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INVERSE PROBLEMS OF PARAMETER RECOVERY IN DIFFERENTIAL EQUATION WITH MULTIPLE CHARACTERISTICS

Inverse problems - the problem of finding the causes of known or given consequences. They arise when the characteristics of an object of interest to us are not available for direct observation. These are, for example, the restoration of the characteristics of the field sources according to their given values at some points, the restoration or interpretation of the original signal from the known output signal, etc. This paper studies the solvability of finding the solution of a differential equation of inverse problems. The work is devoted to the study of the solvability in Sobolev spaces of nonlinear inverse coefficient problems for differential equations of the third order with multiple characteristics. In this paper, alongside with finding the solution of one or another differential equation, it is also required to find one or more coefficients of the equation itself for us to name them inverse coefficient problems. A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter, and not a function of certain independent variables.
Key words: Inverse problems, third-order equations, multiple characteristics, numerical parameter, solvability.

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Сипаттамалары еселі диффененциалдық теңдеулердегі параметрді қалпына келтірудің кері есебі

Кері есептер - белгілі немесе берілген әсерлердің себептерін табу мәселесі. Олар бізді қызықтыратын объектінің сипаттамалары тікелей бақылау үшін қол жетімді болмаған кезде пайда болады. Бұл, мысалы, кейбір нүктелердегі олардың белгіленген мәндеріне сәйкес өріс көздерінің сипаттамаларын қалпына келтіру, белгілі шығыс сигналынан бастапқы сигналды қалпына келтіру немесе интерпретациялау және т.б. Берілген жұмыста біз дифференциалдық теңдеуге қойылған кері есептің шешімділігін зерттейміз. Жұмыс бірнеше сипаттамалары бар үшінші ретті дифференциалдық теңдеулер үшін сызықты

емес кері коэффициентті есептерінің Соболев кеңістігінде шешімділігін зерттеуге арналған. Бұл мақалада белгілі бір дифференциалдық теңдеудің шешімін іздеумен қатар теңдеудің бір немесе бірнеше коэффициенттерін табу да талап етіледі, сондықтан оларды кері коэффициенттік есептер деп атаймыз. Бұл жұмыста зерттелген есептердің айрықша ерекшелігі белгісіз коэффициент белгілі бір тәуелсіз айнымалылардың функциясы емес, сандық параметр болып табылады.
Кілттік сөздер: Кері есептер, үшінші ретті теңдеулер, еселі сипаттауыштар, сандық параметр, шешімділік.

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Обратные задачи восстановления параметров в дифференциальном уравнении с кратными характеристиками

Обратные задачи - задача нахождения причин известных или заданных следствий. Они возникают, когда характеристики интересующего нас объекта недоступны для непосредственного наблюдения. Это, например, восстановление характеристик источников поля в соответствии с их заданными значениями в некоторых точках, восстановление или интерпретация исходного сигнала из известного выходного сигнала и т.д. В данной работе исследуется разрешимость нахождения решения дифференциального уравнения обратных задач. Работа посвящена исследованию разрешимости в пространствах Соболева нелинейных обратных коэффициентных задач для дифференциальных уравнений третьего порядка с множественными характеристиками. В этой статье, наряду с поиском решения того или иного дифференциального уравнения, также требуется найти один или несколько коэффициентов самого уравнения, чтобы мы назвали их обратными коэффициентными задачами. Отличительной особенностью задач, изучаемых в данной работе, является то, что неизвестный коэффициент является числовым параметром, а не функцией определенных независимых переменных.
Ключевые слова: Обратные задачи, уравнения третьего порядка, кратные характеристики, числовой параметр, разрешимость.

1 Introduction

The work is devoted to the study of the solvability in Sobolev spaces of nonlinear inverse coefficient problems for differential equations of the third order with multiple characteristics. In this paper, alongside with finding the solution of one or another differential equation, it is also required to find one or more coefficients of the equation itself for us to name them inverse coefficient problems. Various aspects of the theory of inverse coefficient problems for differential equations are well covered in the world literature - see monographs [1-17]. At the same time, it should be noted that there are not many works devoted to the study of the solvability of inverse problems for differential equations with multiple characteristics - we can only name the works [17-19].

A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter, and not a function of certain independent variables. Similar problems were studied earlier, but only for classical parabolic, hyperbolic and elliptic equations - see papers [20-29]. The inverse problems of determination for differential equations with multiple characteristics, together with the solution of numerical parameters, which are the coefficients of the equation itself, have not been previously studied.

It should be noted that differential equations with constant coefficients are often obtained by mathematical modeling of processes taking place in a homogeneous medium - see papers [30, 31]. If in this case the coefficients characterizing certain properties of the environment are unknown quantities, then we will automatically obtain inverse problems with unknown parameters. All constructions and arguments in this work will be carried out using Lebesgue spaces L_p and Sobolev spaces W_p^l. The necessary information about functions from these spaces can be found in monographs [32-34].

2 Problem statement

Let Ω be an interval on the Ox, Q be a rectangle $\{(x,t):\ x\in\Omega,\ t\in(0,T)\}$ $(0<T<+\infty)$. Further, let $N(x)$ and $f(x,t)$ be given functions defined for $x\in\overline{\Omega}$, $t\in[0,T]$, A and β be given real numbers.

Inverse Problem I: it to find the function $u(x,t)$ and a positive number α connected in the rectangle Q by the equation

$$\alpha u_t - u_{xxx} + \beta u = f(x,t), \tag{1}$$

when the following conditions for the function $u(x,t)$ are met.

$$u(x,0)=0, \quad x\in\Omega; \tag{2}$$

$$u(0,t)=u_x(0,t)=u_{xx}(1,t)=0, \quad t\in(0,T); \tag{3}$$

$$\int\limits_{\Omega} N(x)u(x,T)\,dx = A. \tag{4}$$

Inverse Problem II: it to find the function $u(x,t)$ and a positive number α connected in the rectangle Q by the equation

$$u_t - \alpha u_{xxx} + \beta u = f(x,t), \tag{5}$$

when the conditions (2)–(4) for the function $u(x,t)$ are met .

In inverse problems I and II, conditions (2) and (3) are the conditions of the general initial-boundary value problem for a third-order differential equation with multiple characteristics, while condition (4) is the initial integral overdetermination condition, the presence of which is explained by the presence of an additional unknown value of the coefficient (parameter).

Differential equations (1) and (5) have a simple model form. Possible generalizations of these equations and possible enhancements and generalizations of the obtained results will be described at the end of the work.

3 Solvability of Inverse Problem I.

The study of the solvability of the inverse problem will be carried out by a method based on the transition from the original problem to a new problem for a nonlinear integro–differential equation – see works [26–29].

Let R_1 be a given positive number, $\varphi_1(v)$ be the function

$$\varphi_1(v) = \int\limits_Q N(x)[v_{xxx}(x,t) - \beta v(x,t)]\,dx\,dt.$$

Let us consider the following problem: to find a function $u(x,t)$ which is a solution to the equation in the rectangle Q

$$\frac{R_1 + \varphi_1(u)}{A} u_t - u_{xxx} + \beta u = f(x,t) \tag{6}$$

and conditions (2) and (3) must be satisfied for that function.

In the boundary value problem (6), (2), (3), equation (6) is an integro–differential equation, called in some sources as "loaded" [35, 36].

Let μ_0 be a number from the interval $(0,1)$. Let us consider the following:

$$f_1(x,t) = (T-t)f(x,t),$$

$$K_1 = \frac{T^{1/2}}{\beta}\|N\|_{L_2(\Omega)}\|f_{xxx}\|_{L_2(Q)}, \quad K_2 = \frac{2\beta A T^{1/2}}{(1-\mu_0)}\|N\|_{L_2(\Omega)}\|f_1\|_{L_2(Q)},$$

$$K_0 = \frac{1}{2\mu_0}\left(K_1 + \sqrt{K_1^2 + 4\mu_0 K_2}\right).$$

Theorem 1 *Let the following conditions be satisfied*

$$N(x) \in L_2(\Omega); \;\; \beta > 0, \quad A > 0; \;\; \frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \quad k = \overline{0,3},$$

$$f(0,t) = f_x(0,t) = f_{xx}(1,t) = 0 \quad npu \quad t \in (0,T);$$

$$\exists\, \mu_0 \in (0,1): \;\; K_0 \le R_1.$$

Then the boundary value problem (6), (2), (3) has the following $u(x,t)$ solution, $u(x,t) \in L_2(0,T; W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_1(u)| \le \mu_0 R_1$.

Proof. For the number $\mu = \mu_0 R_1$ let us define a function $G_\mu(\xi)$:

$$G_\mu(\xi) = \begin{cases} \xi, & if \;\; |\xi| \le \mu, \\ \mu, & if \;\; \xi > \mu, \\ -\mu, & if \;\; \xi < -\mu. \end{cases}$$

Let us consider the following problem: to find a function $u(x,t)$ that is a solution to the equation in rectangle Q and such that conditions (2) and (3) are satisfied for it.

$$\frac{R_1 + G_\mu[\varphi_1(u)]}{A} u_t - u_{xxx} + \beta u = f(x,t) \tag{6_μ}$$

Using the regularization method, a priori estimates and the fixed point method we will show that this problem has a regular solution (that is, a solution that has all its derivatives generalized according to S.L. Sobolev).

Let ε be a positive number. Let us consider the following problem: *to find a function $u(x,t)$ which is a solution to the equation*

$$\frac{R_1 + G_\mu(\varphi_1(u))}{A} u_t - u_{xxx} - \varepsilon u_{xxxxxxx} + \beta u = f(x,t) \tag{7}$$

in the rectangle Q and such that conditions (2) and (3), as well as the condition (8) are satisfied

$$u_{xxx}(0,t) = u_{xxxx}(1,t) = u_{xxxxx}(1,t) = 0, \quad t \in (0,T). \tag{8}$$

Let V be the set of functions $v(x,t)$ such that $v(x,t) \in L_2(0,T; W_2^6(\Omega))$, $v_t(x,t) \in L_2(0,T; L_2(\Omega))$, and the function $v(x,t)$ satisfies conditions (2), (3), and (8). Let us give this set the following norm

$$\|v\|_V = \left(\int\limits_Q (v^2 + v_t^2 + v_{xxxxxx}^2)\, dx\, dt \right)^{1/2}.$$

Obviously, the set V with this norm will be a Hilbert space.

For a function $v(x,t)$ from the space V, we will consider the following problem: to find a function $u(x,t)$ which is a solution to the equation

$$\frac{R_1 + G_\mu(\varphi_1(v))}{A} u_t - u_{xxx} - \varepsilon u_{xxxxxxx} + \beta u = f(x,t) \tag{9}$$

and such that conditions (2), (3), and (8) are satisfied for it. In this problem, differential equation (9) is a linear parabolic equation of the sixth order, while boundary conditions (3) and (8) are self-adjoint. Consequently, this problem is solvable in the space V (this fact can be proved directly using the classical Galerkin method with the choice of a special basis).

The solvability in the space V of the boundary value problem (9), (2), (3), (8) means that this problem generates an operator Φ acting from the space V and associating the function $v(x,t)$ from V with the solution $u(x,t)$ of the boundary value problem (9), (2), (3), (8). Let us show that this operator has fixed points in the space V. First, let us note that for the solutions of the boundary value problem (9), (2), (3), (8) there is a priori estimate

$$\|u\|_V \leq K \|f\|_{L_2(Q)} \tag{10}$$

(natural for parabolic equations), where the constant K is determined only by the numbers β, T and ε. It follows from this estimate that the operator Φ takes a closed ball of radius $R^* = K\|f\|_{L_2(Q)}$ of the space V into itself.

Let us show now that the operator Φ is continuous on the space V.

Let $\{v_n(x,t)\}_{n=1}^{\infty}$ be a sequence of functions from the space V converging to a function $v_0(x,t)$. If we put $u_n = \Phi(v_n)$, $u_0 = \Phi(v_0)$, $\overline{v}_n = v_n - v_0$, $\overline{u}_n = u_n - u_0$, then we will have the following equality

$$\frac{R_1 + G_\mu(\varphi_1(v_0))}{A}\overline{u}_{nt} - \overline{u}_{nxxx} - \varepsilon\overline{u}_{nxxxxxxx} + \beta\overline{u}_n = \frac{1}{A}[G_\mu(\varphi_1(v_0)) - G_\mu(\varphi_1(v_n))]. \tag{11}$$

Since the function $G_\mu(\xi)$ is Lipschitz, and the inequality $|G_\mu(\xi)| \leq |\xi|$, then we will have the following estimate

$$|G_\mu(\varphi_1(v_0)) - G_\mu(\varphi_1(v_n))| \leq |\varphi_1(\overline{v}_n)|. \tag{12}$$

Now repeating for equality (11) the proof of estimate (10), taking into account inequality (12) and taking into account that $\varphi_1(\overline{v}_n) \to 0$ as $n \to \infty$ (due to the convergence of the sequence $v_n(x,t)$ in the space V to the function $v_0(x,t)$)), we see that the convergence takes place $\overline{u}_n(x,t) \to 0$ as $n \to \infty$. And this means that the operator Φ is continuous everywhere in the space V.

Let us now show that the operator Φ is compact.

Let $\{v_n(x,t)\}_{n=1}^{\infty}$ be an arbitrary bounded sequence of functions from the space V. Since the embedding $W_2^1(Q) \subset L_2(Q)$ is compact [32–34], it follows from the sequence $\{v_n(x,t)\}_{n=1}^{\infty}$ that a subsequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$, strongly converging in the space $L_2(Q)$ to some function $v_0(x,t)$ belonging to the space V. Let us note that the boundedness in the space V of the sequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ and the strong convergence of the sequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ in the space $L_2(Q)$ implies that the sequence $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$ is fundamental in the space $L_2(Q)$. Indeed, we will have the following equality

$$\int\limits_Q (v_{n_k xxx} - v_{n_l xxx})^2\, dx\, dt = -\int\limits_Q (v_{n_k} - v_{n_l})(v_{n_k xxxxxx} - v_{n_l xxxxxx})\, dx\, dt.$$

It follows from this equality that the sequence $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$

If we put $u_{n_k} = \Phi(v_{n_k})$, $v_{kl}(x,t) = v_{n_k}(x,t) - v_{n_l}(x,t)$, $u_{kl}(x,t) = u_{n_k}(x,t) - u_{n_l}(x,t)$, then we will have the following equality

$$\frac{R_1 + G_\mu(\varphi_1(u_{n_k}))}{A}u_{klt} - u_{klxxx} - \varepsilon u_{klxxxxxx} + \beta u_{kl} = \frac{1}{A}\varphi_1(v_{kl})u_{n_l t}. \tag{11}$$

Repeating for this equality the proof of estimate (10) and taking into account the fundamental nature of the sequences $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ and $\{v_{n_k xxx}(x,t)\}_{k=1}^{\infty}$ in the space $L_2(Q)$, we see that the sequence $\{u_{n_k}(x,t)\}_{k=1}^{\infty}$ is fundamental in the space V.

Thus, from any sequence $\{v_n(x,t)\}_{n=1}^{\infty}$ bounded in the space V, one can extract a subsequence $\{v_{n_k}(x,t)\}_{k=1}^{\infty}$ such that the sequence $\{\Phi(v_{n_k})\}_{k=1}^{\infty}$ is fundamental in the space V. And this means that the operator Φ is compact in the space V

Everything proved above means that for the operator Φ on the ball of radius R^* of the space V, all conditions of Schauder's theorem are satisfied. Consequently, the operator Φ has at least one fixed point in the space V. Obviously, this fixed point will represent the solution of the boundary value problem (7), (2), (3), (8).

Let us show that under the conditions of the theorem for the solutions $u(x,t)$ of the boundary value problem (7), (2), (3), (8), there are a priori estimates uniform in the parameter E, which will allow us to establish the existence of solutions to the boundary value problem (6), (2), (3). We multiply equation (7) by the function $-u_{xxxxxx}(x,t)$ and integrate over the rectangle Q. Applying the formula for integration by parts both on the left and on the right in the obtained equality, we obtain the first estimate uniform in ε

$$\left(\int\limits_Q u_{xxx}^2\,dx\,dt\right)^{1/2} \le \frac{1}{\beta}\left(\int\limits_Q f_{xxx}^2\,dx\,dt\right)^{1/2}. \tag{13}$$

At the next step, we multiply equation (7) by the function $(T-t)u(x,t)$ and integrate over the rectangle Q. Taking into account the inequality $R_1 + G_\mu(\varphi_1(u)) \ge (1-\mu_0)R_1$ and applying Helder's inequality , we obtain that for the solutions $u(x,t)$ of the boundary value problem (7), (2), (3), (8), the second estimate uniform in ε is satisfied

$$\left(\int\limits_Q u^2\,dx\,dt\right)^{1/2} \le \frac{2A}{(1-\mu_0)R_1}\left(\int\limits_Q f_1^2\,dx\,dt\right)^{1/2}. \tag{14}$$

We should note that at the first step, one more estimate uniform in ε is derived

$$\varepsilon\int\limits_Q u_{xxxxxx}^2\,dx\,dt \le \frac{1}{\beta}\int\limits_Q f_{xxx}^2\,dx\,dt. \tag{15}$$

Finally, there is also the obvious last estimate in ε,

$$\int\limits_Q u_t^2\,dx\,dt \le C_0\int\limits_Q f^2\,dx\,dt, \tag{16}$$

where the constant C_0 is determined by the numbers β, μ_0, R_1 и T. Estimates (13) - (16), as well as the reflexivity property of the Hilbert space, allow, after choosing a sequence $\{\varepsilon_m\}_{m=1}^\infty$ of positive numbers that monotonically tends to zero, to go over to the family of solutions to boundary value problems (7), (2), (3), (8) with $\{\varepsilon_m\}_{m=1}^\infty$ to a weakly converging subsequence and then, in the limit, obtain a solution $u(x,t)$ of the boundary value problem (6_μ), (2), (3), and the solution for which the estimates (13) , (14) and (16). For this solution, due to the inequality

$$|\varphi_1(u)| \le T^{1/2}\|N\|_{L_2(\Omega)}\left(\int\limits_Q u_{xxx}^2\,dx\,dt\right)^{1/2} + \beta T^{1/2}\|N\|_{L_2(\Omega)}\left(\int\limits_Q u^2\,dx\,dt\right)^{1/2},$$

of estimates (13) and (14), as well as the condition $K_0 \le R_1$, we will obtain the following estimate

$$|\varphi_1(u)| \le \mu_0 R_1.$$

Therefore, for the found solution $u(x,t)$ of the boundary value problem (6_μ), (2), (3), the equation $G_\mu(\varphi_1(u)) = \varphi_1(u)$ will be satisfied. This means that the found function $u(x,t)$ will be the desired solution of the boundary value problem (6), (2), (3). The theorem is proved.

Theorem 2 *Let the following conditions be satisfied*

$$N(x) \in L_2(\Omega); \;\; \beta > 0, \quad A > 0; \;\; \frac{\partial^k f(x,t)}{\partial x^k} \in L_2(Q), \quad k = \overline{0,3},$$

$$f(0,t) = f_x(0,t) = f_{xx}(1,t) = 0 \quad npu \quad t \in (0,T);$$

$$\exists\, \mu_0 \in (0,1): \;\; K_0 \leq \int\limits_Q N(x) f(x,t)\, dx\, dt.$$

Then inverse problem I has a solution $\{u(x,t), \alpha\}$ *such that* $u(x,t) \in L_2(0,T; W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $\alpha > 0$.

Proof. In the boundary value problem (6), (2), (3), we will take the number $\int\limits_Q N(x) f(x,t)\, dx\, dt$ as the number R_1. According to Theorem 1, this problem has a regular solution $u(x,t)$. Let us define the numberα:

$$\alpha = \frac{R_1 + \varphi_1(u)}{A}. \tag{17}$$

It is obvious that the number will be positive and that the number α and the function $u(x,t)$ are related in the rectangle Q by equation (1). Let us show that the function $u(x,t)$ will satisfy the overdetermination condition (4).

A consequence of equation (1) is the equality

$$\alpha u(x,t) - \int\limits_0^t [u_{xxx}(x,\tau) - \beta u(x,\tau)]\, d\tau = \int\limits_0^t f(x,\tau)\, d\tau.$$

Setting $t = T$ in this equality, then multiplying it by the function $N(x)$ and integrating over Ω, we obtain the ratio

$$\alpha \int\limits_\Omega N(x) f(x,T)\, dx = R_1 + \varphi_1(u).$$

On the other hand, representation (17) gives the equality

$$\alpha A = R_1 + \varphi_1(u).$$

From the two obtained equations and from the positiveness of the number α and we will see that for the solution $u(x,t)$ of the boundary value problem (6), (2), (3) with the above number R_1, the overdetermination condition (4) is satisfied. Consequently, the function $u(x,t)$ and the number α give the desired solution to Inverse Problem I. The theorem is proved.

4 Solvability of Inverse Problem II

We will use a method based on the transition from the studied inverse problem to some direct problem for a nonlinear loaded differential equation again.

