By definition*

$$X^3 = \left(x_1\left(x_1^2-3x_2^2-3x_3^2+3x_4^2\right)+6x_2x_3x_4, x_2\left(x_1^2-x_2^2-3x_3^2+3x_4^2\right)\right.$$

$$\left.-6x_1x_3x_4, x_3\left(3x_1^2-3x_2^2-x_3^2+3x_4^2\right)-6x_1x_2x_4, x_4\left(3x_1^2-3x_2^2-3x_3^2+x_4^2\right)+6x_1x_2x_3\right).$$

Applying formula (5), *we obtain* $\frac{dX^3}{dX} = \left(3\left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right), 3\left(x_1x_2 - x_3x_4\right),\right.$
$\left.3\left(x_1x_3 - x_2x_4\right), 3(x_1x_4 + x_2x_3)\right) = 3 \cdot X^2.$
Find the derivative of the function $sin(X)$. *By definition*
$sin(X) = \left(\frac{1}{2}\left(sin\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right), \frac{1}{2}\left(cos\left(x_1 - x_4\right)\right.\right.$
$sh\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) sh\left(x_2{-}x_3\right)\left.\right), \frac{1}{2}\left(cos\left(x_1 - x_4\right) sh\left(x_2{+}x_3\right) - cos\left(x_1 + x_4\right)\right.$
$sh\left(x_2{-}x_3\right)\left.\right), \frac{1}{2}\left(-sin\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right)\left.\right).$

Applying any one of (5) we obtain the formula
$\frac{dsin(X)}{dX} = \left(\frac{1}{2}\left(cos\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right), \frac{1}{2}\left(-sin\left(x_1 - x_4\right)\right.\right.$
$sh\left(x_2{+}x_3\right) - sin\left(x_1 + x_4\right) sh\left(x_2{-}x_3\right)\left.\right), \frac{1}{2}\left(-sin\left(x_1 - x_4\right) sh\left(x_2{+}x_3\right) + sin\left(x_1 + x_4\right)\right.$
$sh\left(x_2{-}x_3\right)\left.\right), \frac{1}{2}\left(-cos\left(x_1 - x_4\right) ch\left(x_2{+}x_3\right) + cos\left(x_1 + x_4\right) ch\left(x_2{-}x_3\right)\right)\left.\right) = cos\left(X\right).$

Above, we have defined an explicit formula for the four-dimensional exponent. Applying the obtained formulas for determining the derivative, it is easy to make sure that $\frac{dexp(X)}{dX} = exp(X)$.

Definition 6 *7 A four-dimensional function that has a derivative at all points of a certain domain is called regular in this domain.*

Let us now investigate the differential properties of regular functions.

Theorem 5 *Let* $F(X) = (f_1, f_2, f_3, f_4)$, $G(X) = (g_1, g_2, g_3, g_4)$ *be regular functions from the space M5,* $A = (a_1, a_2, a_3, a_4)$ *and* $B = (b_1, b_2, b_3, b_4)$ *are four-dimensional constants. Then the following equalities hold:*

$$1. \frac{d(AF(X) + BG(X))}{dX} = A\frac{dF(X)}{dX} + B\frac{dG(X)}{dX},$$

$$2. \frac{d\left(F(X) \cdot G(X)\right)}{dX} = \frac{dF(X)}{dX} \cdot G\left(X\right) + F\left(X\right) \cdot \frac{dG(X)}{dX}.$$

3. $\frac{d\left(\frac{F(X)}{G(X)}\right)}{dX} = \frac{\frac{dF(X)}{dX} \cdot G(X) - F(X) \cdot \frac{dG(X)}{dX}}{G^2(X)}$*, where* $G(X)$ *non-degenerate function.*

Proof.

1. From the definition of the derivative obviously follows, that $\frac{d(F(X)+G(X))}{dX} = \frac{dF(X)}{dX} + \frac{dG(X)}{dX}$. Applying the first equality from (4) from the definition of multiplication in the space M5, we can write
$\frac{d(AF(X))}{dX} = \left(a_1\frac{\partial f_1}{\partial x_1} - a_2\frac{\partial f_2}{\partial x_1} - a_3\frac{\partial f_3}{\partial x_1} + a_4\frac{\partial f_4}{\partial x_1},\quad a_2\frac{\partial f_1}{\partial x_1} + a_1\frac{\partial f_2}{\partial x_1} - a_4\frac{\partial f_3}{\partial x_1} - a_3\frac{\partial f_4}{\partial x_1},\right.$
$\left. a_3\frac{\partial f_1}{\partial x_1} - a_4\frac{\partial f_2}{\partial x_1} + a_1\frac{\partial f_3}{\partial x_1} - a_2\frac{\partial f_4}{\partial x_1}, a_4\frac{\partial f_1}{\partial x_1} + a_3\frac{\partial f_2}{\partial x_1} + a_2\frac{\partial f_3}{\partial x_1} + a_1\frac{\partial f_4}{\partial x_1}\right) = A\frac{dF(X)}{dX}.$
These two equalities imply the validation of the theorem.

2. The components of the function $W\left(X\right) = F(X) \cdot G(X)$ in the space M5 have the following form

$$w_1 = f_1g_1 - f_2g_2 - f_3g_3 + f_4g_4,$$

$$w_2 = f_2g_1 + f_1g_2 - f_4g_3 - f_3g_4,$$

$$w_3 = f_3g_1 - f_4g_2 + f_1g_3 - f_2g_4,$$

$$w_4 = f_4g_1 + f_3g_2 + f_2g_3 + f_1g_4.$$

Then $\frac{\partial w_1}{\partial x_1} = \frac{\partial f_1}{\partial x_1}g_1 + f_1\frac{\partial g_1}{\partial x_1} - \frac{\partial f_2}{\partial x_1}g_2 - f_2\frac{\partial g_2}{\partial x_1} - \frac{\partial f_3}{\partial x_1}g_3 - f_3\frac{\partial g_3}{\partial x_1} + \frac{\partial f_4}{\partial x_1}g_4 + f_4\frac{\partial g_4}{\partial x_1}$, and $\frac{\partial w_2}{\partial x_2} = \frac{\partial f_2}{\partial x_2}g_1 + f_2\frac{\partial g_1}{\partial x_2} + \frac{\partial f_1}{\partial x_2}g_2 + f_1\frac{\partial g_2}{\partial x_2} - \frac{\partial f_4}{\partial x_2}g_3 - f_4\frac{\partial g_3}{\partial x_2} - \frac{\partial f_3}{\partial x_2}g_4 - f_3\frac{\partial g_4}{\partial x_2}$.

Taking into account that the functions $F(X)$ and $G(X)$ satisfy the Cauchy-Riemann conditions (4), we make sure that $\frac{\partial w_1}{\partial x_1} = \frac{\partial w_2}{\partial x_2}$. In a similar way it is proved that the function $W(X)$satisfies all other equalities of (4), that is, it is a regular function. Proofs of 2 and 3 are carried out by direct verification.

The theorem is proved.

Theorem 6 *Let* $F(X) = F(x_1, x_2, x_3, x_4) = (f_1, f_2, f_3, f_4)$ *be a regular function in certain domain. Then, at all points of this domain, the following equalities hold:*

$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_3^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} - \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_2^2} - \frac{\partial^2 f_i}{\partial x_3^2} = 0,\ \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,\ \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,\ \frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} + \frac{\partial^2 f_i}{\partial x_3^2} + \frac{\partial^2 f_i}{\partial x_4^2} = 0,$$
$$\frac{\partial^2 f_i}{\partial x_1^2} + \frac{\partial^2 f_i}{\partial x_2^2} - \frac{\partial^2 f_i}{\partial x_3^2} - \frac{\partial^2 f_i}{\partial x_4^2} = 0.$$

where $i = 1, 2, 3, 4$.

The proof follows directly from the Cauchy-Riemann conditions.

Definition 7 *The integral of a four-dimensional regular function is the antiderivative of this function of the following form*

$$\int F(X)dX = W(X) + C$$

where $W(X)$ *is any of the antiderivatives,* $C = (C_1, C_2, C_3, C_4)$ *is a four-dimensional arbitrary constant.*

Define the basic properties of the integral [1,2]:

1. $\int (AF(X) \pm BG(X))dX = A\int F(X)dX \pm B\int G(X)dX + C,$

2. $\frac{d\left(\int F(X)dX\right)}{dX} = \int F(X)dX,$

3. $\int \frac{F(X)}{dX}dX = F(X) + C.$

Based on the definitions and properties of the derivative and antiderivative, below is presented the table of four-dimensional functions

1. Derivative

$$\frac{dC}{dX} = \theta; \frac{X^n}{dX} = \frac{X^{n-1}}{n-1}; \frac{LnX}{dX} = \frac{J_1}{X}; \frac{a^X}{dX} = a^X \ lna;$$

$$\frac{sinX}{dX} = cosX \ ; \frac{cosX}{dX} = -sinX \ ; \frac{e^X}{dX} = e^X;$$

$$\frac{tanX}{dX} = \frac{J_1}{cos^2X}; \frac{arcsinX}{dX} = \frac{J_1}{\sqrt{J_1 - X^2}}; \frac{arctanX}{dX} = \frac{J_1}{J_1 + X^2}.$$