Let R_1 be a given positive number, $N_1(x)$ is a function for which the following equalities are satisfied.

$$N_1'''(x) = N(x), \quad N_1(0) = N_1'(0) = N_1''(1) = 0,$$

$\varphi_2(v)$ where

$$\varphi_2(v) = \int\limits_\Omega N_1(x)v(x,T)\,dx + \beta \int\limits_Q N_1(x)v(x,t)\,dx\,dt.$$

Let us consider the following problem: to find a function $u(x,t)$ which is a solution to the equation

$$u_t - \frac{R_1 - \varphi_2(v)}{A} u_{xxx} + \beta u = f(x,t) \tag{18}$$

in rectangle Q and such that conditions (2) and (3) are satisfied. This problem is an auxiliary direct problem for a loaded differential equation.

Let μ_0 be a number from $(0,1)$. We will then have the following

$$f_2(x,t) = (T-t)f_{tt}(x,t),$$

$$M_1 = \sqrt{\frac{2}{\beta}} \|N_1\|_{L_2(\Omega)} \|f_t\|_{L_2(Q)},$$

$$M_2 = \frac{\beta A T^{1/2} \|N_1\|_{L_2(\Omega)}}{(1-\mu_0)} (\|f_t\|_{L_2(Q)} + 2\|f_2\|_{L_2(Q)}),$$

$$M_0 = \frac{1}{2\mu_0}\left(M_1 + \sqrt{M_1^2 + 4\mu_0 M_2}\right).$$

Theorem 3 *Let the following conditions be satisfied:*

$$N(x) \in L_2(\Omega); \;\; \beta > 0, \quad A > 0;$$

$$\frac{\partial^{k+l} f(x,t)}{\partial x^k \partial t^l} \in L_2(Q), \quad k = \overline{0,3}, \quad l = \overline{0,2}, \quad \frac{\partial^{k+l} f(0,t)}{\partial x^k \partial t^l} = 0, \quad k = \overline{0,1},$$

$$l = \overline{0,2}, \quad t \in (0,T), \quad \frac{\partial^{2+l} f(1,t)}{\partial x^2 \partial t^l} = 0, \quad l = \overline{0,2}, \quad t \in (0,T),$$

$$\frac{\partial^l f(x,0)}{\partial t^l} = 0, \quad l = \overline{0,1}, \quad x \in \Omega;$$

$$\exists\, \mu_0 \in (0,1): \;\; M_0 \le R_1.$$

Then the boundary value problem (18), (2), (3) has a solution $u(x,t)$ such that $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_1(u)| \le \mu_0 R_1$.

Proof. Let μ be the number $\mu_0 R_1$, $\widetilde{\varphi}_2(v)$ be the functional.

$$\widetilde{\varphi}_2(v) = \int\limits_Q N_1(x)v(x,t)\,dx\,dt + \beta \int\limits_Q \left(\int\limits_0^t N_1(x)w(x,\tau)\,d\tau \right) dx\,dt.$$

Let us consider the problem: to find a function $w(x,t)$ that is a solution of the equation

$$w_t - \frac{R_1 - G_\mu(\widetilde{\varphi}_2(w))}{A} w_{xxx} + \beta w = f_{tt}(x,t) \tag{19}$$

in the rectangle Q and such that conditions (2) and (3) are satisfied for it. Repeating the proof of the solvability of problem (6_μ), (2), (3), it is easy to show that under the conditions of the theorem this problem has a solution $w(x,t)$ such that $w(x,t) \in L_2(0,T;W_2^3(\Omega))$, $w_t(x,t) \in L_2(Q)$. We define the function $v(x,t)$ as:

$$v(x,t) = \int\limits_0^t w(x,\tau)\,d\tau.$$

Since $f_t(x,0) = 0$, then for the function $v(x,t)$ in the rectangle Q the equation

$$v_t - \frac{R_1 - G_\mu(\varphi_2(v))}{A} v_{xxx} + \beta v = f_t(x,t), \tag{20}$$

will be satisfied and conditions (2) and (3) will also be satisfied. Let us show that the required a priori estimates hold for the functions $w(x,t)$ and $v(x,t)$. Let us multiply equation (19) by the function $(T-t)w(x,t)$ and integrate over the rectangle Q. After simple transformations, we obtain the estimate

$$\left(\int\limits_Q w^2\,dx\,dt\right)^{1/2} \le 2\left(\int\limits_Q f_2^2\,dx\,dt\right)^{1/2}. \tag{21}$$

At the next step, we multiply equation (20) by the function $v(x,t)$ and integrate over the rectangle Q. The consequence of the obtained equality will be the second estimate

$$\int\limits_\Omega v^2(x,T)\,dx \le \frac{2}{\beta}\int\limits_Q f_t^2\,dx\,dt. \tag{22}$$

Next, we multiply equation (20) by the function $-v_{xxx}(x,t)$ and integrate over the rectangle Q. Using the inequality $R_1 - G_\mu(\varphi_1(v)) \ge (1-\mu_0)R_1$, applying Helder's inequality and taking into account the estimate (21), we obtain inequality

$$\left(\int\limits_Q v_{xxx}^2\,dx\,dt\right)^{1/2} \le \frac{A}{(1-\mu_0)R_1}\left(\|f_t\|_{L_2(Q)} + 2\|f_2\|_{L_2(Q)}\right). \tag{23}$$

Inequalities (22) and (23) make it possible to estimate $|\varphi_2(v)|$:

$$|\varphi_2(v)| \le \|N_1\|_{L_2(\Omega)}\left(\int\limits_\Omega v^2\,dx\right)^{1/2} + \beta T^{1/2}\|N_1\|_{L_2(\Omega)}\left(\int\limits_Q v^2\,dx\,dt\right)^{1/2} \le$$

$$\leq \sqrt{\frac{2}{\beta}}\|N_1\|_{L_2(\Omega)}\|f_t\|_{L_2(Q)} + \beta T^{1/2}\|N_1\|_{L_2(\Omega)}\left(\int\limits_Q v_{xxx}^2\,dx\,dt\right)^{1/2} \leq$$

$$\leq M_1 + \frac{M_2}{R_1}. \tag{24}$$

The resulting estimate and the inequality $M_0 \leq R_1$ from the conditions of the theorem mean that the inequality $|\varphi_2(v)| \leq \mu_0 R_1$ is satisfied, and then the equality $G_\mu(\varphi_2(v)) = \varphi_2(v)$ is satisfied. The last equality means that the solution $v(x,t)$ to equation (20) is a solution to the equation

$$v_t - \frac{R_1 - \varphi_2(v)}{A}v_{xxx} + \beta v = f_t(x,t).$$

Let us define the function $u(x,t)$ in the following form:

$$u(x,t) = \int\limits_0^t v(x,\tau)\,d\tau.$$

Since $f(x,0) = 0$, the function $u(x,t)$ will be a solution to equation (18). The function $u(x,t)$ belongs to the required class, conditions (2) and (3) are fulfilled for it, as well as the inequality $|\varphi_2(u_t)| \leq \mu_0 R_1$ is satisfied. The theorem is proved. We will set the following

$$R_1 = \int\limits_\Omega N_1(x)f(x,T)\,dx.$$

Theorem 4 *Let R_1 R1 be positive, and all conditions of Theorem 3 are satisfied for it and for a given function $f(x,t)$ and for β number. Then inverse problem II has a solution $\{u(x,t),\alpha\}$ such that $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $\alpha > 0$.*

Proof. For the indicated number R_1, let us consider the boundary value problem (18), (2), (3). According to Theorem 3, this problem has a solution $u(x,t)$ such that $u(x,t) \in L_2(0,T;W_2^3(\Omega))$, $u_t(x,t) \in L_2(Q)$, $|\varphi_2(u_t)| \leq \mu_0 R_1$. We will define the number α:

$$\alpha = \frac{R_1 - \varphi_2(u_t)}{A}. \tag{25}$$

It is obvious that this number and the function $u(x,t)$ will be related in the rectangle Q by equation (5), and that the number α will be positive. Further, the function $u(x,t)$ will satisfy the overdetermination condition (4) - this is proved quite similarly as the proof of the fulfillment of the overdetermination condition in Theorem 2. Therefore, the solution $u(x,t)$ to the boundary value problem (18), (2), (3) and the number defined by formula (25) give the desired solution to Inverse Problem II.

The theorem is proved.

5 Remarks and additions

1. As mentioned above, the paper considers model equations with multiple characteristics. It is easy to carry out all the reasoning and obtain theorems on the solvability of inverse problems of I and II types for more general equations. For example, in equations (1) and (5) the coefficient β can be a function of the variables x and t, there can be lower terms (derivatives with respect to the variable x of the first and second orders), the coefficient of the third derivative in equation (1) and the coefficient of the derivative with respect to the time variable in equation (5) can be functions of independent variables. The number of calculations and conditions in similar more general problems increases significantly, but the essence of the results on solvability remains the same.

2. Along with inverse problems I and II, it is not difficult, practically repeating all reasoning and calculations, to study the solvability of inverse problems with the setting for $x = 1$ the value of the solution (and not the second derivative of the solution). All statements of Theorems 1–4 on the existence of solutions remain valid, only the condition $f_{xx}(1,t) = 0$ will need to be replaced by the condition $f(1,t) = 0$ (we only specify that in the regularized problem for equation (7) additional boundary conditions will have the form $u_{xxx}(0,t) = u_{xxx}(1,t) = u_{xxxx}(1,t) = 0$ при $t \in (0,T)$).

3. It is easy to establish that the conditions $K_0 \leq R_1$ of Theorems 1 and 2, μ_0 of Theorems 3 and 4 are satisfied for a fixed value of ?0 and for a given function $f(x,t)$ if the number β is large, but the number A is small.

4. Theorems 1 and 3 on the solvability of initial-boundary value problems for "loaded" differential equations with multiple characteristics (as well as similar theorems with the replacement of the condition $u_{xx}(1,t) = 0$ by the condition $u(1,t) = 0$) have, in our opinion, an independent value.

6 Conclusion

In this paper we investigate the inverse problem for differential equations with multiple characteristics. The paper shows the results of the study of solvability in Sobolev spaces. Inverse problems arise when the characteristics of the object of interest are not available for direct observation. A distinctive feature of the problems studied in this paper is that the unknown coefficient is a numerical parameter rather than a function of some independent variables. Similar problems have been studied before, but only for classical parabolic, hyperbolic, and elliptic equations. It should be noted that inverse problems of determination for differential equations with multiple characteristics, together with the solution of numerical parameters that are coefficients of the equation itself, have not been studied before.

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ONE RESULT ON BOUNDEDNESS OF THE HILBERT TRANSFORM

In mathematics and in signal theory, the Hilbert transform is an important linear operator that takes a real-valued function and produces another real-valued function. The Hilbert transform is a linear operator which arises from the study of boundary values of the real and imaginary parts of analytic functions. Also, it is a widely used tool in signal processing. The Cauchy integral is a figurative way to motivate the Hilbert transform. The complex view helps us to relate the Hilbert transform to something more concrete and understandable. Moreover, the Hilbert transform is closely connected with many operators in harmonic analysis such as Laplace and Fourier transforms which have numerous application in partial and ordinary differential equations. In this paper, we study boundedness properties of the classical (singular) Hilbert transform acting on Marcinkiewicz spaces. More precisely, we obtain if and only if condition for boundedness of the Hilbert transform in Marcinkiewicz function spaces.
Key words: Symmetric (quasi-)Banach function space, Hilbert transform, Calderón operator, Marcinkiewicz space.

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Гильберт түрлендіруінің шенелгендігі туралы бір нәтиже

Математикада және сигнал теориясында Гильберт түрлендіруі нақты мәнді функцияны қабылдайтын және оған басқа бір нақты мәнді функцияны сәйкес қоятын маңызды сызықтық оператор болып табылады. Гильберт түрлендіруі - аналитикалық функциялардың нақты және жорамал бөліктерінің шекаралық мәндерін зерттеу нәтижесінде пайда болатын сызықтық оператор. Сондай-ақ, ол сигналды өңдеуде кеңінен қолданылатын құрал болып табылады. Коши интегралы Гильберт түрлендіруін қолдану үшін маңызды рөл атқарады. Комплекс тұрғыда бізге Гильберт түрлендіруін нақтырақ және түсінікті нәрсемен байланыстыруға болады. Сонымен қатар, Гильберт түрлендіруі гармоникалық талдаудың көптеген операторларымен тығыз байланысты, мысалы, қарапайым және дербес туындылы дифференциалдық теңдеулерде көп қолданылатын Лаплас және Фурье түрлендірулері. Бұл жұмыста біз Марцинкевич кеңістігіндегі функцияларға әсер ететін классикалық (сингулярлық) Гильберт түрлендіруінің шенелгендік қасиеттерін зерттедік. Дәлірек айтқанда, Марцинкевич функционалдық кеңістіктеріндегі Гильберт түрлендіруінің шенелген болу шартын алдық.

Түйін сөздер: Симметриялық(квази-) Банах кеңістігі, Гильберт түрлендіруі, Кальдерон операторы, Марцинкевич кеңістігі.

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Один результат об ограниченности преобразования Гильберта

В математике и теории сигналов преобразование Гильберта является важнейшим линейным оператором, который переводит функцию действительной переменной в другую функцию действительной переменной. Преобразование Гильберта - линейный оператор, возникающий при изучении граничных значений действительной и мнимой частей аналитических функций. Кроме того, это широко используемый инструмент в обработке сигналов. Интеграл Коши - образный способ мотивировать преобразование Гильберта. Комплексное представление помогает нам связать преобразование Гильберта с чем-то более конкретным и понятным. Более того, преобразование Гильберта тесно связано со многими операторами гармонического анализа, такими как преобразования Лапласа и Фурье, которые находят многочисленные применения в обыкновенных дифференциальных уравнениях и в уравнениях с частными производными. В данной работе изучаются свойства ограниченности классического (сингулярного) преобразования Гильберта, действующего на пространствах Марцинкевича. Точнее, мы получили необходимое и достаточное условие ограниченности преобразования Гильберта в функциональных пространствах Марцинкевича.

Ключевые слова: Симметричные(квази-) банаховы пространства, преобразование Гильберта, оператор Кальдерона, пространство Марцинкевича.

1 Introduction

The Hilbert transform is one of the powerful operators in the field of signal theory. Given a locally integrable measurable function f, its Hilbert transform, denoted by $\mathcal{H}(f)$, is calculated through the integral in the sense of principal value

$$(\mathcal{H}f)(t) = p.v.\frac{1}{\pi}\int_{\mathbb{R}}\frac{f(s)}{t-s}ds = \lim_{\varepsilon\to 0+}\frac{1}{\pi}\int_{|s-t|\geq\varepsilon}\frac{f(s)}{t-s}ds.$$

The Hilbert transform is named after German mathematician David Hilbert (1862-1943). Its first use dates back to 1905 in Hilbert's work concerning analytical functions in connection to the Riemann problem. In 1928 it was proved by Marcel Riesz (1886-1969) that the Hilbert transform is a bounded linear operator on $L_p(\mathbb{R})$ for $1 < p < \infty$. This result was generalized for the Hilbert transform in several dimensions (and singular integral operators in general) by Antoni Zygmund (1900-1992) and Alberto Calderón (1920-1998). Our investigation concerns with boundedness of the Hilbert transform in so called rearrangement invariant Banach function spaces which received a lot of attention since Boyd's pioneer work in 1966 [1] (see also [2]). We also refer the reader to recent papers [6–11] and references there in. In this note, we study boundedness of the Hilbert transform from one Marcinkiewicz space to another.

2 Materials and methods

2.1 Symmetric Banach function spaces and their Köthe dual spaces

Let (I, m) denote the measure space $I = \mathbb{R}_+, \mathbb{R}$, where $\mathbb{R}_+ := (0, \infty)$ and $\mathbb{R}$ is the set of real numbers, equipped with Lebesgue measure m. Let $L(I, m)$ be the space of all measurable real-valued functions on I equipped with Lebesgue measure m, i.e. functions which coincide almost everywhere are considered identical. Define $L_0(I)$ to be the subset of $L(I, m)$ which consists of all functions f such that $m(\{t : |f(t)| > s\})$ is finite for some $s > 0$.

For $f \in L_0(I)$ (where $I = \mathbb{R}_+$ or $\mathbb{R}$), we denote by f^* the decreasing rearrangement of the function $|f|$. That is,

$$f^*(t) = \inf\{s \geq 0 : \ m(\{|f| > s\}) \leq t\}, \quad t > 0.$$

Definition 1 *We say that* $(E(I), \|\cdot\|_{E(I)})$ *is a symmetric (quasi-)Banach function space on* I, *if the following conditions hold:*

(a) $E(I)$ *is a subset of* $L_0(I)$;

(b) $(E(I), \|\cdot\|_{E(I)})$ *is a (quasi-)Banach space;*

(c) *If* $f \in E(I)$ *and if* $g \in L_0(I)$ *are such that* $g^*(t) \leq f^*(t), t > 0$ *then* $g \in E(I)$ *and* $\|g\|_{E(I)} \leq \|f\|_{E(I)}$.

It is well known that $L_p(I)$, $(0 < p \leq \infty)$ is a classical example of symmetric (quasi-)Banach space of functions

We say that $g \in L_0(I)$ is submajorized by $f \in L_0(I)$ in the sense of Hardy–Littlewood–Pólya (written $g \prec\prec f$) if

$$\int_0^t g^*(s)ds \leq \int_0^t f^*(s)ds, \quad t \geq 0.$$

Let E be a symmetric Banach function space on I with Lebesgue measure m the Köthe dual space $E^\times$ on I is defined by

$$E(I)^\times = \left\{ g \in L_0(I) : \int_0^\infty |f(t)g(t)|dt < \infty, \ \forall f \in E(I) \right\}.$$

The space $E^\times$ is Banach with the norm

$$\|g\|_{E(I)^\times} := \sup \left\{ \int_0^\infty |f(t)g(t)|dt : f \in E(I), \ \|f\|_{E(I)} \leq 1 \right\}.$$

If E is a symmetric Banach function space, then $(E^\times, \|\cdot\|_{E^\times})$ is also a symmetric Banach function space (cf. [3, Section 2.4]). For more details on Köthe duality we refer to [3,5].

2.2 Lorentz and Marcinkiewicz spaces

For the function $\varphi(t) := \log(1+t)$, $t > 0$, the Lorentz space $\Lambda_{\log}(I)$ is defined by setting

$$\Lambda_{\log}(I) := \left\{ f \in L_0(I) : \int_{\mathbb{R}_+} \frac{f^*(s)}{1+s} ds < \infty \right\}$$

equipped with the norm

$$\|f\|_{\Lambda_{\log}(I)} := \int_{\mathbb{R}_+} \frac{f^*(s)}{1+s} ds.$$

Definition 2 *[4, Definition II. 1.1, p. 49] A function* φ *on the semiaxis* $[0, \infty)$ *is said to be quasiconcave if*

(i) $\varphi(t) = 0 \Leftrightarrow t = 0$;

(ii) $\varphi(t)$ *is positive and increasing for* $t > 0$;

(iii) $\frac{\varphi(t)}{t}$ *is decreasing for* $t > 0$.

Observe that every nonnegative concave function on $[0, \infty)$ that vanishes only at origin is quasiconcave. The reverse, however, is not true. But, we may replace, if necessary, a quasiconcave function φ by its smallest concave majorant $\widetilde{\varphi}$ such that

$$\frac{1}{2}\widetilde{\varphi} \leq \varphi \leq \widetilde{\varphi}$$

(see [3, Proposition 5.10, p. 71]). Let $\phi : [0, \infty) \to [0, \infty)$ be a quasiconcave function for which $\lim\limits_{t \to 0+} \phi(t) = 0$ (or simply $\phi(0+) = 0$). Define the Marcinkiewicz space $M_\phi(I)$ as follows

$$M_\phi(I) := \left\{ f \in L_0(I) : \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds < \infty \right\}$$

with the norm

$$\|f\|_{M_\phi(I)} := \sup_{t>0} \frac{1}{\phi(t)} \int_0^t f^*(s)ds.$$

2.3 Weak-L_1 and $L_{1,\infty} + L_\infty$ spaces

Define the weak-L_1 space $L_{1,\infty}(I)$ by setting

$$L_{1,\infty}(I) = \{f \in L_0(I) : \sup_{t>0} tf^*(t) < \infty\}$$

and equip it with the quasi-norm

$$\|f\|_{L_{1,\infty}(I)} = \sup_{t>0} tf^*(t), \quad f \in L_{1,\infty}(I).$$

The space $L_{1,\infty}(I)$ is a quasi-Banach symmetric space.

Equip the vector space $L_0(I)$ on I with the topology of convergence in measure. The space $(L_{1,\infty} + L_\infty)(I) = L_{1,\infty}(I) + L_\infty(I)$ consists of functions for which

$$\begin{aligned} \|f\|_{(L_{1,\infty}+L_\infty)(I)} &= \inf\{\|f_1\|_{L_{1,\infty}(I)} + \|f_2\|_{L_\infty(I)} : f = f_1 + f_2, \\ &\quad f_1 \in L_{1,\infty}(I),\ f_2 \in L_\infty(I)\} < \infty. \end{aligned}$$

2.4 Calderón operator and Hilbert transform

For a function $f \in \Lambda_{\log}(\mathbb{R}_+)$, define the Calderón operator $S : \Lambda_{\log}(\mathbb{R}_+) \to (L_{1,\infty} + L_\infty)(\mathbb{R}_+)$ as follows

$$(Sf)(t) := \frac{1}{t} \int_0^t f(s)ds + \int_t^\infty f(s)\frac{ds}{s}, \ \ t > 0. \tag{1}$$

For more details on this operator, see for instance [3], [7]. If $f \in \Lambda_{\log}(\mathbb{R})$, then the classical Hilbert transform $\mathcal{H}$ is defined by the principal-value integral

$$(\mathcal{H}f)(s) := p.v.\frac{1}{\pi} \int_{\mathbb{R}} \frac{f(\eta)}{s - \eta} d\eta, \ \ \forall f \in \Lambda_{\log}(\mathbb{R}), \tag{2}$$

(see, e.g. [3, Chapter III. 4]).

We use standart methods of integration theory and the theory of rearrangement invariant Banach function spaces. We also use some results on Köthe duality of rearrangement invariant Banach function spaces.

3 Main results

Let $\phi : [0,\infty) \to [0,\infty)$ be a quasi-concave function. Define a function ψ by the following formula

$$\psi(t) := \inf_{t<s} \frac{s}{\phi(s)\log(\frac{t}{s})}. \tag{3}$$

We need the following lemmas.

Lemma 1 *If a function $\phi : [0,\infty) \to [0,\infty)$ is quasi-concave, then the function ψ defined by the formula (3) is also quasi-concave.*

Proof. First, it is easy to see that $\psi(x) = 0$ if and only if $x = 0$. By changing variables in (3), we obtain

$$\psi(u) := \inf_{\omega>1} \frac{u\omega}{\phi(u\omega)(1+\log(\omega))}$$

Since ϕ is quasi-concave, it follows that

$$\begin{aligned}\psi(u_1) &:= \inf_{w>1} \frac{u_1 w}{\phi(u_1 w)(1+\log(w))} = \inf_{w>1} \frac{1}{\frac{\phi(u_1 w)}{u_1 w}\cdot(1+\log(w))} \\ &\leq \inf_{w>1} \frac{1}{\frac{\phi(u_2 w)}{u_2 w}\cdot(1+\log(w))} = \inf_{w>1} \frac{u_2 w}{\phi(u_2 w)(1+\log(w))} = \psi(u_2), \quad 0<u_1<u_2,\end{aligned}$$

which shows that ψ is increasing for any $u>0$. Similarly, we can prove that $\frac{\psi(u)}{u}$ is decreasing for any $u>0$, thereby completing the proof.

Let E and F be symmetric spaces on $\mathbb{R}_+$ with the Fatou property and let $E^\times$ and $F^\times$ their Köthe dual spaces on $\mathbb{R}_+$, respectively.

Lemma 2 *The operator S is self-adjoint with respect to L_1-pairing in the following sense*

$$\int_{\mathbb{R}_+} (Sf)(s)g(s)ds = \int_{\mathbb{R}_+} f(s)(Sg)(s)ds, \tag{4}$$

for all non-negative functions $x, y \in \Lambda_{\log}(\mathbb{R}_+)$.

If $E(\mathbb{R}_+), F^\times(\mathbb{R}_+) \subseteq \Lambda_{\log}(\mathbb{R}_+)$, then $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$ if and only if $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$, and we have

$$\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)} = \|S\|_{F^\times(\mathbb{R}_+)\to E^\times(\mathbb{R}_+)}. \tag{5}$$

Proof. The equality (4) follows from the formula (6.31) in [4, Chapter II.7, p.138].

If $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$, then $S^\times : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$. Since $S^\times = S$ by (4), it follows that $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$.

Conversely, if $S : F^\times(\mathbb{R}_+) \to E^\times(\mathbb{R}_+)$, then $S^\times : E^{\times\times}(\mathbb{R}_+) \to F^{\times\times}(\mathbb{R}_+)$. Since $E(\mathbb{R}_+)$ and $F(\mathbb{R}_+)$ have Fatou property, we have $E^{\times\times}(\mathbb{R}_+) = E(\mathbb{R}_+)$ and $F^{\times\times}(\mathbb{R}_+) = F(\mathbb{R}_+)$. Therefore, again using $S^\times = S$ we obtain that $S : E(\mathbb{R}_+) \to F(\mathbb{R}_+)$. In this case, we have

$$\begin{aligned}\|S\|_{E(\mathbb{R}_+)\to F(\mathbb{R}_+)} &= \sup_{\|f\|_{E(\mathbb{R}_+)}\leq 1} \sup_{\|g\|_{F^\times(\mathbb{R}_+)}\leq 1} \left|\int_{\mathbb{R}_+} (Sf)(s)g(s)ds\right| \\ &\overset{(4)}{=} \sup_{\|g\|_{F^\times(\mathbb{R}_+)}\leq 1} \sup_{\|f\|_{E(\mathbb{R}_+)}\leq 1} \left|\int_{\mathbb{R}_+} f(s)(Sg)(s)ds\right| = \|S\|_{F^\times(\mathbb{R}_+)\to E^\times(\mathbb{R}_+)},\end{aligned}$$

which completes the proof.

Theorem 1 *Let ϕ and φ be increasing concave functions on $[0,\infty)$ vanishing at the origin. Suppose $M_\phi(\mathbb{R}_+) \subset \Lambda_{\log}(\mathbb{R}_+)$. We have*

$$S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$$

if and only if

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'.$$

Proof. Since $M_\phi(\mathbb{R}_+)^\times = \Lambda_\phi(\mathbb{R}_+)$, $M_\varphi(\mathbb{R}_+)^\times = \Lambda_\varphi(\mathbb{R}_+)$, and both spaces have the Fatou property, it follows from Lemma 2 that

$$S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$$

if and only if

$$S : \Lambda_\varphi(\mathbb{R}_+) \to \Lambda_\phi(\mathbb{R}_+).$$

But, by Lemma 8 and Lemma 9 in [8] the latter one is equivalent to

$$\|S\chi_{(0,u)}\|_{\Lambda_\phi(\mathbb{R}_+)} \le c_{\phi,\varphi}\varphi(u), \quad u > 0,$$

which is, in fact, equivalent to

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'. \tag{6}$$

Indeed, we have

$$\begin{aligned} \|S\chi_{(0,u)}\|_{\Lambda_\phi(\mathbb{R}_+)} &= \int_0^\infty S\chi_{(0,u)}(s)d\phi(s) = \int_0^\infty S\chi_{(0,u)}(s)\phi'(s)ds \\ &= \int_0^\infty \chi_{(0,u)}(s)(S\phi')(s)ds = \int_0^u (S\phi')(s)ds, \quad u > 0. \end{aligned} \tag{7}$$

For the left hand side of (6), since $\varphi(+0) = 0$, we have

$$c_{\phi,\varphi} \cdot \varphi'(u) = c_{\phi,\varphi} \cdot \int_0^u \varphi(s)ds, \quad u > 0. \tag{8}$$

Combining (7) and (8), we obtain (6), thereby completing the proof.

Corollary 1 *Let the assumptions of Theorem 1 hold. Then the Hilbert transform*

$$\mathcal{H} : M_\phi(\mathbb{R}) \to M_\varphi(\mathbb{R})$$

is bounded if and only if

$$S(\phi') \prec\prec c_{\phi,\varphi} \cdot \varphi'.$$

Proof. Following the argument in [8, Corollary 13] mutatis mutandi, we obtain that the Hilbert transform $\mathcal{H} : M_\phi(\mathbb{R}) \to M_\varphi(\mathbb{R})$ is bounded if and only if $S : M_\phi(\mathbb{R}_+) \to M_\varphi(\mathbb{R}_+)$. Hence, the assertion follows from Theorem 1.

Proposition 1 *Let $\phi : [0,\infty) \to [0,\infty)$ be a quasi-concave function and let ψ be the function defined by the formula* (3)*. For every $u > 0$, there exists $y \in M_\phi(\mathbb{R}_+)$ such that $\|y\|_{M_\phi(\mathbb{R}_+)} \leq c_{abs} \cdot \psi(u)$ and*

$$\chi_{(0,u)} \leq S\mu(y).$$

Proof. Choose $w > 1$ such that

$$c_{abs} \cdot \psi(u) \geq \frac{uw}{\phi(uw)(1+\log(w))}, \quad u > 0. \tag{9}$$

By (1), we have

$$(S\chi_{(0,uw)})(u) = 1 + \log(w), \quad w > 1. \tag{10}$$

Set $y = \mu(y) = \frac{\chi_{(0,uw)}}{1+\log(w)}$. Then by (10), we obtain

$$1 = (Sy)(u) \leq (Sy)(t), \quad t < u,$$

and, therefore, $\chi_{(0,u)} \leq S\mu(y)$.

On the other hand, by (9) we obtain

$$\begin{aligned}
\|y\|_{M_\phi(\mathbb{R}_+)} &= \sup_{t>0} \frac{1}{\phi(t)} \int_0^t \mu(s,y)ds = \frac{1}{1+\log(w)} \cdot \sup_{t>0} \left\{ \frac{1}{\phi(t)} \int_0^{\min\{t,uw\}} ds \right\} \\
&= \frac{1}{1+\log(w)} \cdot \sup_{t>0} \left\{ \frac{1}{\phi(t)} \min\{t, uw\} \right\} \\
&= \frac{1}{1+\log(w)} \cdot \max \left\{ \sup_{t<uw} \frac{t}{\phi(t)}, \sup_{t\geq uw} \frac{uw}{\phi(t)} \right\} \\
&= \frac{1}{1+\log(w)} \cdot \frac{uw}{\phi(uw)} \leq c_{abs} \cdot \psi(u).
\end{aligned}$$

This concludes the proof.

4 Conclusions

In this paper, we studied boundedness properties of the classical Hilbert transform acting on Marcinkiewicz spaces. We obtained if and only if condition for boundedness of the Hilbert transform from one Marcinkiewicz space into another. We obtained results by using standart methods of integration theory and the theory of rearrangement invariant Banach function spaces. The results can be further used to identify the optimal range of the Hilbert transform acting on Marcinkiewicz spaces.

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ON UNIVERSAL NUMBERINGS OF GENERALIZED COMPUTABLE FAMILIES

The paper investigates the existence of universal generalized computable numberings of different families of sets and total functions. It was known that for every set A such that $\emptyset' \leq_T A$, a finite family $\mathcal{S}$ of A-c.e. sets has an A-computable universal numbering if and only if $\mathcal{S}$ contains the least set under inclusion. This criterion is not true for infinite families. For any set A there is an infinite A-computable family of sets $\mathcal{S}$ without the least element under inclusion that has an A-computable universal numbering; moreover, the family $\mathcal{S}$ consist pairwise not intersect sets. If A is a hyperimmune set, then an A-computable family F of total functions which contains at least two elements has no A-computable universal numbering. And if $\deg_T(A)$ is hyperimmune-free, then every A-computable finite family of total functions has an A-computable universal numbering. In this paper for a hyperimmune-free oracle A we show that any infinite effectively discrete family of sets has an A-computable universal numbering. It is also proved that if family $\mathcal{S}$ contains all co-finite sets and does not contain at least one co-c.e. set, then this family has no Σ_2^{-1}-computable universal numbering.
Key words: Rogers semilattice, Ershov hierarchy, computable numbering, universal numbering, hyperimmune set.

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Жалпыланған есептелімді үйірлердің универсал нөмірлеулері

Бұл жұмыста жиындардан және барлық жерде анықталған функциялардан құралған әр түрлі үйірлердің универсал жалпыланған есептелімді нөмірлеулері болуы туралы сұрақтар зерттеледі. Бұрыннан белгілі кез келген A жиыны үшін, бұл жерде $\emptyset' \leq_T A$, A-e.c. $\mathcal{S}$ ақырлы үйірінде A-есептелімді универсал нөмірлеу болады, сонда тек сонда ғана $\mathcal{S}$ үйірі ішкі жиын қатынасы бойынша ең кіші элементті қамтыса. Бұл критерий шексіз үйірлер үшін орындалмайды. Кез келген A жиыны үшін A-есептелімді универсал нөмірлеуі бар ішкі жиын қатынасы бойынша ең кіші элементті қамтымайтын жиындардан құралған шексіз A-есептелімді $\mathcal{S}$ үйірі табылады, сонымен қатар $\mathcal{S}$ үйірінде барлық элементтер жұптық қиылыспайды. Егер A– гипериммунды жиын болса, онда кем дегенде екі элементі бар барлық жерде анықталған функциялардан құралған A-есептелімді $\mathcal{F}$ үйірінде A-есептелімді универсал нөмірлеуі болмайды. Ал егер $\deg_T(A)$ гипериммунды-бос болса, онда әрбір барлық жерде анықталған функциялардан құралған үйірде A-есептелімді универсал нөмірлеуі болады. Бұл жұмыста біз A-гипериммунды-бос оракулымен есептелетін жиындардан құралған кез келген шексіз эффективті дискретті үйірлерде A-есептелімді универсал нөмірелуі болатынын көрсеттік. Сондай-ақ, егер $\mathcal{S}$ үйірі барлық толықтауы ақырлы жиындарды қамтыса және кем дегенде бір толықтауы рекурсив саналымды жиынды қамтымаса, онда бұл үйірдің универсал Σ_2^{-1}-есептелімді нөмірлеуі болмайтыны дәлелденді.

Түйін сөздер: Роджерс жарты торы, Ершов иерархиясы, есептелімді нөмірлеу, универсал нөмірлеу, гипериммунды жиын.

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Об универсальных нумерациях обобщенно вычислимых семейств

В работе исследуется вопросы существования универсальных обобщенно-вычислимых нумерации различных семейтв множеств и всюду определенных функций. Было известно, что для любого множество A такого, что $\emptyset' \leq_T A$, конечное семейство A -в.п. множеств $\mathcal{S}$ имеет A -вычислимую универсальную нумерацию тогда и только тогда, когда $\mathcal{S}$ содержит наименьшее по включению множество. Данный критерий не верно для бесконечных семейств. Для любого множества A существует бесконечное A-вычислимое семейство множеств $\mathcal{S}$ без наименьшего по включению элемента имеющий A-вычислимую универсальную нумерацию, более того в семействе $\mathcal{S}$ все элементы попарно не пересекаются. Если A — гипериммунное множество, тогда A-вычислимое семейство F всюду определенных функций содержащий не менее двух элементов не имеет универсальной A-вычислимой нумерации. А если $\deg_T(A)$ гипериммунно-свободно, тогда каждое A-вычислимое конечное семейство всюду определенных функций имеет A-вычислимую универсальную нумерацию. В данной работе показывается, что любое бесконечное эффективно дискретное семейство множеств с гипериммунно-свободным оракулом A имеет A-вычислимую универсальную нумерацию. Также доказывается, что если семейство $\mathcal{S}$ содержит все ко-конечные множества и не содержит хотябы один ко-в.п. множество, тогда данное семейство не имеет универсальную Σ_2^{-1}-вычислимую нумерацию.

Ключевые слова: полурешётка Роджреса, иерархия Ершова, вычислимая нумерация, универсальная нумерация, гипериммунное множество.

1 Introduction

In this paper, we consider some issues, mainly related to universal numberings. The interest in the study of such numberings is due to the fact that the universal computable numbering of any family contains information about all its computable numberings. Until now, various results have been obtained on generalized computable numberings in the arithmetic hierarchy and in the Ershov hierarchy. Most of the results in numbering theory are related to the study of properties of Rogers semilattices. Recall some definitions of the theory of numberings (see, for example, [1], [2] for details). Any surjective mapping α of the set ω of natural numbers onto a nonempty set S is called a *numbering* of S. Let α and β be numberings of S. We say that a numbering α is *reducible* to a numbering β (written $\alpha \leq \beta$) if there exists a computable function f such that $\alpha(n) = \beta(f(n))$ for any $n \in \omega$. We say that the numberings α and β are *equivalent* (written $\alpha \equiv \beta$) if $\alpha \leq \beta$ and $\beta \leq \alpha$.

Let A be a set of natural numbers, and let $\mathcal{S}$ be a family of A-computably enumerable (in Ershov hierarchy Σ^0_{n+1}-c.e.) set. We say that a numbering α of S is A-*computable* (in Ershov hierarchy Σ^0_{n+1}-*c.*) if its universal set $\{< x, n > | x \in \alpha(n)\}$ is A-c.e. (in Ershov hierarchy Σ^0_{n+1}-c.e.). Let $Com(\mathcal{S})$ be the set of all A-computable (in Ershov hierarchy Σ^0_{n+1}-c.) numberings of the family $\mathcal{S}$. The numbering reducibility relation is a preorder on $Com^A(\mathcal{S})$, (in Ershov hierarchy $Com^0_{n+1}(\mathcal{S})$-c.) which in the usual way induces some quotient structure $\mathcal{R}(\mathcal{S})$ (in Ershov hierarchy $\mathcal{R}^0_{n+1}(\mathcal{S})$), which is an upper semilattice and is called the *Rogers semilattice* of A-computable (in Ershov hierarchy Σ^0_{n+1}-c.) numberings of the family $\mathcal{S}$.

An A-computable numbering of a family $\mathcal{S}$ is *universal* if any A-computable numbering of S is reducible to that numbering. Denote by $A^{<x>}$ the set $\{y :< x, y >\in A\}$. An A-c.e. set with Godel number e is denoted by W^A_e.

A computable family $\mathcal{S}$ is called *effectively discrete* if there exists such a strongly computable sequence of finite sets such that [3]:

(1) for any $A \in \mathcal{S}$ there exists $T_i \subseteq A_i$;
(2) $T_i \subseteq T_j \Rightarrow T_i = T_j$;
(3) $T_i \subseteq A_j \in A$ and $T_i \subseteq A_p \in A$ then $A_j = A_p$.

A function f dominates a function g if $f(x) \geq g(x)$ for all x. A degree a is *hyperimmune* if there is a function $f \leq_T a$ which is not dominated by any recursive function: otherwise a is *hyperimmune-free* [4].

The Ershov hierarchy consists of finite and infinite levels. The finite levels of the hierarchy consists of the n-c.e. or Σ^{-1}_n sets for $n \in \omega$.

A set A is Σ^{-1}_n set, if either $n = 0$ and $A = \emptyset$, or $n > 0$ and there are c.e. sets $R_0 \supseteq R_1 \supseteq .... \supseteq R_{n-1}$ such that [5]

$$A = \bigcup_{i=0}^{[\frac{n-1}{2}]} (R_{2i} \setminus R_{2i+1}) \quad \text{(Here if } n \text{ is odd number then } R_n \neq \emptyset)$$

The class of Σ^{-1}_1 sets coincide with the class c.e. sets, Σ^{-1}_2 sets can be written as $R_1 \setminus R_2$, where $R_1 \supseteq R_2$ c.e. sets, therefore they are also called d-c.e. sets.

The n-c.e. sets are exactly those sets constitute the level Σ^{-1}_n of the Ershov hierarchy.

2 Literature review

In [6] the theory of numbering was studied for the first time and the concepts of computable numbering were proposed for constructive languages describing a numbered family of objects that were called generalized computable. Also, in fact, in [7] [also [8–10]] the work of Badaev and Goncharov was started to study the numberings A-computable, where A is a given oracle. Hyperimmune-free degrees have been studied extensively beginning with the work of Miller and Martin [11], and Jockusch and Soare 12. Despite the abundance of work dedicated to the generalized computable numberings, there are a number of open questions about Rogers semilattice of generalized computable families.

3 Material and methods

3.1 The formulation of the problem

Let us pass now to recalling the available results about the existence of A-computable universal numberings. The following results about criteria for finite and infinite families of A-c.e. sets with $\emptyset' \leq_T A$ were obtained in [7].

Theorem 1. *[7] For every set A such that $\emptyset' \leq_T A$, a finite family $\mathcal{S}$ of A-c.e. sets has an A-computable universal numbering if and only if $\mathcal{S}$ contains the least set under inclusion.*

Proof. We will give a brief of the proof [13].

Let $\mathcal{S} = \{A_1, A_2, ..., A_n\}$ be an A-computable family. Choose a family of finite sets $F_1, ..., F_n$ with the following property: for all $1 \leq i,\ j \leq n$, we have

$$F_i \subseteq F_j \Leftrightarrow A_i \subseteq A_j \Leftrightarrow F_i \subseteq A_j$$

Suppose that A_1 is a least element of $\mathcal{S}$ and $F_1 = \emptyset$. Let $\mathfrak{C}$ be the set of all chains, i.e. strictly increasing sequences $F_{i_1} \subset ... \subset F_{i_k}$. There is a maximal chain $\mathcal{C} = \{i_0 < i_1 < ... < i_k\}$ and denote by $A_{\mathcal{C}}$ the set of the family corresponding to $F_{\mathcal{C}}$ (i.e. $A_{\mathcal{C}} = A_i$ if and only if $F_{\mathcal{C}} = F_i$). It is easy (see [14] for details) to build a A-computable numbering α such that, for every e,

(1) We found maximal i_k with $F_{i_k} \subseteq W_e^A$. Then, clearly, $\alpha(e) = A_{i_k}$;

(2) We found number i_k such that $F_{i_k} \subseteq W_e^A$. If $W_e^A \in (\mathcal{S})$ then $W_e^A = \alpha(e)$.