2. Antiderivative

$$\int \theta dX = C; \int X^n dX = \frac{X^{n+1}}{n+1} + C, \ n \neq -1; \int \frac{dX}{X} = LnX + C; \int a^X dX = \frac{a^X}{lna} + C;$$

$$\int \sin X \ dX = -\cos X \ + C; \int \cos X \ dX = \sin X \ + C; \int e^X dX = e^X + C;$$

$$\int \frac{dX}{cos^2X} = \tan X \ + C; \int \frac{dX}{\sqrt{J_1 - X^2}} = \arcsin X \ + C; \int \frac{dX}{J_1 + X^2} = \arctan X \ + C.$$

where $\theta = (0, 0, 0, 0)$ is four-dimensional zero, $J_1 = (1, 0, 0, 0)$ is four-dimensional unit.

3 Conclusion

In this article, functions of a four-dimensional variable in the space M5 and their properties, as well as continuity and differentiability have been investigated. The types of elementary functions, such as sine, cosine, hyperbolic sine and cosine, exponential, logarithmic, exponential and power functions are defined using spectral values. Theorems on the continuity and differentiability of functions of a four-dimensional variable in the space M5 are proved. The regularity of functions of four-dimensional variables is proved, and the Cauchy-Riemann conditions for the differentiability are defined. This work is of an overview type and is a continuation of research paper [3]. The results of the study show that the analysis of functions of a four-dimensional variable, their properties, continuity and differentiability of functions have an analogy with the studies in work [1]. Furthermore, the obtained results show that the theory of functions of a four-dimensional variable of space M5 is a generalization of the theories of real and complex analyzes.

References

[1] Abenov M.M. Chetirehmernaya matematika. Metody i prilozheniya. Nauchnaya monographia [Four-dimensional mathematics: Methods and applications. Scientific monograph]. Almaty.: Publishing House Kazakh University, 2019. - 176.

[2] Abenov M.M., Gabbassov M.B. Anyzotropnie chetirehmernie prostranstva ili novie kvaternioni [Anisotropic four-dimensional spaces or new quaternions]. Preprint, Nur-Sultan. 2020.

[3] Rakhymova A.T., Gabbassov M.B., Shapen K.M., "On one space of four-dimensional numbers," *Journal of Mathematics, Mechanics and Computer Science (Vol 4)* (2020): 199-225.

[4] Kudryavsev L.D. Kurs matematicheskogo analyza [Mathematical Analysis Course]. – M.: Vishaya shkola, 1981. -687.

[5] Lavrent'ev M.A., Shabat B.V. Metody teorii funksii kompleksnogo peremennogo [Methods of the theory of functions of a complex variable]. M.: Nauka, 1965. -716.

[6] Bitsadze A.V. Osnovi teorii analyticheskih funksii kompleksnogo peremennogo [Fundamentals of the theory of analytic functions of a complex variable]. M.: Nauka, 1984. -280.

[7] Kolmogorov A.N., Fomin S.V. Elemenri teorii funksii i funksional'nogo analyza [Elements of function theory and functional analysis]. M.: Nauka, 1989. -624.

[8] Fikhtengol'ts G.M. Osnovi matematicheskogo analyza [The Fundamentals of Mathematical Analysis]. M.: Nauka, 1968. -441.

[9] Il'in V.A., Sadovnichii V.A., Sendov B.Kh. Matematicheskii analyz. Nachal'nii kurs [Mathematical analysis. Initial course]. M.: Izdatel'stvo MGU, 1985. -660.

[10] Polovinin E.S. Teoriya funksii kompleksnogo preremennogo: uchebnik [Theory of functions of complex variables: book]. M.: MFTI, 2014. -253.

[11] Sidorov V.Yu., Fedoryuk M.I., Shabunin M. Leksii po teorii funksii kompleksnogo peremennogo [Theory of functions of complex variables]. M.: Nauka,1982. -488.

[12] Markushevich A.I. Teotiya analyticheskih funksii [Theory of analytical functions]. M.: Nauka, 1967. -491.

[13] C. Caratheodory, *The Theory of Functions of a Complex Variable*. Vol. 1 (Chelsea publishing company, 1954), -319.

[14] M Giaquinta and G Modica, *Mathematical analysis. An introduction to functions of several variable* (Birkhauser. Boston, 2009), -348.

[15] S.L. Green, *The theory and use of the complex variable. An introduction* (New York: SIR ISAAC Pitman & Sons, 1939), -136.

[16] R. Remmert, *Theory of complex functions* (New York: Springer, 1989), - 480.

МАЗМҰНЫ – СОДЕРЖАНИЕ – CONTENTS

4-бөлім	**Раздел 4**	**Section 4**
Қолданбалы математика	**Прикладная математика**	**Applied Mathematics**