Assume now $\rho(e) = W_e^A$. Thus, for any A-computable numbering β of the family $\mathcal{S}$, there exists a computable function $f(x)$ such that $\beta(x) = \rho(f(x))$ for all x. By constructing A-computable numbering α implies that $\beta(x) = \rho(f(x)) = \alpha(f(x))$, and therefore, α is universal A-computable numbering of the family $\mathcal{S}$.

Suppose that the family $\mathcal{S}$ has no least element. Let

$$\mathcal{S}_0 \leftrightharpoons \mathcal{S} \cup \{\emptyset\}$$

and by the above argument let α_0 be universal in $Com^A(\mathcal{S}_0)$. Let us define a A-computable function $f : \omega \rightarrow \{x | \alpha_0(x) \neq \emptyset\}$.

Then $\alpha = \alpha_0 \circ f \in Com^A(\mathcal{S})$. Let now $\beta \in Com^A(\mathcal{S})$, and define

$$\beta_0 \leftrightharpoons \begin{cases} \emptyset & \text{if } x = 0, \\ \beta(x-1) & \text{otherwise} \end{cases}$$

and let h be a computable function such that $\beta_0 = \alpha_0 \circ h$. Hence, for every x

$$\beta(x) = \beta_0(x+1) = \alpha_0(h(x+1)).$$

But $h(x+1) \in range(f)$. Let

$$k(x) \leftrightharpoons \mu y(f(y) = h(x+1)).$$

It follows that $\beta(x) = \alpha_0(f(k(x)))$, i.e. $\beta = \alpha \circ k$. Since k is A-computable, it follows that $\beta \leq \alpha$, hence α is A-universal in $Com^A(\mathcal{S})$. □

Badaev and Goncharov also showed that for $\emptyset' \leq_T A$ the presence of the least set under inclusion is neither necessary nor sufficient for an infinite family of A-c.e., sets to have an A-computable universal numbering [7].

Corollary 1. *[7] For every set A, there is an A-computable family that contains the least set under inclusion but has no A-computable universal numbering.*

The following theorem gives a negative answer to the question of Podzorov in [15]: Is it true that if the Rogers semilattice of a family of arithmetical sets has the greatest element then the family itself has the least set under inclusion?

Theorem 2. *[7] For every set A, there is an infinite A-computable family $\mathcal{S}$ of sets with pairwise disjoint elements such that $\mathcal{S}$ has an A-computable universal numbering.*

Also, in [7] the following questions were posed: Does the statement of Theorem 1 for finite families of sets remain valid if $\emptyset <_T A <_T \emptyset'$ or A is Turing incomparable with $\emptyset'$?

In [8–10], gives answers with hyperimmune and hyperimmune-free oracles. If $\emptyset <_T A <_T \emptyset'$ or $\emptyset' \leq_T A$, then $\deg_T(A)$ is hyperimmune.

Theorem 3. *[8] Let A be a hyperimmune set. If A-computable family F of total functions contains at least two elements, then F has no universal A-computable numbering.*

Theorem 4. *[8] If $\deg_T(A)$ is hyperimmune-free. Then every A-computable finite family of total functions has an A-computable universal numbering.*

Theorem 5. *[9] Let $\deg_T(A)$ is hyperimmune-free. Then any finite family of A-c.e. sets has a universal A-computable numbering.*

Corollary 2. *[9] For a set A the following conditions are equivalent.*

(1) $\deg_T(A)$ *is hyperimmune;*
(2) there exists a finite family of A-c.e. sets which does not have universal A-computable numberings;
(3) every finite family of A-c.e. sets without a least element under inclusion has no universal A-computable numberings.

After these results, the unsolved question was whether there exists an infinite family of total functions with a hyperimmune-free oracle that has an A-computable universal numbering. In [10] gives a positive answer to this question and it is proved.

Theorem 6. *[10] There exists an infinite A-computable family $\mathcal{F}$ of total functions, where Turing degree of the set A is hyperimmune-free, such that $\mathcal{F}$ has an A-computable universal numbering.*

In [16], some generalization of this result was obtained for an infinite A-computable effectively discrete family of total functions.

Theorem 7. *[16] Let $\mathcal{S}$ be an infinite A-computable effectively discrete family of total functions, where A—hyperimmune-free, then family $\mathcal{S}$ has an A-computable universal numbering.*

4 Main results

In this section, we have proved the results for a family of sets.

Theorem 8. *Let $\mathcal{S}$ be an infinite A-computable effectively discrete family of sets, where A—hyperimmune-free, then family $\mathcal{S}$ has an A-computable universal numbering.*

Proof. The following proof is based on the ideas of [9, 10]. Let $\mathcal{S}$ be an infinite A-computable effective discrete family of sets, say $\mathcal{S} = \{A_i\}_{i\in\omega}$. Then it is possible to find a strongly computable sequence of finite sets $T_i | i \in \omega$ such that

(1) $T_i \subseteq A_i$;
(2) $T_i \subseteq T_j \Rightarrow T_i = T_j$;
(3) $T_i \subseteq A_j \in A$ and $T_i \subseteq A_p \in A$ then $A_j = A_p$.

We define an A-computable numbering β as follows: for every e,

$$\beta(< e, x, s >) = \begin{cases} A_k & \text{if } (\exists k < s)[T_k \subseteq (W^A_{e,s})^{<x>}], \\ A_0 & \text{otherwise} \end{cases}$$

It is clear that β is a A-computable numbering of the family $\mathcal{S}$. Now let α be an arbitrary A-computable numbering of $\mathcal{S}$. We need to show that $\alpha \leq \beta$. Fix an index e for which $\alpha(x) = (W^A_{e,s})^{<x>}$ with any x.

Let g be A-computable function, and define

$$g(x) = \mu_s[(\exists k < s)[T_k \subseteq (W^A_{e,s})^{<x>}].$$

Since A is hyperimmune-free sets, it is follows that there exists a computable function f such that $g(x) \leq f(x)$ for all x. It means that for all e and x satisfy the following

$$\beta < e, x, f(x) >= A_k = (W^A_{e,f(x)})^{<x>} = \alpha(x)$$

Hence, β is A-computable universal numbering $Com^A(\mathcal{S})$.

□

Theorem 9. *Let $\mathcal{A}$ be a family of all co-finite sets. If $\mathcal{S}$ be a Σ^{-1}_2-computable family such that $\mathcal{A} \subseteq \mathcal{S}$ and there is co-c.e. set B that $B \notin \mathcal{S}$, then $\mathcal{R}^{-1}_2(\mathcal{S})$ has no universal numbering.*

Proof. Let $\nu \in \mathcal{R}^{-1}_2(\mathcal{S})$ be arbitrary numbering. We will construct a numbering $\mu \in Com^{-1}_2(\mathcal{S})$ which not reduced to ν. Let $\mu(2x) = \nu(x)$ and $\mu(2x+1)$ defined as following. In construction of $\mu(2x+1)$ we additional constructed function r_s.

Stage 0. Put $\mu_0(2x+1) = \omega$ and $r_0 = 1$.

Stage s+1. We will consecutively implement the following stages:

(1) If $\varphi_{x,s}(2x+1)\uparrow$ then go to the next stage
(2) If $\varphi_{x,s}(2x+1)\downarrow = y$, and $\mu_s(2x+1) \upharpoonright r_s = \nu_s(y) \upharpoonright r_s$ then $r_{s+1} = r_s + 1$ and $\mu_{s+1}(2x+1) \upharpoonright r_s = B_s \upharpoonright r_s$
(3) If $\varphi_{x,s}(2x+1)\downarrow = y$ and $\mu_s(2x+1) \upharpoonright r_s \neq \nu_s(y) \upharpoonright r_s$ then $\mu_{s+1}(2x+1) = \mu_s(2x+1)$ and $r_{s+1} = r_s$.

The description of the construction is over. Let

$$\nu(2x+1) = \bigcap_s \mu_s(2x+1).$$

Lemma 1. *If for x case (2) in construction is hold infinitely often then $\nu(y) = B$.*

Proof. Let for x case (2) is hold infinitely often and $\nu(y) \neq B$, where $y = \varphi_x(2x+1)$. Consider two case

(1) Let $\exists z$, $z \in \nu(y)$ and $z \notin B$.
Every stage when case (2) is hold the function r_s will be increasing. Then we can find stage s^* such that $z \in \nu_{s^*}(y)$ and $r_{s^*} \geq z$. Since $z \notin B$ then $z \notin B_s$ for every $s \in \omega$. So for every $s^{'} > s^*$, we have $z \notin \mu_{s'}(2x+1) \restriction r_{s'}$ and $z \in \nu_{s'}(y)$. So the condition of case (2) is not hold for any $s^{'} > s^*$. It is contradicted that case (2) is hold infinitely often.

(2) Let $\exists z$, $z \notin \nu(y)$ and $z \in B$.
Suppose that there is a step s^* such that $z \notin \nu_{s^*}(y)$, $r_{s^*} \geq z$ and $z \notin B$. Let's choose the smallest step $s^{'}$ and $\forall s^{'} > s^*$ and $z \notin \mu_{s'}(2x+1) \restriction r_{s'}$. We come to a contradiction with properties of the case (2) in construction, that $\mu_{s'}(2x+1) \restriction r_{s'} \neq \nu_{s'}(y)$ such that $z \notin B_{s'} \restriction r_{s'}$.

□

Lemma 2. *If* $\varphi_x(2x+1)\downarrow$ *then* $\lim_s(r_s)$ *exists and* $\mu(2x+1) \neq \nu(\varphi_x(2x+1))$.

Proof. Since $B \notin \mathcal{S}$ then from Lemma 1 there is stage $s^{'}$ that for all $s > s^{'}$ which hold only condition of case (3). It means that the function r_s has a limit and let it will be r. Then $\mu(2x+1) \restriction r \neq \nu(y) \restriction r$. Consequently, $\mu(2x+1) \neq \nu(\varphi_x(2x+1))$. □

Lemma 3. μ *is* Σ_2^{-1}*-computable numbering of family* $\mathcal{S}$.

Proof. $\theta_\mu = \{(x,y)\colon y \in \mu(x)\}$. Let A_1, A_2 computable enumerable sets such that $\theta_\nu = A_1 \setminus A_2$. Then

$$B_1 = \{(2x,y)\colon (x,y) \in A_1\} \cup \{(2x+1,y)\colon x,y \in \omega\},$$

$$B_2 = \{(2x,y)\colon (x,y) \in A_2\} \cup \{(2x+1,y)\colon \exists s[y \notin \mu_s(2x+1)]\}.$$

It is not difficult to see $\theta_\mu = B_1 \setminus B_2$ and B_1, B_2 are computably enumerable sets. Consequently, $\theta_\mu \in \Sigma_2^{-1}$.

Now show that $\mu(x) \in \mathcal{S}$ for all x. If $x = 2k$ then $\mu(2k) = \nu(k)$. So $\nu(k) \in \mathcal{S}$ then $\mu(x) \in \mathcal{S}$. If $x = 2k+1$ then $\mu(2k+1) = \omega$ if $\varphi_k(2k+1)\uparrow$ and it is gives co-finite. If $\varphi_k(2k+1)\downarrow$ by Lemma 2 there exists $\lim_s(r_s)$ and by constraction $\mu(2k+1)$ contain $[\lim_s(r_s), \infty)$. Consequently, $\mu(2k+1)$ is co-finite. Since family $\mathcal{A} \subseteq \mathcal{S}$ contain all co-finite sets, it means that $\mu(x) \in \mathcal{S}$. □

If $\mu \leq \nu$ then there exists computable function f such that $\mu(x) = \nu(f(x))$ for all x. Let $f = \varphi_e$ for some e. Since f is total, $\varphi_e(2e+1)\downarrow$. From Lemma 2 $\mu(2e+1) \neq \nu(\varphi_e(2e+1))$. It is contradiction. □

Corollary 3. *If* $\mathcal{S}$ *is the family of all c.e. sets, then* $\mathcal{R}_2^{-1}(\mathcal{S})$ *has no universal numbering.*

5 Conclusion

In conclusion, we proved that if an infinite A-computable effectively family of sets, where A is a hyperimmune set, then the family has an A-computable universal numbering. It was also proved that in $\mathcal{R}_2^{-1}(\mathcal{S})$ has no universal numbering if $\mathcal{A}$ be a family of all co-finite sets and if $\mathcal{S}$ be a Σ_2^{-1}-computable family such that $\mathcal{A} \subseteq \mathcal{S}$ and there is co-c.e. set B that $B \notin \mathcal{S}$.

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Initial-boundary value problem for the time-fractional degenerate diffusion equation

In this paper the initial-boundary value problems for the one-dimensional linear time-fractional diffusion equations with the time-fractional derivative ∂_t^α of order $\alpha \in (0,1)$ in the variable t and time-degenerate diffusive coefficients t^β with $\beta \geq 1-\alpha$ are studied. The solutions of initial-boundary value problems for the one-dimensional time-fractional degenerate diffusion equations with the time-fractional derivative ∂_t^α of order $\alpha \in (0,1)$ in the variable t, are shown. The second section present Dirichlet and Neumann boundary value problems, and in the third section has shown the solutions of the Dirichlet and Neumann boundary value problem for the one-dimensional linear time-fractional diffusion equation. The solutions of these fractional diffusive equations are presented using the Kilbas-Saigo function $E_{\alpha,m,l}(z)$. The solution of the problems is discovered by the method of separation of variables, through finding two problems with one variable. The existence and uniqueness to the solution of the problem are confirmed. In addition, the convergence of the solution has been proven using the estimate for the Kilbas-Saigo function $E_{\alpha,m,l}(z)$ from [13] and Parseval's identity.
Key words: Time-fractional diffusion equation, the method of separation variables, the Kilbas-Saigo function.

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Бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін бастапқы шеттік есебі

Бұл жұмыста t^β, $\beta \geq 1-\alpha$ диффузиялық коэффициенттері бар бір өлшемді сызықты $\alpha \in (0,1)$ үшін ∂_t^α бөлшек ретті туындылы өзгешеленген диффузия теңдеулеріне қойылған бастапқы - шеттік есептері қарастырылған. Бір өлшемді сызықты t айнымалысына тәуелді $\alpha \in (0,1)$ үшін ∂_t^α бөлшек ретті туындылы өзгешеленген диффузия теңдеулеріне қойылған бастапқы - шеттік есептерінің шешімдері көрсетілген. Екінші бөлімінде Дирихле және Нейман шеттік есептері берілген, ал үшінші бөлімінде бір өлшемді сызықты бөлшек ретті туындылы өзгешеленген диффузия теңдеулері үшін Дирихле және Нейман шеттік есептерінің шешімдері көрсетілген. Бұл бөлшек ретті туындылы өзгешеленген диффузиялық теңдеулердің шешімдері $E_{\alpha,m,l}(z)$ Килбас-Сайго функциясы арқылы берілген. Есептердің шешімдері айнымалысын ажырату әдісін қолданып, бір айнымалысы бар екі есепті шешу арқылы табылған. Есептің шешімінің бар болуы мен жалғыздығы дәлелденген. Шешімнің жинақтылығы Килбас-Сайго $E_{\alpha,m,l}(z)$ функциясының [13] көрсетілгендей бағалауын және Парсевал теңдігін қолдану арқылы дәлелденді.

Түйін сөздер: Бөлшек ретті туындылы өзгешеленген диффузия теңдеуі, айнымалыларын ажырату әдісі, Килбас-Сайго функциясы.

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Начально-краевая задача для дробных вырожденных диффузионных уравнений

В данной работе рассматриваются начально-краевые задачи для одномерных дробных вырожденных линейных диффузионных уравнений с дробной производной ∂_t^α порядка $\alpha \in (0,1)$ по переменной t и с вырождающимися коэффициентами диффузии t^β при $\beta \geq 1-\alpha$. Показаны решения начально-краевых задач для одномерных уравнений вырождающейся диффузии с дробной по времени производной ∂_t^α порядка $\alpha \in (0,1)$ по переменной t. Во второй части даны краевые задачи Дирихле и Неймана, а в третьей части показаны решения краевых задач Дирихле и Неймана для одномерного дробного вырожденного линейного диффузионного уравнения. Решения этих дробных диффузионных уравнений представлены с помощью функции Килбаса-Сайго $E_{\alpha,m,l}(z)$. Решение задач получено с помощью метода разделения переменных, путем нахождения двух задач с одной переменной. Доказаны существование и единственность решения задач. Сходимости решения доказано с помощью оценки функции Килбаса-Сайго $E_{\alpha,m,l}(z)$ из [13] и тождество Парсеваля.

Ключевые слова: Дробно-вырожденное диффузионное уравнение, метод разделения переменных, функция Килбаса-Сайго.

1 Introduction

Over the past several millennia, fractional partial differential equations have begun to play an important role. They are used in modeling anomalous phenomena and in the theory of complex systems [1-6].

In the book [7], it is written about various applications of differential equations of fractional order in chemistry, technology, physics, etc. It contains research related to the equation of fractional diffusion in time. It is obtained from the classical diffusion equation by replacing the first-order time derivative with a fractional derivative.

In [8-11], the correctness and numerical modeling of thermal and wave equations with nonlocal conditions in time were studied. In [12] paper, authors consider the initial-boundary value problems of Dirichlet and Neumann for the diffusion equation in a variable coefficient. In [14], Nakhusheva proved a positive maximum principle for a nonlocal parabolic equation with Riemann– Liouville derivative. In [16], Luchko proved the maximum principle for the generalized diffusion equation with a fractional time derivative using the maximum principle for the Caputo fractional derivative. Then the maximum principle was applied to show some results of uniqueness and existence for the initial-boundary value problem of the fractional diffusion equation. In [15] Luchko studied initial-boundary value problems for a generalized diffusion equation with a distributed order. And [17] studied initial-boundary value problems for a fractional-fold diffusion equation in time. Thus, he obtained results on the existence of generalized solutions in [15, 17]. In [18], the generalized solution of the initial-boundary value problem for the diffusion equation with fractional time was shown as a regular solution. In [19], Gorenflo and Mainardi studied the one-dimensional diffusion-wave equation with fractional time.

In this paper, the initial-boundary value problems of Dirichlet and Neumann for the time-fractional diffusion equation in a variable coefficient are considered. The solution of the problems has been found by using the Kilbas-Saigo function and by the method of separation of variables. The existence and uniqueness of the solution are also proved.

2 Material and methods

2.1 Cauchy-Dirichlet problem

Let us consider the one-dimensional time-fractional diffusion equation

$$\partial_t^\alpha u(x,t) - t^\beta u_{xx}(x,t) = 0, \ \ (x,t) \in (0,1) \times (0,\infty), \tag{1}$$

with the Dirichlet boundary condition

$$u(0,t) = u(1,t) = 0, \ \ t \geq 0, \ \ x \in [0,1], \tag{2}$$

and the Cauchy initial condition

$$u(x,0) = \phi(x), \ \ x \in [0,1], \tag{3}$$

where ∂_t^α is the time-fractional derivative of order $\alpha \in (0,1)$ in the variable t and $\beta \geq 1-\alpha$

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \partial_s u(x,s) ds.$$

Let $X_k(x) = sin\pi kx$ are the orthonormal eigenfunctions and $\lambda_k = (\pi k)^2$ the corresponding eigenvalues of the Sturm-Liouville operator with the Dirichlet boundary and the Cauchy initial conditions.

$H^2(0,1)$ is a Hilbert space defined by the initial-boundary

$$H^2(0,1) = \{u : \ \ u \in L^2(0,1); \ \ u_{xx} \in L^2(0,1)\},$$

endowed with the norm

$$||u||^2_{H^2(0,1)} = \sum_{k=1}^{\infty} \lambda_k^2 |(u, X_k(x))|^2 < \infty.$$

Definition 1. The solution of problem (1)-(3) is $u(x,t) \in C(L^2(0,1), R_+)$, such that satisfies $t^{-\beta}\partial_t^\alpha u, \ \ u_{xx} \in C(L^2(0,1), R_+)$.

2.2 Cauchy-Neumann problem

Let us consider the time-fractional diffusion equation

$$\partial_t^\alpha u(x,t) - t^\beta u_{xx}(x,t) = 0, \ \ (x,t) \in (0,1) \times (0,\infty), \tag{4}$$

with the Neumann boundary condition

$$u_x(0,t) = u_x(1,t) = 0, \ \ t \geq 0, \ \ x \in [0,1], \tag{5}$$

supplemented with the initial data

$$u(x,0) = \phi(x), \ \ x \in [0,1], \tag{6}$$

where ∂_t^α is the time-fractional fractional derivative of order $\alpha \in (0,1)$ in the variable t and $\beta \geq 1-\alpha$.

Definition 2. The solution of problem (4)-(6) is $u(x,t) \in C(L^2(0,1), R_+)$, which satisfies $t^{-\beta}\partial_t^\alpha u, \ \ u_{xx} \in C(L^2(0,1), R_+)$.

3 Main results

Theorem 1 *Let $\phi(x) \in H^2(0,1)$, then the unique solution of problem (1)-(3) is the function $u(x,t) \in C(L^2(0,1), R_+)$, which has the form*

$$u(x,t) = \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \tag{7}$$

where

$$\phi_k = 2\int_0^1 \phi(x) sin\pi kx dx,$$

and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function([5] defined as

$$E_{\alpha,m,l}(z) = \sum_{k=1}^{\infty} c_k z^k, \ \ c_0 = 1, \ \ c_k = \Pi_{j=0}^{k-1} \frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \ \ k \geq 1. \tag{8}$$

And for the function $E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha})$ the following estimate holds ([13])

$$E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \leq \frac{1}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \pi^2 k^2 t^{\beta+\alpha}}, \ \ t > 0. \tag{9}$$

Theorem 2 *Let $\phi(x) \in H^2(0,1)$, then the unique solution of problem (4)-(6) is the function $u(x,t) \in C(L^2(0,1), R_+)$, which given by*

$$u(x,t) = \phi_0 + \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) cos\pi kx, \tag{10}$$

where $\phi_0 = \int_0^1 \phi(x)dx$ and $\phi_k = 2\int_0^1 \phi(x) cos\pi kx dx, \ \ k \in N$ and $E_{\alpha,m,l}(z)$ is the Kilbas-Saigo function, which is defined by the formula (8)-(9).

3.1 Proofs

Proof of Theorem 1

The existence of a solution. Since the Sturm-Liouville operator has eigenvalues $\{\lambda_k \geq 0, \ \ k \in N\}$ on $L^2(0,1)$ and the corresponding ortonormal eigenfunctions $\{X_k(x), \ \ k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we can give the solution of problem (1)-(3) as follows

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x), \ \ (x,t) \in (0,1) \times R_+, \tag{11}$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k X_k(x), \ \ x \in (0,1), \tag{12}$$

where

$$\phi_k = 2\int_0^1 \phi(x) X_k(x) dx.$$

Substituting (11) in to the diffusion equation (1)-(3), we obtain the next problems

$$\partial_t^{\alpha} T_k(t) + \lambda_k t^{\beta} T_k(t) = 0, \ \ t > 0, \tag{13}$$

$$T_k(0) = \phi_k. \tag{14}$$

$$X_k''(x) + \lambda_k X_k(x) = 0, \tag{15}$$

$$X_k(0) = X_k(1) = 0. \tag{16}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Dirichlet problem (15)-(16) are $X_k(x) = sin\pi kx$ and $\lambda_k = (\pi k)^2$, respectively.

The general solution of the Cauchy problem (13)-(14) is

$$T_k(t) = \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}), \tag{17}$$

where

$$\phi_k = 2\int_0^1 \phi(x) sin\pi kx dx.$$

Substituting $X_k(x) = sin\pi kx$ orthonormal eigenfunctions and (17) to (11), we obtain the solution of problem (1)-(3) as

$$u(x,t) = \sum_{k=1}^{\infty} \phi_k E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \ \ (x,t) \in (0,1) \times [0,\infty). \tag{18}$$

Convergence of the solution. Using (9) to (17), we get

$$T_k(t) \le \frac{|\phi_k|}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \pi^2 k^2 t^{\beta+\alpha}}.$$

By Parseval's identity, it follows from (18) that

$$\begin{aligned}
\sup_{t\ge 0} ||u(\cdot,t)||^2_{L^2(0,1)} &= \sup_{t\ge 0} \sum_{k=1}^{\infty} |\phi_k|^2 \left| E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) \right|^2 ||sin\pi kx||^2_{L^2(0,1)} \\
&\le \sup_{t\ge 0} \sum_{k=1}^{\infty} \frac{|\phi_k|^2}{\left(1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \pi^2 k^2 t^{\beta+\alpha}\right)^2} \\
&\le \sup_{t\ge 0} \frac{1}{\left(1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \pi^2 t^{\beta+\alpha}\right)^2} \sum_{k=1}^{\infty} |\phi_k|^2 \\
&\le \sum_{k=1}^{\infty} |\phi_k|^2 = ||\phi(\cdot)||^2_{L^2(0,1)}.
\end{aligned} \tag{19}$$

Applying the operators $\partial_t^{\alpha} u$ and u_{xx} to the identity (18) we obtain

$$\partial_t^{\alpha} u(x,t) = \sum_{k=1}^{\infty} \phi_k \partial_t^{\alpha} E_{\alpha, 1+\frac{\beta}{\alpha}, \frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx$$

$$= -t^{\beta} \sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \ \ (x,t) \in (0,1)\times[0,\infty), \tag{20}$$

and

$$u_{xx}(x,t) = \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin'' \pi kx$$

$$= -\sum_{k=1}^{\infty} \pi^2 k^2 \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) sin\pi kx, \ \ (x,t) \in (0,1)\times[0,\infty). \tag{21}$$

Applying (19)-(21) we get

$$\sup_{x\geq 0} ||t^{-\beta}\partial_t^{\alpha} u(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty} \lambda_k^2 |\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty,$$

and

$$\sup_{x\geq 0} ||u_{xx}(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=1}^{\infty} \lambda_k^2 |\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty.$$

Uniqueness of the solution. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to problem (1)-(3). We choose $u(x,t) = u_1(x,t) - u_2(x,t)$ in such a way, that $u(x,t)$ satisfies the diffusion equation (1) and boundary, initial condition (2), (3), respectively. Let us consider

$$T_k(t) = \int_0^1 u(x,t) sin\pi kx dx, \ \ k \in N, t \in [0,\infty). \tag{22}$$

Applying the operator ∂_t^{α} to the left-side of (22) equation by using (1) we obtain

$$\partial_t^{\alpha} T_k(t) = \int_0^1 \partial_t^{\alpha} u(x,t) sin\pi kx dx$$

$$= t^{\beta} \int_0^1 u_{xx}(x,t) sin\pi kx dx$$

$$= t^{\beta} \int_0^1 u(x,t) sin''\pi kx dx$$

$$= -t^{\beta}\pi^2 k^2 \int_0^1 u(x,t) sin\pi kx dx$$

$$= -t^{\beta}\pi^2 k^2 T_k(t), \ \ k \in N, t \in [0,\infty).$$

Due to (2) and (3) we have

$$T_k(0) = 0.$$

From the equation we get that $T_k(0) = 0$, which means $u(x,t) \equiv 0$. Hence $u_1(x,t) = u_2(x,t)$, therefore the diffusion problem (1)-(3) has a unique solution.

Proof of Theorem 2

The existence of a solution. Since the Sturm-Liouville operator has eigenvalues $\{\lambda_k \geq 0,\ \ k \in N\}$ on $L^2(0,1)$ and the corresponding ortonormal eigenfunctions $\{X_k(x),\ \ k \in N\}$ in $L^2(0,1)$ and $\phi(x) \in H^2(0,1)$, then we write the solution of problem (4)-(6) as follows

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x),\ \ (x,t) \in (0,1) \times R_+, \tag{23}$$

$$\phi(x) = \sum_{k=1}^{\infty} \phi_k X_k(x),\ \ x \in (0,1), \tag{24}$$

where

$$\phi_k = 2\int_0^1 \phi(x)X_k(x)dx$$

and

$$\phi_0 = \int_0^1 \phi(x)dx.$$

Substituting (23) in to the diffusion equation (4)-(6), we obtain the next problems

$$\partial_t^\alpha T_k(t) + \lambda t^\beta T_k(t) = 0,\ \ t > 0, \tag{25}$$

$$T_k(0) = \phi_k. \tag{26}$$

$$X_k''(x) + \lambda_k X_k(x) = 0, \tag{27}$$

$$X_k'(0) = X_k'(1) = 0. \tag{28}$$

The orthonormal eigenfunctions and the corresponding eigenvalues of the Neumann problem (27)-(28) are $X_k(x) = cos\pi kx$ and $\lambda_k = (\pi k)^2$, respectively.

The general solution of the Cauchy problem (25)-(26) is

$$T_k(t) = \phi_0 + \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}), \tag{29}$$

where

$$\phi_k = 2\int_0^1 \phi(x)cos\pi kxdx$$

and

$$\phi_0 = \int_0^1 \phi(x)dx.$$

Substituting $X_k(x) = cos\pi kx$ orthonormal eigenfunctions and (29) to (23), we obtain the solution of problem (4)-(6) as

$$u(x,t) = \phi_0 + \sum_{k=1}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2 k^2 t^{\beta+\alpha}) cos\pi kx,\ \ (x,t) \in (0,1) \times [0,\infty). \tag{30}$$

Convergence of the solution. Using (9) to (29), we get

$$T_k(t) \leq |\phi_0| + \frac{|\phi_k|}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2 k^2 t^{\beta+\alpha}}.$$

By Parseval's identity, it follows from (30) that

$$\sup_{t\geq 0}||u(\cdot,t)||^2_{L^2(0,1)} = \sup_{t\geq 0}\sum_{k=0}^{\infty}|\phi_k|^2\left|E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})\right|^2||cos\pi kx||^2_{L^2(0,1)}$$

$$\leq \sup_{t\geq 0}\sum_{k=0}^{\infty}\frac{|\phi_k|^2}{\left(1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2k^2t^{\beta+\alpha}\right)^2}$$

$$\leq \sup_{t\geq 0}\frac{1}{\left(1+\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\pi^2t^{\beta+\alpha}\right)^2}\sum_{k=0}^{\infty}|\phi_k|^2$$

$$\leq \sum_{k=0}^{\infty}|\phi_k|^2 = ||\phi(\cdot)||^2_{L^2(0,1)}. \tag{31}$$

Applying the operators $\partial_t^\alpha u$ and u_{xx} to the identity (30) we get

$$\partial_t^\alpha u(x,t) = \sum_{k=0}^{\infty}\phi_k\partial_t^\alpha E_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx$$

$$= -t^\beta\sum_{k=0}^{\infty}\pi^2k^2\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx, \ \ (x,t)\in(0,1)\times[0,\infty), \tag{32}$$

and

$$u_{xx}(x,t) = \sum_{k=0}^{\infty}\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos''\pi kx$$

$$= -\sum_{k=0}^{\infty}\pi^2k^2\phi_kE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}(-\pi^2k^2t^{\beta+\alpha})cos\pi kx, \ \ (x,t)\in(0,1)\times[0,\infty). \tag{33}$$

Applying (31)-(33) we get

$$\sup_{x\geq 0}||t^{-\beta}\partial_t^\alpha u(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=0}^{\infty}\lambda_k^2|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty,$$

and

$$\sup_{x\geq 0}||u_{xx}(\cdot,t)||^2_{L^2(0,1)} \leq \sum_{k=0}^{\infty}\lambda_k^2|\phi_k|^2 = ||\phi(\cdot)||^2_{H^2(0,1)} < \infty.$$

Uniqueness of the solution. Suppose that $u_1(x,t)$ and $u_2(x,t)$ are solutions to problem (4)-(6). We choose $u(x,t) = u_1(x,t) - u_2(x,t)$ in such a way, that $u(x,t)$ satisfies the diffusion equation (4) and boundary, initial condition (5), (6), respectively. Let us consider

$$T_k(t) = \int_0^1 u(x,t)cos\pi kxdx, \ \ k\in N, t\in[0,\infty). \tag{34}$$

Applying ∂_t^α to left-side (34) equation by using (4) we obtain

$$\partial_t^\alpha T_k(t) = \int_0^1 \partial_t^\alpha u(x,t) cos\pi kx dx$$

$$= t^\beta \int_0^1 u_{xx}(x,t) cos\pi kx dx$$

$$= t^\beta \int_0^1 u(x,t) cos'' \pi kx dx$$

$$= -t^\beta \pi^2 k^2 \int_0^1 u(x,t) cos\pi kx dx$$

$$= -t^\beta \pi^2 k^2 T_k(t), \quad k \in N, t \in [0,\infty).$$

Due to (5) and (6) we have

$$T_k(0) = 0.$$

From the equation we get that $T_k(0) = 0$, which means that $u(x,t) \equiv 0$. Hence $u_1(x,t) = u_2(x,t)$, therefore the diffusion problem (4)-(6) has a unique solution.

4 Conclusions

In this research considered the initial-boundary value problems of Dirichlet and Neumann for the time-fractional diffusion equation in a variable coefficient. The solution of the problems has been found by using the Kilbas-Saigo function and by the method of separation of variables. The existence, uniqueness and convergence of solution were confirmed.

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The Cauchy problems for q-difference equations with the Caputo fractional derivatives

The fractional differential equations play important roles due to their numerous applications and also for the important role they play not only in mathematics but also in other sciences. In the present research work, we build up the explicit solutions to linear fractional q-differential equations with the q-Caputo fractional derivative of real order $\alpha > 0$. To speak more precisely, we will achieve our main results we use that this Cauchy type q-fractional problem is equivalent to a corresponding Volterra q-integral equation. After that, by using the method of successive approximations is applied to solve the Volterra q-integral equation we construct the the explicit solutions to linear fractional q-differential equations. In the same way we have the more general homogeneous fractional q-differential equation with the Caputo fractional q-derivative of real order $\alpha > 0$ and we give other The (Mittag-Leffler) q-function. Finally, some examples are presented to illustrate our main results in cases where we can even give concrete formulas for these explicit solutions.

Key words: Cauchy type q-fractional problem, existence, uniqueness, q-derivative, q-calculus, fractional calculus, fractional derivative, Caputo fractional derivatives.

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Капуто бөлшек туындысы бар q-айрымдық теңдеулер үшін Коши есептері

Бөлшек туындылары бар теңдеулер өздерінің көп салаларда қолданылуына байланысты маңызды рөл атқарады, сонымен қатар олар тек математикада ғана емес, сонымен қатар басқа ғылымдарда да маңызды рөл атқарады. Бұл зерттеу жұмысында біз $\alpha > 0$ нақты ретті Капутоның q-бөлшек туындысы бар бөлшек-сызықтық q- дифференциалдық теңдеулердің нақты шешімдерін құрамыз. Нақтырақ айтсақ, біз осы Коши типтес q-бөлшек есептің сәйкес Вольтердің q-интегралдық теңдеуіне эквивалентті болатындығын қолдана отырып, негізгі нәтижелерге қол жеткіземіз. Осыдан кейін Вольтердің q-интегралдық теңдеуінің шешіміне тізбектеп жуықтау әдісін қолдана отырып, бөлшек-сызықтық q-дифференциалдық теңдеулердің нақты шешімдерін құрамыз. Сол сияқты бізде $\alpha > 0$ нақты ретті Капутоның бөлшек q-туындысы бар жалпы біртекті бөлшек q-дифференциалдық теңдеу бар және біз басқа q-функциясын береміз (Миттаг-Леффлер). Соңында, біз осы нақты шешімдерге нақты формулалар бере алатын жағдайларда негізгі нәтижелерімізді көрсететін бірнеше мысалдар келтірілген.
Түйін сөздер: Коши типтес q-бөлшек есеп, бар болу, жалғыз болу, q-туынды, q-есептеу, бөлшек есептеу, бөлшек туынды, Капутоның бөлшек туындысы.

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Задачи Коши для q-разностных уравнений с дробными производными Капуто

Уравнения с дробными производными играют важную роль из-за их многочисленных применений, а также из-за той важной роли, которую они играют не только в математике, но и в других науках. В данной исследовательской работе мы строим явные решения дробно-линейных q-дифференциальных уравнений с q-дробной производной Капуто действительного порядка $\alpha > 0$. Точнее говоря, мы достигнем наших основных результатов, используя то, что эта q-дробная задача типа Коши эквивалентна соответствующему q-интегральному уравнению Вольтерра. После этого, применяя метод последовательных приближений к решению q-интегрального уравнения Вольтерра, строим явные решения дробно-линейных q-дифференциальных уравнений. Таким же образом у нас есть более общее однородное дробное q-дифференциальное уравнение с дробной q-производной Капуто действительного порядка $\alpha > 0$, и мы даем другую q-функцию (Миттаг-Леффлера). Наконец, представлены некоторые примеры, иллюстрирующие наши основные результаты в тех случаях, когда мы даже можем дать конкретные формулы для этих явных решений.
Ключевые слова: q-дробная задача типа Коши, существование, единственность, q-производная, q-исчисление, дробное исчисление, дробная производная, дробные производные Капуто.

1 Introduction

The fractional calculus is the field of mathematics that investigates the integration and differentiation of real or complex orders. The fractional differential equations based on the Caputo fractional derivative require initial conditions for integer order derivatives. Consequently, fractional differential equations have grasped the interest of many researchers working in diverse applications [1]- [10]. Recently, there has been a significant development in ordinary and partial differential equations involving fractional derivatives and a huge amount of papers and also some books devoted to this subject in various spaces have appeared, see e.g. the monographs of T. Sandev and Z. Tomovski [7], A.A. Kilbas et al. [8], R. Hilfer [9], K.S. Miller and the B. Ross [11], the papers [12], [13], [14], [15], [16], [17], [18], [19] and [20] and the references therein.

The origin of the q-difference calculus can be traced back to the works in [21, 22] by F. Jackson and R.D. Carmichael [23] from the beginning of the twentieth century. For more interesting theory results and scientific applications of the q-difference calculus, we cite the monographs [24–26] and the references therein. In the last decades, the fractional q-difference calculus has been proposed by W. Al-salam [27] and R.P. Agarwal [28] and P.M. Rajkovic', S.D. Marinkovic', and M.S. Stankovic [29]. Recently, many researchers got much interested in looking at fractional q-differential equations (FDEs) as new model equations for many physical problems. For example, some researchers obtained q-analogues of the integral and differential fractional operators properties such as the q-Laplace transform and q-Taylor's formula [30], q-Mittage Leffler function [28] and so on.

However, the theory of the q-difference equations with constant and variable coefficients is still in the initial stages and many aspects of this theory need to be explored. For some recent developments on the subject, see e.g. [31], [32], [33], [34] and the references therein. To the best of our knowledge, the theory of the Cauchy problem for linear, homogeneous and nonhomogeneous q-difference equations based the basic Caputo fractional derivative is yet to be developed.

Motivated by this, we discuss to construct the explicit solution to linear fractional q-differential equation with the Caputo fractional q-derivative ${}^cD^{\alpha}_{q,0+}$ order of $\alpha > 0$ in the following form (see Definition 2.2):

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) \;=\; f(x)\,, \quad 0 \le a < x \le b, \alpha > 0; \lambda \in \mathbb{R}, \tag{1}$$

with the initial conditions

$$y^{(k)}(0+) = b_k, b_k \in \mathbb{R}, k = 0, 1, 2, \ldots, n = -[-\alpha], \tag{2}$$

where $f \in C_{q,\lambda}[a, b]$ (see 2.8) with $0 \le \gamma \le 1$, $\gamma \le \alpha$ and where $[\alpha]$ denotes the smallest integer greater or equal to α. Moreover, we consider the Cauchy problem for the following more general homogeneous fractional q-differential equation than

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) \;=\; 0, \quad 0 \le a < x \le b, \alpha > 0; \lambda \in \mathbb{R}, \tag{3}$$

with the initial conditions

$$y^{k}(0+) = b_k, \quad b_k \in \mathbb{R}, k = 0, 1, 2, \ldots, n = -[-\alpha], \tag{4}$$

with $\beta > -\alpha$.

In Section 3 of this paper we construct explicit solutions to linear fractional q-differential equations with the Caputo fractional q-derivative ${}^cD^{\alpha}_{q,a+}f$ of order $\alpha > 0$ given by Definition 2.2. in the space $C^{\alpha,n-\alpha}_{q,\gamma}[0, a]$, denned in (13). The main result in this Section is Theorems 2.1 but in order to prove this result we need to prove two results (Theorem 3.1 and 3.3) of independent interest.

The paper is organized as follows: The main results are presented and proved in subsection 3 and the announced examples are given in subsection 4. In order to not disturb these presentations we include in Section 1 some necessary Preliminaries.

2 Materials and methods

2.1 Preliminaries

First we recall some elements of q-calculus, for more information see e.g. the books [24], [26] and [34]. Throughout this paper, we assume that $0 < q < 1$ and $0 \le a < b < \infty$.

Let $\alpha \in \mathbb{R}$. Then a q-real number $[\alpha]_q$ is defined by

$$[\alpha]_q = \frac{1 - q^{\alpha}}{1 - q},$$

where $\lim\limits_{q \to 1} \frac{1-q^{\alpha}}{1-q} = \alpha$.

We introduce for $k \in \mathbb{N}$:

$$(a; q)_0 = 1,\ (a; q)_n = \prod_{k=0}^{n} \left(1 - q^k a\right),\ (a; q)_{\infty} = \lim_{n \to \infty} (a, q)_n,\ (a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(q^{\alpha} a; q)_{\infty}}.$$

For any two real numbers α and β, we have

$$(a - b)^{\alpha}_q \left(a - q^{\alpha} b\right)^{\beta}_q = (a - b)^{\alpha+\beta}_q\,. \tag{5}$$

The q-analogue of the binomial coefficients $[n]_q!$ are defined by

$$[n]_q! = \begin{cases} 1, & \text{if n} = 0, \\ [1]_q \times [2]_q \times \cdots \times [n]_q, & \text{if n} \in \mathrm{N}, \end{cases}$$

The gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

for any $x > 0$. Moreover, it yields that

$$\Gamma_q(x)[x]_q = \Gamma_q(x+1).$$

The q-analogue differential operator $D_q f(x)$ is

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)},$$

and the q-derivatives $D_q^n(f(x))$ of higher order are defined inductively as follows:

$$D_q^0(f(x)) = f(x), \quad D_q^n(f(x)) = D_q\left(D_q^{n-1} f(x)\right), (n = 1, 2, 3, \dots)$$

The q-integral (or Jackson integral) $\int_0^a f(x) d_q x$ is defined by

$$\int_0^a f(x) d_q x := (1-q)a \sum_{m=0}^{\infty} q^m f(aq^m)$$

and

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x,$$

for $0 < a < b$. Notice that

$$\int_a^b D_q f(x) d_q x = f(b) - f(a).$$

For any $t, s > 0$ the definition of q-Beta function is that:

$$B_q(t,s) := \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)} := \int_0^1 x^{t-1}(qx;q)_{s-1} d_q x \tag{6}$$

The (Mittag-Leffler) q-function $E_{q,\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta,a}\left[zx^\alpha (a/x;q)_\alpha; q\right] := \sum_{k=0}^{\infty} \frac{z^k x^{k\alpha}(a/x;q)_{k\alpha}}{\Gamma_q(\alpha k + \beta)} \tag{7}$$

and

$$E_{\alpha,m,l}\left[z;q\right] := \sum_{k=0}^{\infty} c_k z^k \tag{8}$$

where c_0 and $c_k = \prod\limits_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]}(k \in \mathbb{N})$.

A q-analogue of the classical exponential function e^x is

$$e_q^x := \sum_{j=0}^{\infty} \frac{x^j}{[j]!}. \tag{9}$$

Moreover, the multiple q-integral $\left(I_{q,a+}^n f\right)(x)$ is

$$\begin{aligned}\left(I_{q,a+}^n f\right)(x) &= \int_a^x \int_a^t \int_a^{t_{n-1}} \ldots \int_a^{t_2} d_q t_1 d_q t_2 \ldots d_q t_{n-1} d_q t \\ &= \frac{x^{n-1}}{\Gamma_q(n)} \int_a^x (qt/x;q)_{n-1} f(t) d_q t.\end{aligned}$$

Definition 1 *The Riemann-Liouville q-fractional integrals $I_{q,a+}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$\left(I_{q,0+}^{\alpha} f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t) d_q t.$$

Definition 2 *The Caputo fractional q-derivative ${}^cD_{q,a+}^{\alpha} f$ of order $\alpha > 0$ is defined as*

$$\left({}^cD_{q,a+}^{\alpha} f\right)(x) := \left(I_{q,a+}^{[\alpha]-\alpha} D_{q,a+}^{[\alpha]} f\right)(x).$$

Notice that

$$\left(I_{q,a+}^{\alpha} x^{\lambda}(a/x;q)_{\lambda}\right)(x) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)} x^{a+\lambda}(a/x;q)_{\alpha+\lambda}, \tag{10}$$

for $\lambda \in (-1, \infty)$.

For $1 \le p < \infty$ we define the space $L_q^p = L_q^p[a,b]$ by

$$L_q^p[a,b] := \left\{ f : \left(\int_a^b |f(x)|^p d_q x \right)^{\frac{1}{p}} < \infty \right\}.$$

Let $\alpha > 0$, $\beta > 0$ and $1 \le p < \infty$. Then the q–fractional integration has the following semigroup property

$$\left(I_{q,a+}^{\alpha} I_{q,a+}^{\beta} f\right)(x) = \left(I_{q,a+}^{\alpha+\beta} f\right)(x), \tag{11}$$

for all $x \in [a,b]$ and $f(x) \in L_q^p[a,b]$.

Let $0 < a < b < \infty$, $0 \leq \lambda \leq 1$ and $n \in \mathbb{N}$. Then we introduce the spaces $C_{q,\lambda}[a,b]$ and $C_q^n[a,b]$ of functions f given on $[a,b]$, such that the functions with the norms,respectively

$$\|f\|_{C_{q,\lambda}[a,b]} := \max_{x\in[a,b]} \left|x^{\lambda}\left(qa/x;q\right)_{\lambda} f(x)\right| < \infty. \tag{12}$$

$$\|f\|_{C_q^n[a,b]} := \sum_{k=0}^{n} \max_{x\in[a,b]} \left|D_q^k f(x)\right| < \infty.$$

and the space $C_{q,\lambda}^{\alpha,n}[0,a]$ defined for $n - q < \alpha \leq n$, $n \in \mathbb{N}$ by

$$C_{q,\lambda}^{\alpha,n}[0,a] := \{f(x) : f(x) \in C_q^n[a,b], \left({}^{c}D_{q,a+}^{\alpha} f\right)(x) \in C_{q,\lambda}[a,b]\}. \tag{13}$$

In the classical case several authors have considered such problems even in linear cases, see e.g. [8, subection 4.1.3] and the references therein.

Theorem 2.1 *(See [35, Theorem 8.1]) Let $n-1 < \alpha \leq n; n \in \mathbb{N}$, G be an open set in $\mathbb{R}$ and $f(.,.) : (0,a] \times G \to \mathbb{R}$ be a function such that $F(x,y(x)) = f(x) + \lambda y(x) \in L_q^1[0,a]$ for any $y \in G$. If $y(x) \in L_q^1[0,a]$, then $y(t)$ satisfies a.e. the relations (1)-(2) if and only if $y(x)$ satisfies a.e. the integral equation*

$$y(x) := \sum_{k=0}^{n-1} \frac{b_k}{[k]_q!} x^{\alpha-k} + \left(I_{q,0}^{\alpha} f(t,y(t))\right)(x), \forall x \in (0,a]. \tag{14}$$

3 The Main results

3.1 The Cauchy Problems for q-difference equation with the Caputo fractional q-derivative

In this section we construct the explicit solution to linear fractional q-difference equations (1) and with the initial conditions (2). From here we obtain the following result.

Theorem 3.1 *Let $n-1 < \alpha < n (n \in \mathbb{N})$ and $0 \leq \gamma < 1$ be such that $\gamma \geqq \alpha$. Also let $\lambda \in \mathbb{R}$. If $f(x) \in C_{q,\gamma}[0,a]$, the Cauchy problem (1)-(2) has a unique solution $y(x) \in C_{q,\gamma}^{\alpha,n-\alpha}[0,a]$ and this solution is given by*

$$\begin{aligned} y(x) &:= \sum_{k=0}^{n-1} b_k x^k E_{\alpha,k+1,0}\left(\lambda x^{\alpha}; q\right) \\ &+ \int_0^x x^{\alpha-1}\left(qt/x;q\right)_{\alpha-1} E_{\alpha,\alpha,t}\left(\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right) f(t) d_q t. \end{aligned} \tag{15}$$

Proof. First, we solve the Volterra q-integral equation (14), we apply the method of successive approximations by setting

$$y_0(x)=\sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k$$

and

$$\begin{aligned} y_i(x) &= y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}y_{n-1}(t)d_qt \\ &+ \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f(t)d_qt \end{aligned} \tag{16}$$

Using Definition 2.1 and (10) and (16) we find $y_1(x)$:

$$y_1(x)=y_0(x)+\lambda\left(I^{\alpha}_{q,0+}y_0\right)(x)+\left(I^{\alpha}_{q,0+}f\right)(x)$$

that is,

$$\begin{aligned} y_1(x) &= \sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k+\lambda\sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}\left(I^{\alpha}_{q,0+}t^k\right)(x)+\left(I^{\alpha}_{q,0+}f\right)(x) \\ &= \sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k+\lambda\sum_{k=1}^{n-1}\frac{b_kx^{\alpha+k}}{\Gamma_q(\alpha+k+1)}+\left(I^{\alpha}_{q,0+}f\right)(x) \\ &= \sum_{k=0}^{n-1}b_k\sum_{m=0}^{1}\frac{\lambda^m x^{m\alpha+k}}{\Gamma_q(\alpha m+k+1)}+\left(I^{\alpha}_{q,0+}f\right)(x). \end{aligned} \tag{17}$$

Similarly, using Definition 2.1 and (10), (11) and (17) we have for $y_2(x)$ that

$$\begin{aligned} y_2(x) &= y_0(x)+\lambda\left(I^{\alpha}_{q,0+}y_1\right)(x)+\left(I^{\alpha}_{q,0+}f\right)(x) \\ &= \sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k+\sum_{k=0}^{n-1}b_k\sum_{m=0}^{1}\frac{\lambda^{m+1}}{\Gamma_q(\alpha m+k+1)}\left(I^{\alpha}_{q,0+}t^{m\alpha+k}\right)(x) \\ &+ \lambda\left(I^{\alpha}_{q,0+}I^{\alpha}_{q,0+}f(t)\right)(x)+\left(I^{\alpha}_{q,0+}f\right)(x) \\ &= \sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k+\sum_{k=0}^{n-1}b_k\sum_{m=0}^{1}\frac{\lambda^{m+1}}{\Gamma_q(\alpha(m+1)+k+1)}x^{\alpha(m+1)+k} \\ &+ \lambda\left(I^{2\alpha}_{q,0+}f(t)\right)(x)+\left(I^{\alpha}_{q,0+}f\right)(x) \\ &= \sum_{k=0}^{n-1}\frac{b_k}{[k]_q!}x^k+\sum_{k=0}^{n-1}b_k\sum_{m=0}^{1}\frac{\lambda^{m+1}}{\Gamma_q(\alpha(m+1)+k+1)}x^{\alpha(m+1)+k} \\ &+ \frac{\lambda x^{2\alpha-1}}{\Gamma(2\alpha)}\int_0^x f(t)\,(qt/x;q)_{2\alpha-1}\,d_qt+\left(I^{\alpha}_{q,0+}f\right)(x). \end{aligned} \tag{18}$$

Now for $\alpha m-1=\alpha(m-1)+\alpha-1$, using (5) we get

$$x^{\alpha m-1}(qt/x;q)_{\alpha m-1}=x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha(m-1)}(q^{\alpha}t/x;q)_{\alpha(m-1)}. \tag{19}$$

Thus combined with (18) and (19). gives

$$\begin{aligned}
y_2(x) &= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=1}^{2} \frac{\lambda^{m-1} x^{\alpha m-1}(qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{1} \frac{\lambda^{m} x^{\alpha (m+1)-1}(qt/x;q)_{\alpha (m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{2} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{1} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt.
\end{aligned}$$

Continuing this process, we derive the following relation for $y_i(x)$:

$$\begin{aligned}
y_i(x) &= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=1}^{i} \frac{\lambda^{m-1} x^{\alpha m-1}(qt/x;q)_{\alpha m-1}}{\Gamma_q(\alpha m)}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha (m+1)-1}(qt/x;q)_{\alpha (m+1)-1}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt \\
&= \sum_{k=0}^{n-1} b_k \sum_{m=0}^{i} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} \\
&+ \int_0^x \left[\sum_{m=0}^{i-1} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha(m+1))}\right] f(t)d_qt.
\end{aligned}$$

Taking the limit as $i\to\infty$, we obtain the following explicit solution $y(x)$ to the q-integral equation (16):

$$y(x) = \sum_{k=0}^{n-1} b_k \sum_{m=0}^{\infty} \frac{\lambda^m x^{\alpha m+k}}{\Gamma_q(\alpha m+k+1)} + \int_0^x \left[\sum_{m=0}^{\infty} \frac{\lambda^{m} x^{\alpha-1}(qt/x;q)_{\alpha-1}x^{\alpha m}(q^{\alpha}t/x;q)_{\alpha m}}{\Gamma_q(\alpha m+\alpha)}\right] f(t)d_qt.$$

On the basis of Theorem 2.1 an explicit solution to the Volterra q-integral equation (14) and hence to the Cauchy type problem (1)-(2).

Theorem 3.2 *Let $n-1<\alpha<n$ $(n\in N)$ and let $0\leqq\gamma<1$ be such that $\gamma\leqq\alpha$. Also let $\lambda\in\mathbb{R}$. If $f(x)\in C_{q,\gamma}[0,a]$, the Cauchy problem (1)-(2) has a unique solution $y(x)\in C^{\alpha,n-\alpha}_{q,\gamma}[0,a]$ and this solution is given by (15).*

In particular, if $\gamma=0$ and $f(x)\in C_q[0,a]$, then the solution $y(x)$ in (15) belongs to the space $C^{\alpha,n-\alpha}_{q}[0,a]$ defined in (13).

The Cauchy problem 2 involving the homogeneous q-difference equation (1)

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x)-\lambda y(x)=0 \quad (0\leqq x\leqq a; n-1<\alpha<n; n\in\mathbb{N};\lambda\in\mathbb{R}) \tag{20}$$

has a unique solution $y(x)\in C^{\alpha,n-\alpha}_{q,\gamma}[0,a]$ of the from

$$y(x)=\sum_{k=0}^{n-1} b_k x^k E_{\alpha,k+1,0}\left[\lambda x^{\alpha};q\right]. \tag{21}$$

3.2 The Cauchy problem for the more general homogeneous fractional q-difference equation with the Caputo fractional q-derivative

Now we consider the Cauchy problem for the more general homogeneous fractional q-difference equation (3) with the initial conditions (4).

Theorem 3.3 *Let $n-1<\alpha<n$; $(n\in\mathbb{N})$,and let $0\leq\gamma<1$,be such that $\gamma\leq\alpha$. Also let $\lambda\in\mathbb{R}$ and $\beta\geqq 0$. If $f\in C_{q,\gamma}[0,a]$, then the Cauchy problem (3)-(4) has a unique solution $y(x)$ in the space $C^{\alpha}_{q,n-\alpha}[0,a]$ and this solution is given by*

$$y(x) \;=\; \sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^j E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta+j)}{\alpha}}\left[\lambda x^{\alpha+\beta};q\right]. \tag{22}$$

Proof. With $\beta>-\alpha$. Note again that, in accordance with Theorem 3.1, the problem (3)-(4) is equivalent in the space $C_{q,n-1}[0,a]$ to the following Volterra q-integral equation of the second kind:

$$y(x)=\sum_{j=0}^{n-1}\frac{b_j}{\Gamma_q(j+1)}x^j+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y(t)d_qt. \tag{23}$$

Similarity, we again apply the method of successive approximations to solve this q-integral equation (23). We assume that $y_0(x)=\sum\limits_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^j$ and

$$y_m(x)=y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1}y_{m-1}(t)d_qt. \tag{24}$$

Using the same arguments as above, by using (5), (6) and (24) we find $y_1(x)$:

$$\begin{aligned}
y_1(x) &= y_0(x)+\frac{\lambda x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x t^{\beta}\left(qt/x;q\right)_{\alpha-1} y_0(x)d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_jx^{\alpha-1}}{[j]_q!}\int_0^x t^{\beta+j}\left(qt/x;q\right)_{\alpha-1} d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_jx^{\alpha+\beta+j}}{[j]_q!}\int_0^1 y^{\beta+j}\left(qy;q\right)_{\alpha-1} d_qy \\
&= \sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^j+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_jx^{\alpha+\beta+j}}{[j]_q!}B_q\left(\beta+j+1,\alpha\right) \\
&= \sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^j \\
&+ \lambda\sum_{j=0}^{n-1}\frac{b_jx^{\alpha+\beta+j}}{\Gamma_q(j+1)}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}. \qquad (25)
\end{aligned}$$

Similarly, for $m=2$ using (6), (24) and taking (25) into account, we derive

$$\begin{aligned}
y_2(x) &= y_0(x)+\lambda\left(I^{\alpha}_{q,0+}y_1\right)(x) \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^{\alpha-1}\int_0^x t^{\beta+j}\left(qt/x;q\right)_{\alpha-1} d_qt \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}x^{\alpha-1}\int_0^x t^{\alpha+2\beta+j}\left(qt/x;q\right)_{\alpha-1} d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^{\beta+j+\alpha}\int_0^1 y^{\beta+j}\left(qy;q\right)_{\alpha-1} d_qy \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^{2\alpha+2\beta+j}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}\int_0^1 y^{\alpha+2\beta+j}\left(qy;q\right)_{\alpha-1} d_qt \\
&= y_0(x)+\frac{\lambda}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^{\beta+j+\alpha}B_q\left(\beta+j+1,\alpha\right) \\
&+ \frac{\lambda^2}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}\frac{b_j}{[j]_q!}x^{2\alpha+2\beta+j}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}B_q\left(\alpha+2\beta+j+1,\alpha\right) \\
&= \sum_{j=0}^{n-1}\frac{b_jx^j}{[j]_q!}\left[1+\lambda x^{\alpha+\beta}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}\right] \\
&+ \sum_{j=0}^{n-1}\frac{b_jx^j}{[j]_q!}\left[\lambda^2x^{2\alpha+2\beta}\frac{\Gamma_q(\beta+j+1)}{\Gamma_q(\alpha+\beta+j+1)}\frac{\Gamma_q(\alpha+2\beta+j+1)}{\Gamma_q(2\alpha+2\beta+j+1)}\right].
\end{aligned}$$

The same arguments as in Section 3.1 lead to the following expression for $y_m(x)$ $m\in\mathbb{N}$:

$$y_m(x) = \sum_{j=0}^{n-1}\frac{b_j}{\Gamma_q(j+1)}t^j\left[1+\sum_{k=1}^{m}d_k\left(\lambda t^{\alpha+\beta}\right)^k\right], \tag{26}$$

where

$$d_k=\prod_{r=1}^{k}\frac{\Gamma_q[r(\alpha+\beta)-\alpha+j+1]}{\Gamma_q[r(\alpha+\beta)+j+1]};\quad (k\in\mathbb{N}). \tag{27}$$

Taking the limit as $m\to\infty$, we obtain the following explicit solution $y(x)$ to the q-integral equation (24) and hence to the Cauchy type problem (3)-(4):

$$y(x) = \sum_{j=0}^{n-1}\frac{b_j}{\Gamma_q(j+1)}t^j\left[1+\sum_{k=1}^{\infty}d_k\left(\lambda t^{\alpha+\beta}\right)^k\right], \tag{28}$$

According to the relations (8), we rewrite this solution in terms of the generalized Mittag-Leffler q-function $E_{q,\alpha,m,l(z)}$:

$$y(x) = \sum_{j=0}^{n-1}\frac{b_j}{\Gamma_q(j+1)}t^jE_{q;\alpha,1+\frac{\beta}{\alpha},\frac{(\beta+j)}{\alpha}}\left[\lambda t^{\alpha+\beta};q\right]. \tag{29}$$

If $\beta\geq 0$, then $f[x,y]=\lambda(t)^{\beta}$ satisfies the Lipschitz condition for any $x_1,x_2\in(a,b]$ and any $y\in G$, where G is any open set of $\mathbb{C}$. If $\gamma\geq n-\alpha$, then, by Property 3.l(b) and Remark 3.18, there exists a unique solution to the Cauchy type problem (1)-(2) in the space $C^{\alpha}_{n-\alpha}$, and thus this solution has the form (29). This leads to the following result.

4 A Set of Examples

Example 1 *Let $b\in\mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x)-\lambda y(x)=f(x), y(0+)=b, \tag{30}$$

with $0<\alpha<1$ and $\lambda\in\mathbb{R}$ has the form:

$$\begin{aligned} y(x) &= bE_{\alpha,0}\left[\lambda x^{\alpha};q\right] \\ &+ x^{\alpha-1}\int_0^x(qt/x;q)_{\alpha-1}E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha};q\right]f(t)d_qt \end{aligned} \tag{31}$$

while the solution to the problem

$$\left({}^cD^{\alpha}_{q,0+}y\right)(x)-\lambda y(x)=0, y(0+)=b, \tag{32}$$

is given by

$$y(x) = bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] \tag{33}$$

In particular, the Cauchy type problem

$$\left({}^{c}D^{\frac{1}{2}}_{q,0+}y\right)(x) - \lambda y(x) = f(x), y(0+) = b, \tag{34}$$

has the solution given by

$$\begin{aligned} y(x) &= bE_{\frac{1}{2},0}\left[\lambda x^{\frac{1}{2}}; q\right] \\ &+ \int_0^x E_{\frac{1}{2},\frac{1}{2},t}\left[\lambda x^{\frac{1}{2}}(q^{\frac{1}{2}}t/x;q)_{\frac{1}{2}}; q\right] \frac{f(t)}{x^{\frac{1}{2}}(qt/x;q)_{\frac{1}{2}}} d_qt \end{aligned} \tag{35}$$

and the solution to the problem

$$\left({}^{c}D^{\frac{1}{2}}_{q,0+}y\right)(x) - \lambda y(x) = 0, y(0+) = b, \tag{36}$$

is given by

$$y(x) = bE_{\frac{1}{2},0}\left[\lambda x^{\frac{1}{2}}; q\right] \tag{37}$$

Example 2 *Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = f(x), y(0+) = b, y'(0+) = d, \tag{38}$$

with $1 < \alpha < 2$ and $\lambda \in \mathbb{R}$ has the form:

$$\begin{aligned} y(x) &= bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] + dxE_{\alpha,2,t}\left[\lambda x^{\alpha}; q\right] \\ &+ x^{\alpha-1}\int_0^x (qt/x;q)_{\alpha-1}E_{\alpha,\alpha,t}\left[\lambda x^{\alpha}(q^{\alpha}t/x;q)_{\alpha}\right] f(t)d_qt \end{aligned} \tag{39}$$

In particular, the solution to the problem $(1 < \alpha < 2)$

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda y(x) = 0, y(0+) = b, y'(0+) = d, \tag{40}$$

is given by

$$y(x) = bE_{\alpha,0}\left[\lambda x^{\alpha}; q\right] + dxE_{\alpha,2,t}\left[\lambda x^{\alpha}; q\right]. \tag{41}$$

Example 3 *let $b \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, \tag{42}$$

with $0 < \alpha < l$, $\beta \in \mathbb{R}(\beta > -\alpha)$ and $\lambda \in \mathbb{R}$ is given by

$$y(x) \;=\; bE_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta)}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right]. \tag{43}$$

In particular, the Cauchy type problem

$$\left({}^{c}D^{\frac{1}{2}}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, \tag{44}$$

with $\beta > -\frac{1}{2}$ is given by

$$y(x) \;=\; bE_{\frac{1}{2},2\beta+1,2\beta}\left[\lambda x^{\beta+\frac{1}{2}}; q\right]. \tag{45}$$

Example 4 *Let $b, d \in \mathbb{R}$. Then the solution to the Cauchy type problem*

$$\left({}^{c}D^{\alpha}_{q,0+}y\right)(x) - \lambda x^{\beta} y(x) = 0, y(0+) = b, y'(0+) = d, \tag{46}$$

with $1 < \alpha < 2$, $\beta > -\alpha$ and $\lambda \in \mathbb{R}$ has the form

$$\begin{aligned} y(x) \;&=\; bE_{\alpha,1+\frac{\beta}{\alpha},\frac{\beta}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right] \\ &+\; dxE_{\alpha,1+\frac{\beta}{\alpha},1+\frac{(\beta+1)}{\alpha}}\left[\lambda x^{\alpha+\beta}; q\right]. \end{aligned} \tag{47}$$

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CONVOLUTIONS GENERATED BY THE DIRICHLET PROBLEM OF THE STURM-LIOUVILLE OPERATOR

This paper is devoted to approximations of the product of two continuous functions on a finite segment by some special convolutions. The accuracy of the approximation depends on the length of the segment on which the functions are defined. These convolutions are generated by the Sturm-Liouville boundary value problems. The paper indicates that each boundary value problem for a second order differential equation generates its own individual convolution and its own individual Fourier transform. At that the Fourier transform of the convolution is equal to the product of the Fourier transforms. The latter property makes it possible to approximately solve nonlinear Burgers-type equations by first replacing the nonlinear term with a convolution of two functions. Similar methods of studying nonlinear partial differential equations can be found in the works of A. Y. Kolesov, N. H. Rozov, V. A. Sadovnichy.
In this paper, we construct a concrete convolution generated by the Dirichlet boundary value problem for twofold differentiation. The properties of the constructed convolution and their connection with the corresponding Fourier transform are derived. In the final part of the paper, the convergence of convolution is proved $(g(x)\sin(x)) * (f(x)\sin(x))$ defined on a segment $C[0, b]$ to the product $g(x)f(x)$ with b tending to zero for any two continuous functions $f(x)$ and $g(x)$.
Key words: approximation, convolution, boundary value problems, Dirichlet problem, Fourier transform.

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Штурм-Лиувилль операторының Дирихле есебінен туындайтын үйірткілер

Бұл жұмыс ақырлы кесіндіде анықталған арнайы үйірткілері бар екі үзіліссіз функцияның көбейтіндісінің аппроксимациясына арналған. Берілген функцияның жуықтау дәлдігі кесіндінің ұзындығына байланысты. Бұл үйірткілер Штурм-Лиувилль шеттік есебінен туындайды. Жұмыста екінші ретті дифференциалдық теңдеу үшін әрбір шеттік есептің өзінің жеке үйірткісі мен Фурье түрлендіруінің туындатылатынын көрсетеді. Сонымен қатар, бұл үйірткіден алынған Фурье түрлендіруі Фурье түрлендірулерінің көбейтіндісіне тең. Соңғы қасиет екі функцияның үйірткісінің сызықты емес мүшесінің алдын-ала алмастыру арқылы Бюргерс типті сызықты емес теңдеулерді жуықтап шешуге мүмкіндік береді. Дербес туындылары бар сызықты емес дифференциалдық теңдеулерді зерттеудің ұқсас әдістерін А.Ю. Колесов, Н.Х. Розов, В.А. Садовничий-лердің еңбектерінен табуға болады.
Жұмыста екі еселенген дифференциал үшін Дирихле шеттік есебінен туындаған нақты үйірткі құрылады. Құрылған үйірткінің қасиеттері және олардың Фурье түрлендірулерімен байланысы көрсетілген. Жұмыстың соңғы бөлімінде $C[0, b]$ кесіндісінде анықталған $(g(x)\sin(x)) * (f(x)\sin(x))$ үйірткісі үшін кез келген екі үзіліссіз $f(x)$, $g(x)$ функцияларының $g(x)f(x)$ көбейтісінің b нөлге ұмтылғандағы жинақтылығы дәлелденген.
Түйін сөздер: жуықтау, үйірткі, шеттік есеп, Дирихле есебі, Фурье түрлендіруі.

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Свертки, порождаемые задачей Дирихле оператора Штурма-Лиувилля

Настоящая работа посвящена аппроксимаций произведения двух непрерывных на конечном отрезке функций некоторыми специальными свертками. Точность приближения зависит от длины отрезка на котором задаются функций. Эти свертки порождаются краевыми задачами Штурма-Лиувилля. В работе указывается, что каждая краевая задача для дифференциального уравнения второго порядка порождает свою индивидуальную свертку и свое индивидуальное преобразование Фурье. Причем преобразование Фурье от свертки равно произведению преобразований Фурье. Последнее свойство позволяет приближенно решать нелинейные уравнения типа Бюргерса, предварительно заменив нелинейный член сверткой двух функций. Подобные методы исследования нелинейных дифференциальных уравнений с частными производными можно найти в работах А. Ю. Колесова, Н. Х. Розова, В. А. Садовничего.
В работе строится конкретная свертка, порожденная краевой задачей Дирихле для двухкратного дифференцирования. Выведены свойства построенной свертки и связь их с соответствующим преобразованием Фурье. В заключительной части работы доказана сходимость свертки $(g(x)\sin(x)) * (f(x)\sin(x))$ определенной на отрезки $C[0,b]$ к произведению $g(x)f(x)$ при b стремящемся к нулю для любых двух непрерывных функций $f(x)$ и $g(x)$.
Ключевые слова: приближение, свертка, краевые задачи, задача Дирихле, преобразование Фурье.

1 Introduction

In this paper, we are interested in the approximation of nonlinear terms of differential operators by some special convolutions. To motivate our research, let's consider the Burgers equation for simplicity

$$\frac{\partial u(t,x)}{\partial t} + u(t,x)\frac{\partial u(t,x)}{\partial x} = \nu\frac{\partial^2 u(t,x)}{\partial x^2},\ 0 < x < b,\ t > 0 \tag{1}$$

on a finite segment $(0, b)$ with kinematic viscosity ν. Replacement

$$\xi = x\sqrt{\nu} \tag{2}$$

equation (1) leads to the form

$$\frac{\partial v(t,\xi)}{\partial t} + v(t,x)\sqrt{\nu}\frac{\partial v(t,\xi)}{\partial \xi} = \frac{\partial^2 v(t,\xi)}{\partial \xi^2},\ 0 < \xi < b\sqrt{\nu},\ t > 0. \tag{3}$$

Whereas as kinematic viscosity ν and the Reynolds number are mutually inverse, then there is a critical viscosity value ν_{cr}. When $\nu > \nu_{cr}$ the fluid flow will be steadily laminar. Movement at $\nu < \nu_{cr}$ becomes unstably turbulent. Thus, for small values ν there is a movement of the liquid acquiring a turbulent character. If $\nu \to 0$, then the length of the interval $[0, b\sqrt{\nu}]$ becomes a small quantity. In this case, it becomes possible to approximate of the nonlinear term $v(t,x)\dfrac{\partial v(t,\xi)}{\partial \xi}$ by some special convolution $v_1 * \left(\dfrac{\partial v}{\partial \xi}s_0(\xi)\right)$. Here $v_1(t,\xi) = v(t)s_0(\xi)$, where $s_0(\xi)$ - fixed function. By convolution we mean some two-dimensional, associative, bilinear operation consistent with the corresponding Fourier transform. In other words, if F is Fourier transform, then the equality

$$F(f * g) = Ff \cdot Fg \tag{4}$$

is rightly for the convolution we introduced. Then for small values ν equation (3) can be approximated by its approximation

$$\frac{\partial W(t,\xi)}{\partial t} + (W(t,\xi)s_0(\xi)) * \left(\frac{\partial W(t,\xi)}{\partial t}s_0(\xi)\right) = \frac{\partial^2 W(t,\xi)}{\partial \xi^2}. \tag{5}$$

Property (4) makes it possible to solve equation (5) efficiently by the method of separation of variables. Similar schemes used in works [3]-[14].

A wide set of convolutional operations generate boundary value problems for linear differential operators. In mathematical physics, the solution of an inhomogeneous equation $Au = f$ is written as a convolution of two functions $u = \varepsilon * f$, where ε is the corresponding fundamental solution [3]. Under the convolution is understood to be the bilinear a (possibly noncommutative) operation without the right annulators. When there is an inverse operator A^{-1}, then the convolution associated with the linear operator A has nonzero divisors. If the operator A corresponds to a boundary value problem in a bounded domain, then the convolution may depend on its boundary conditions. For example, the convolution corresponding to the operator B_1 in the function space $L_2(0,1)$ has the following form

$$(f *_{B_1} g)(x) = \int_0^x f(x-t)g(t)dt + \frac{1}{h}\int_x^1 f(1+x-t)g(t)dt.$$

Here the operator B_1 corresponds to the boundary value problem

$$-i\frac{dy}{dt} = f(x),\ 0 < x < 1,\ y(1) = hy(0).$$

The resolvent of the B_1 operator has a convolutional representation

$$(B_1 - \lambda I)^{-1}f(x) = (\varepsilon_\lambda *_{B_1} f)(x),\ where\, \varepsilon_\lambda(t) = ih\frac{e^{i\lambda t}}{h - e^{i\lambda}}. \tag{6}$$

The convolution $*_{B_1}$ defined by formula (6) depends on the boundary parameter h. A more difficult example is given [4]. In the Hilbert space $L_2[0,1]$, we define the operator B_2 generated by the differential expression $lu = -\dfrac{d^2u(x)}{dx^2}$, $0 < x < 1$, and the domain of definition

$$D(B_2) = \left\{u \in W_2^2[0,1] : u(0) = 0,\ u^{'}(1) = u(1)\right\}.$$

The spectral properties of operator are studied in detail in the work of N. I. Ionkin [5]. The convolution generated by the operator B_2 is defined by the formula

$$(g *_{B_2} f)(x) = \frac{1}{2}\int_x^1 g(1+x-t)f(t)dt + \int_{1-x}^1 g(x-1+t)f(t)dt +$$

$$+\int_0^x g(x-t)f(t)dt - \frac{1}{2}\int_0^{1-x} g(1-x-t)f(t)dt + \frac{1}{2}\int_0^x g(1+t-x)f(t)dt.$$

In this case, the resolvent of operator B_2 has the convolutional representation

$$(B_2 - \lambda I)^{-1} f(x) = (\varepsilon_\lambda *_{B_2} f)(x), where\, \varepsilon_\lambda(t) = \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}(cos(\sqrt{\lambda}-1))}.$$

In [6, 7, 8, 9] we can find convolutions generated by first-order differential operators with integral boundary conditions. In the works of M. V. Ruzhansky and his co-authors [4, 10, 11, 12], convolutions generated by

1. Operators whose root elements form a Riesz basis in the corresponding space.

2. Riesz basis of the Hilbert space are investigated.

In [8], the construction of explicit convolution formulas uses representation of the Green function. Usually, the Green function $G(x, t)$ is a two-place function, while the fundamental solution $\varepsilon(t)$ is a one-place function. When deriving an explicit convolution formula, it is necessary to express the two-place function $G(x, t)$ linearly in terms of the one-place function $\varepsilon(t)$, and it is allowed to use integration and differentiation operations [13].

In the future, we will need a convolution generated by the periodic problem. For the operator B_3 corresponding to the periodic problem

$$-y - f(x),\ 0 < x < 1,\ y(0) = y(1),\ y'(0) = y'(1)$$

convolution $*_{B_3}$ has the following form

$$(f*_{B_3}g)(x) = \int_0^x f(x-t)g(t)dt + \int \_x^1 f(t-x)g(t)dt + \int_0^x f(1+t-x)g(t)dt + \int_x^1 f(1+x-t)g(t)dt.$$

The resolvent of the $*_{B_3}$ operator has a convolutional representation

$$(B_3 - \lambda I)^{-1} f(x) = (\varepsilon_\lambda *_{B_3} f)(x),$$

where

$$\varepsilon_\mu(x) = -\frac{\sin(\sqrt{\lambda}x)}{2\sqrt{\lambda}(1-cos(\sqrt{\lambda}))}.$$

In the future, the convolution $*_{B_3}$ is re denoted by $*$.

2 Integral representation of the solution to the Dirichlet problem of the Sturm-Liouville operator

The main result of this section is stated in the following lemma.

Lemma 1 *In the function space $L_2(0, b)$ is studied the Dirichlet problem for the Sturm-Liouville equation*

$$-y''(x) = \lambda y(x) + f(x),\ 0 < x < b \tag{7}$$

$$y(0)=0,\ y(b)=0. \tag{8}$$

The solution to problem (7)-(8) at $0<x<b$ *has the representation*

$$y(x)=\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt+\int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt+\frac{1}{2\sin\sqrt{\lambda}b}\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t-\xi)f(t)dt\right\}. \tag{9}$$

In the functional space $L_2(0,b)$, we denote by B the Sturm-Liouville operator, which corresponds to the Dirichlet problem (7)-(8).Then the right part of formula (9) determines of the resolvent of operator B.

Proof of Lemma 1. First, let us check that the right part of relation (7) satisfies boundary conditions (8). For this, it is necessary to denote the right part of relation (9) by $u(x)$. Then direct substitution into $u(x)$ the values $x=0$ and $x=b$ leads to the equalities

$$u(0)=0,$$

$$u(b)=\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^b \sin\sqrt{\lambda}\tau d\tau\int_\tau^b f(t)dt-\int_0^b \sin\sqrt{\lambda}\tau d\tau\int_\tau^b f(t)dt\right\}=0.$$

Now need to check that the function $u(x)$ is a solution of equation (7). For this we calculate the corresponding derivatives.

$$u'(x)=-\frac{1}{2\sin\sqrt{\lambda}b}\sqrt{\lambda}\left\{\int_0^x \cos\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt-\int_0^x \cos\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\}+$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_x^b \sin\sqrt{\lambda}(b-t+x)f(t)dt+\int_x^b \sin\sqrt{\lambda}(b-t-x)f(t)dt\right\},$$

$$u''(x)=-\frac{1}{2\sin\sqrt{\lambda}b}\lambda\int_{-x}^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt-$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt-\int_0^x \sin\sqrt{\lambda}(b-x+\tau)\int_\tau^x f(t)dt\right\}-f(x).$$

The identity is used here

$$\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt-\int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt=\int_{-x}^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt.$$

Then the equality follows required $u''(x)=-\lambda u(x)-f(x)$. Thus, Lemma 1 is completely proved.

3 Convolutions generated by the Dirichlet problem of the Sturm-Liouville operator

In this paragraph, the convolution formula is given, which corresponds to the Dirichlet problem of the Sturm-Liouville operator.

For any two functions $f(x), g(x) \in L_2(0,b)$ introduce a convolution, which at $0 < x < \dfrac{b}{2}$ is determined by the formula

$$\begin{aligned}(g*f)(x) &= \int_0^x g(b-x+\tau)d\tau \int_\tau^x f(t)dt + \int_0^x g(b-x-\tau)d\tau \int_\tau^x f(t)dt+ \\ &+\int_0^x d\xi \int_x^b g(b-t+\xi)f(t)dt + \int_0^x d\xi \int_x^{\frac{b}{2}} g(b-t-\xi)f(t)dt- \\ &-\int_0^x d\xi \int_{\frac{b}{2}}^{b-\xi} g(b-t-\xi)f(t)dt - \int_0^x d\xi \int_{b-\xi}^b g(t+\xi-b)f(t)dt,\end{aligned} \tag{10}$$

and at $\dfrac{b}{2} < x < b$ is determined by the formula

$$\begin{aligned}(g*f)(x) &= \int_0^x g(b-x+\tau)d\tau \int_\tau^x f(t)dt + \int_0^{b-x} g(b-x-\tau)d\tau \int_\tau^x f(t)dt- \\ &-\int_{b-x}^x g(x+\tau-b)d\tau \int_\tau^x f(t)dt + \int_0^x d\xi \int_x^b g(b-t+\xi)f(t)dt+ \\ &+\int_0^{\frac{b}{2}} d\xi \int_x^{b-\xi} g(b-t-\xi)f(t)dt - \int_0^{\frac{b}{2}} d\xi \int_{b-\xi}^b g(t+\xi-b)f(t)dt - \int_{\frac{b}{2}}^x d\xi \int_x^b g(t+\xi-b)f(t)dt.\end{aligned} \tag{11}$$

The convolution introduced by us is linear for each argument and has associativity properties, at the same time, this convolution is not commutative.

Definition 1 *We will say that * convolution is generated by the operator B if its resolvent $(B-\lambda I)^{-1}$ has the following convolutional representation*

$$(B-\lambda I)^{-1}f(x) = (\varepsilon_\lambda * f)(x),$$

where ε_λ is the corresponding fundamental solution.

In functional space $L_2(0,b)$ the Sturm-Liouville operator corresponding to the Dirichlet problem (7) - (8) is denoted by B.

Lemma 2 *The convolution given by formulas (10)-(11) is generated by the operator B.*

Proof of Lemma 2. According to Lemma 1, the resolvent of operator B is given using the right side of formula (9). In this paragraph, we will rewrite the right part of formula (9) in a convenient form for further research:

If $0 < x < \frac{b}{2}$, then

$$u(x) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt + \int_0^x \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\} +$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt + \int_0^x d\xi\int_x^{\frac{b}{2}} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt\right\} -$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x d\xi\int_{\frac{b}{2}}^{b-\xi} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt + \int_0^x d\xi\int_{b-\xi}^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt\right\}.$$

If $\frac{b}{2} < x < b$, then

$$u(x) = \frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^x \sin\sqrt{\lambda}(b-x+\tau)d\tau\int_\tau^x f(t)dt + \int_0^{b-x} \sin\sqrt{\lambda}(b-x-\tau)d\tau\int_\tau^x f(t)dt\right\} -$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_{b-x}^x \sin\sqrt{\lambda}(x+\tau-b)d\tau\int_\tau^x f(t)dt - \int_0^x d\xi\int_x^b \sin\sqrt{\lambda}(b-t+\xi)f(t)dt\right\} +$$

$$+\frac{1}{2\sin\sqrt{\lambda}b}\left\{\int_0^{\frac{b}{2}} d\xi\int_x^{b-\xi} \sin\sqrt{\lambda}(b-t-\xi)f(t)dt - \int_0^{\frac{b}{2}} d\xi\int_{b-\xi}^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt\right\} -$$

$$-\frac{1}{2\sin\sqrt{\lambda}b}\int_{\frac{b}{2}}^x d\xi\int_x^b \sin\sqrt{\lambda}(t+\xi-b)f(t)dt.$$

It is easy to notice that the value of the resolvent $(B-\lambda I)^{-1}f(x)$ coincides with $u(x)$. On the other hand, the solution has a convolutional representation

$$u(x) = (g * f)(x),$$

where $g(x) = \dfrac{\sin\sqrt{\lambda}x}{2\sin\sqrt{\lambda}b}$. Lemma 2 is completely proved.

4 The Fourier transform generated by the operator B

The poles of the resolvent $(B-\lambda I)^{-1}$ determine the eigenvalues of the operator B. Since

$$(B-\lambda I)^{-1}f(x) = (g * f)(x), \tag{12}$$

where $g(x) = \dfrac{\sin\sqrt{\lambda}x}{2\sin\sqrt{\lambda}b}$ is the appropriate fundamental solution. It follows from the representation (12) that the poles of the resolvent $(B-\lambda I)^{-1}$ are zeros of the function $\sin\sqrt{\lambda}b = 0$. It follows that zeros have the form

$$\lambda_k = \frac{\pi^2}{b^2}k^2, k = 0, \pm 1, ... \tag{13}$$

A more detailed analysis shows that λ_0 is the eliminable singular point of the resolvent $(B-\lambda I)^{-1}$.

Thus, the resolvent $(B-\lambda I)^{-1}$ has only simple poles $\lambda_k = \dfrac{\pi^2}{b^2}k^2, k = 1, 2, ...$ In order to find their corresponding eigenfunctions of the operator B, we need to calculate

$$\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(x)\sin\frac{\pi kt}{b}dt.$$

The direct calculation of the residue at the point $\lambda = \lambda_k$ leads to the formula

$$\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(x)\sin\frac{\pi kt}{b}dt =$$

$$= \frac{(-1)^{2k+1}2\pi k}{b^2}\left\{\int_0^x \sin\frac{\pi k}{b}(x-\tau)d\tau\int_\tau^x f(t)dt + \int_0^x \sin\frac{\pi k}{b}(x+\tau)d\tau\int_\tau^x f(t)dt\right\} +$$

$$+\frac{(-1)^{2k+1}2\pi k}{b^2}\left\{\int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(t-\xi)f(t)dt + \int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(t+\xi)f(t)dt\right\} =$$

$$= -\frac{2\pi k}{b^2}\left[\int_0^x f(t)dt\left\{\int_0^t \sin\frac{\pi k}{b}(x-\tau)d\tau + \int_0^t \sin\frac{\pi k}{b}(x+\tau)d\tau\right\}\right] -$$

$$-\frac{2\pi k}{b^2}\left[\int_x^b f(t)dt\left\{\int_0^x \sin\frac{\pi k}{b}(t-\xi)d\xi + \int_0^x \sin\frac{\pi k}{b}(t+\xi)d\xi\right\}\right] =$$

$$= -\frac{2\pi k}{b^2}\left[\int_0^x f(t)dt\int_0^t 2\sin\frac{\pi k}{b}x\cos\frac{\pi k}{b}\tau d\tau + \int_x^b f(t)dt\int_0^x 2\sin\frac{\pi k}{b}t\cos\frac{\pi k}{b}\xi d\xi\right] =$$

$$= -\frac{2\pi k}{b^2}\left[\frac{2b}{\pi k}\int_0^x f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt + \frac{2b}{\pi k}\int_x^b f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt\right] =$$

$$= -\frac{4}{b}\int_0^b f(t)\sin\frac{\pi kx}{b}\sin\frac{\pi kt}{b}dt = -\frac{4}{b}\sin\frac{\pi kx}{b}\int_0^b f(t)\sin\frac{\pi kt}{b}dt.$$

It follows from this that the system of eigenfunctions of the operator B has the form $\{\sin\frac{\pi x}{b}, \sin\frac{2\pi x}{b}, \sin\frac{3\pi x}{b}, ...\}$, corresponding Fourier coefficients are calculated by the formulas

$$b_k(f) = -\underset{\lambda_k}{res}(B-\lambda I)^{-1}f(x) = \frac{4}{b}\int_0^b f(t)\sin\frac{\pi k}{b}tdt, k = 1, 2, ...$$

5 The relation of the Fourier transform and convolution generated by the operator B

Take two functions f and g from $L_2(0,b)$ and decompose them according to the system of eigenfunctions $\{\sin\frac{\pi x}{b}, \sin\frac{2\pi x}{b}, \sin\frac{3\pi x}{b}, ...\}$, as a result , we have

$$f(x)=\sum_{k=1}^{\infty} b_k(f)\sin\frac{\pi kx}{b}$$

$$g(x)=\sum_{j=1}^{\infty} b_j(g)\sin\frac{\pi jx}{b}.$$

Now calculate the convolution $(g*f)(x)$, to do this, we formulate an auxiliary statement.

Lemma 3 *For any k and j, the equality is true*

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=0, k\neq j,$$

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=1, k=j.$$

Proof of Lemma 3. By definition, at $0<x<\frac{b}{2}$

$$\sin\frac{\pi kx}{b}*\sin\frac{\pi jx}{b}=\int_0^x \sin\frac{\pi k}{b}(b-x+\tau)d\tau\int_\tau^x \sin\frac{\pi jt}{b}dt+$$

$$+\int_0^x \sin\frac{\pi k}{b}(b-x-\tau)d\tau\int_\tau^x \sin\frac{\pi jt}{b}dt+\int_0^x d\xi\int_x^b \sin\frac{\pi k}{b}(b-t+\xi)\sin\frac{\pi jt}{b}dt+$$

$$+\int_0^x d\xi\int_x^{\frac{b}{2}} \sin\frac{\pi k}{b}(b-t-\xi)\sin\frac{\pi jt}{b}dt-\int_0^x d\xi\int_{\frac{b}{2}}^{b-\xi} \sin\frac{\pi k}{b}(b-t-\xi)\sin\frac{\pi jt}{b}dt-$$

$$-\int_0^x d\xi\int_{b-\xi}^{b} \sin\frac{\pi k}{b}(t+\xi-b)\sin\frac{\pi jt}{b}dt.$$

For $\frac{b}{2}<x<b$, the statement of Lemma 3 is checked similarly. Lemma 3 is fully proved. Immediately follows by Lemma 3

$$(g*f)(x)=\sum b_k(f)b_k(g).$$

6 Approximation of multiplication by convolution

Let f, g be two arbitrary functions both defined and continuous on the segment $[0, b]$. Denote by $g_1(x)$ and $f_1(x)$

$$g_1(x) \equiv g(x)\sin(x),\ f_1(x) \equiv f(x)\sin(x).$$

Theorem 1 *For any two functions f and g continuous on $[0, b]$ is the limiting relation rightly*

$$\lim_{b\to 0}\left[(g_1 * f_1)(x) - g(x)f(x)\right] = 0, \forall x \in [0, b].$$

Consider the difference

$$R(x) \equiv (g_1 * f_1)(x) - g(x)f(x)$$

Proof of Theorem 1. Introduce the notation

$$R_1(x) = \int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt,$$

$$R_2(x) = \int_0^x g(b-x-\tau)\sin(b-x-\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x-\tau)d\tau \int_\tau^x f(x)\sin(t)dt,$$

$$R_3(x) = \int_0^x d\xi \int_x^b g(b-t+\xi) sin(b-t+\xi) f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_x^b f(x)\sin(b-t+\xi)\sin(t)dt,$$

$$R_4(x) = \int_0^x d\xi \int_x^{\frac{b}{2}} g(b-t-\xi)\sin(b-t-\xi)f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_x^{\frac{b}{2}} f(x)\sin(b-t-\xi)\sin(t)dt,$$

$$R_5(x) = \int_0^x d\xi \int_{\frac{b}{2}}^{b-\xi} g(b-t-\xi)\sin(b-t-\xi)f(t) sin(t)dt - \int_0^x g(x)d\xi \int_{\frac{b}{2}}^{b-\xi} f(x)\sin(b-t-\xi)\sin(t)dt,$$

$$R_6(x) = \int_0^x d\xi \int_{b-\xi}^b g(t+\xi-b)\sin(t+\xi-b)f(t)\sin(t)dt - \int_0^x g(x)d\xi \int_{b-\xi}^b f(x)\sin(t+\xi-b)\sin(t)dt.$$

Note that

$$R(x) = R_1(x) + R_2(x) + R_3(x) + R_4(x) + R_5(x) + R_6(x).$$

For the upper estimate of $R(x)$, it is necessary to estimate the values of $R_1(x)$, $R_2(x)$, ... , $R_6(x)$ from above. Now we evaluate the module of the function $|R_1(x)|$ from above

$$|R_1(x)| = \left|\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(t)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt\right| =$$

$$= |\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\, dt +$$

$$+ \int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt - \int_0^x g(x)\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt|,$$

$$|R_1(x)| = |\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\, dt -$$

$$- \int_0^x (g(b-x+\tau) - g(x))\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt| \leq$$

$$\leq \left|\int_0^x g(b-x+\tau)\sin(b-x+\tau)d\tau \int_\tau^x (f(t)\sin(t) - f(x)\sin(t))\,dt\right| +$$

$$+\left|\int_0^x (g(b-x+\tau) - g(x))\sin(b-x+\tau)d\tau \int_\tau^x f(x)\sin(t)dt\right| \leq$$

$$\leq \int_0^x |g(b-x+\tau)|\,|\sin(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,|\sin(t)|\,dt +$$

$$+\int_0^x |g(b-x+\tau)-g(x)|\,|\sin(b-x+\tau)|\,d\tau \int_\tau^x |f(x)|\,|\sin(t)|\,dt \leq$$

$$\leq \int_0^x |g(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,dt + \int_0^x |g(b-x+\tau)-g(x)|\,d\tau \int_\tau^x |f(x)|\,dt.$$

If $0 < x < \dfrac{b}{2}$

$$|R_1(x)| \leq \int_0^x |g(b-x+\tau)|\,d\tau \int_\tau^x |f(t)-f(x)|\,dt + \int_0^x |g(b-x+\tau)-g(x)|\,d\tau \int_\tau^x |f(x)|\,dt.$$

Since $f, g \in C[0,b]$, then $|f(x)| \leq M_f$, $|g(x)| \leq M_g$. Therefore

$$|R_1(x)| \leq M_g \int_0^x d\tau \int_\tau^x |f(t)-f(x)|\,dt + M_f \int_0^x |g(b-x+\tau)-g(x)|\,(x-\tau)d\tau.$$

We consider that the length of the segment $[0,b]$ very small, then the inequalities are fulfilled $|f(t)-f(x)| < \varepsilon$, $|g(b-x+\tau)-g(x)| < \varepsilon$, $\forall\, 0 < \tau < x < \dfrac{b}{2}$.

$$|R_1(x)| \leq \varepsilon M_g \int_0^x d\tau \int_\tau^x dt + \varepsilon M_f \int_0^x (x-\tau)d\tau = \varepsilon(M_g + M_f)\frac{x^2}{2},$$

if the value b enough small. In an anological way , the values are estimated from above $|R_2(x)|$, $|R_3(x)|$, $|R_4(x)|$,$|R_5(x)|$,$|R_6(x)|$. Thus Theorem 1 is proved.

7 Conclusion

The article presents the convolution generated by the Dirichlet problem for the operator of twofold differentiation. This convolution makes it possible to approximate nonlinear expressions depending on two continuous functions. The accuracy of the approximation depends on the length of the segment on which these two functions are defined. Replacing nonlinear expressions with convolution allows applying the Fourier method to nonlinear partial differential equations.

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2-бөлім **Раздел 2** **Section 2**

Механика **Механика** **Mechanics**

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Non-local mathematical models for aggregation processes in dispersive media

Particles aggregation is widespread in different technological processes and nature, and there are many approaches to modeling this phenomenon. However, the time non-locality effects with witch these processes are often accompanied leave to be none well elaborated at present. This problem is justified especially in reference to nano-technological processes. The paper is devoted to the non-local modification of Smoluchowski equation that is the key point for describing influence of synchrony and asynchrony delays in aggregation processes for clusters of different orders. The main scientific contribution consists in deriving the non-linear wave equation describing the evolution of different orders clusters concentration under aggregation processes in polydispersed systems with following for the mentioned non-locality. The practical significance lies in the fact that the results obtained can serve as the basis for the engineering calculation of the kinetics of aggregation in polydisperse nano-systems. The research methodology is based on mathematical modeling with the help of the relaxation transfer kernels approach. Succeeding analysis of aggregation processes on the base of submitted ideology can be directed to generalizing master equations with allowing for space non-locality too. The submitted approach opens up fresh opportunities for detailed study of influence of relaxation times hierarchy on the intensity of aggregation and gelation processes in non-crystalline media containing dispersed solid phase.

Key words: aggregation, dispersive systems, non-local model, kinetic equation, relaxation times.

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Дисперсті ортадағы агрегация процестерінің локальды емес математикалық модельдері

Бөлшектердің агрегациясы әртүрлі технологиялық процестер мен табиғатта кең таралған және бұл құбылысты модельдеудің көптеген тәсілдері бар. Алайда, бұл процестер жиі жүретін уақытқа жергілікті емес әсері қазіргі уақытта жеткіліксіз зерттелген. Бұл мәселе әсіресе нанотехнологиялық процестерге қатысты негізделген. Мақала Смолуховский теңдеуінің локалды емес модификациясына арналған, ол әр түрлі ретті кластерлер үшін агрегация процестеріндегі синхрондылық пен асинхрондылықтың кідірістерінің әсерін сипаттаудың кілттік мезеті болып табылады. Негізгі ғылыми үлес - бұл жергілікті емес жағдайды ескере отырып, полидисперсті жүйелердегі агрегация процестеріндегі әртүрлі тапсырыс кластерлерінің шоғырлану эволюциясын сипаттайтын сызықты емес толқындық теңдеуді шығару. Практикалық маңыздылығы - алынған нәтижелер полидисперсті наносистемалардағы агрегация кинетикасын инженерлік есептеу үшін негіз бола алады. Зерттеу әдістемесі релаксация ядросының тәсілін қолдана отырып, математикалық модельдеуге негізделген. Ұсынылған идеологияға негізделген агрегаттау процестерін кейінгі талдау кеңістіктің жергілікті еместігін ескере отырып, негізгі теңдеулерді жалпылауға бағытталуы мүмкін. Ұсынылған тәсіл релаксация иерархиясының дисперсті қатты фазасы бар кристалды емес ортадағы агрегация және гель түзілу процестерінің қарқындылығына әсерін егжей-тегжейлі зерттеуге жаңа мүмкіндіктер ашады.

Түйін сөздер: агрегация, дисперсиялық жүйелер, локальды емес модель, кинетикалық теңдеу, релаксация уақыты.

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Нелокальные математические модели процессов агрегации в дисперсивных средах

Агрегация частиц широко распространена в различных технологических процессах и природе, и существует множество подходов к моделированию этого явления. Однако эффекты нелокальности во времени, с которыми часто сопровождаются эти процессы, в настоящее время недостаточно проработаны. Эта проблема оправдана, особенно применительно к нанотехнологическим процессам. Статья посвящена нелокальной модификации уравнения Смолуховского, которая является ключевым моментом для описания влияния задержек синхронности и асинхронности в процессах агрегации для кластеров разного порядка. Основной научный вклад заключается в выводе нелинейного волнового уравнения, описывающего эволюцию концентрации кластеров различных порядков при процессах агрегации в полидисперсных системах с учетом указанной нелокальности. Практическая значимость заключается в том, что полученные результаты могут послужить основой для инженерного расчета кинетики агрегации в полидисперсных наносистемах. Методология исследования основана на математическом моделировании с помощью подхода ядер переноса релаксации. Последующий анализ процессов агрегирования на основе представленной идеологии может быть направлен на обобщение основных уравнений с учетом также нелокальности пространства. Представленный подход открывает новые возможности для детального изучения влияния иерархии времен релаксации на интенсивность процессов агрегации и гелеобразования в некристаллических средах, содержащих дисперсную твердую фазу.

Ключевые слова: агрегация, дисперсионные системы, нелокальная модель, кинетическое уравнение, времена релаксации.

1 Introduction and problem set up

Particles aggregation is widespread in different chemical technological processes, metallurgy pharmaceutical industry. Because of that many approaches to modeling this phenomenon are offered [1]. However, a lot of key issues in the aggregation processes description leave to be none elaborated up to day [2, 3, 4]. One of the important and pracically non elaborated problems is time non-locality of aggregation processes [5, 6, 7]. However, it is impossible to describe the influence of characteristic relaxation times on aggregates formation kinetics without allowing for the non- locality aspect [8, 9, 10]. It is justified especially in reference to nano-technological processes.

This paper is devoted to the non-local modification of Smoluchowski equation for particles aggregation kinetics. There are not discussed some especially physical problems as, for example, particles nucleation, etc. But we try to understand and to emphasize some difficulties emerging in description of the non-locality applying to the aggregation kinetic equations.

The following non-local modification of the Smoluchowski equation for aggregation in the uniform system can be offered as the principal ansatz with allowance for the general case of asynchrony delays for clusters of different orders formation:

$$\frac{dC_i}{dt}=\frac{1}{2}\sum_{j=1}^{i-1}\int_0^t\int_0^t N_{j,i-j}C_j(t_1)C_{i-j}(t_2)dt_1dt_2-\sum_{j=1}^{\infty}\int_0^t\int_0^t N_{i,j}C_i(t_1)C_j(t_2)dt_1dt_2 \tag{1}$$

C_i denotes the concentration of i-mers, and aggregation kernels $N_{i,j}$ are functions of the delay times $t-t_1$ and $t-t_2$.

Form (1) follows from our detail consideration of relaxation kernels method applying to heat and mass transfer problems [2].

The general linearized model for the aggregation matrix can be obtained from the model equation for transfer kernels which is submitted in [3, 4]:

$$r_i\frac{\partial N_{i,j}}{\partial s_i}+r_j\frac{\partial N_{i,j}}{\partial s_j}+\frac{\partial f^0_{i,j}}{\tau_{i,j}}N_{i,j}=0 \tag{2}$$

where $s_i=t-t_1, \qquad s_j=t-t_2$.

In equation (2) the coefficients r_i on a level with relaxation time $\tau_{i,j}$ play a part of control parameters of clusters "inertness", the parameter f answers for media and particles characteristics.

Thus the aggregation matrix, satisfying equation (2) and coming up to the condition of fast relaxation in time $t >> \tau_{i,j}$, can be written as

$$N_{i,j}=\eta^0_{i,j}exp\Big(-\frac{f^0_{i,j}}{2\tau_{i,j}}\left(\frac{s_i}{r_i}+\frac{s_j}{r_j}\right)\Big). \tag{3}$$

The master equation reads

$$\frac{dC_i}{dt}=\frac{1}{2}\sum_1\eta_{j,i-j}exp(-(g^{(j)}_{j,i-j}+g^{(i-j)}_{j,i-j}t)I_1I_2-\sum_2\eta_{i,j}exp(-(g^{(i)}_{i,j}+g^{(j)}_{i,j}t)I_3I_4 \tag{4}$$

Here

$$g^{(i)}_{m,n}=\frac{a_{m,n}}{2r_i};\qquad g^{(j)}_{m,n}=\frac{a_{m,n}}{2r_j};\qquad a_{m,n}=\frac{f^0_{m,n}}{\tau_{m,n}}$$

$$I_1=\int_0^t exp(g^{(i-j)}_{j,i-j}s)C_{i-j}(s)ds;\qquad I_2=\int_0^t exp(g^{(j)}_{j,i-j}s)C_j(s)ds;$$

$$I_3=\int_0^t exp(g^{(j)}_{i,j}s)C_j(s)ds;\qquad I_4=\int_0^t exp(g^{(i)}_{i,j}s)C_i(s)ds.$$

At the present time the clear way to rigorous reducing general governing equation (4) to an ODE form is unknown [8, 11].

So, let's assume at the beginning $r_i=r_j=1$ and $\frac{f^0_{i,j}}{\tau_{i,j}}\equiv a_{i,j}=a=const.$

Thus we have

$$\frac{dC_i}{dt}=\frac{1}{2}exp(-at)\sum_1\eta_{j,i-j}I_1I_2-exp(-at)I_3\sum_2\eta_{i,j}I_2. \tag{5}$$

Here $\sum_1$ means $\sum_{j=1}^{i-1}$; $\sum_2$ means $\sum_{j=1}^{\infty}$; $I_1 = \int_0^t exp(as/2)C_{i-j}(s)ds$;

$$I_2 = \int_0^t exp(as/2)C_j(s)ds; \quad I_3 = \int_0^t exp(as/2)C_i(s)ds$$

.

The main contribution of this work lies in that the features of the influence of the nonlocality on the aggregation process in cases of synchronous and asynchronous delays in the formation of different orders clusters have been highlighted and separately considered.

2 Materials and methods

2.1 Asymptotic analysis for asynchrony case

In order to simplify the problem it is supposed that for small relaxation times the Laplace method in the neighborhood of the time point t can be used. Immediate substitution of the integrals expansions into equation (5) requires multiplying asymptotic sequences. Such procedure is badly conditioned, as it may lead to utter loss while checking orders of approximation.

Therefore, we rearrange the equations to the form which is free from a product of integrals:

$$\frac{d^2C_i}{dt^2} + a\frac{dC_i}{dt} = \frac{1}{2}exp\left(-\frac{at}{2}\right)\sum_1 \eta_{j,i-j}(C_jI_1 + C_{i-j}I_2) -$$
$$-exp\left(-\frac{at}{2}\right)\left[C_i\sum_2 \eta_{i,j}I_2 + I_3\sum_2 \eta_{i,j}C_j\right]. \tag{6}$$

Using then Laplace method we obtain the asymptotic relations in which the orders of equations and approximations are concerted:

$$I_1^{(1)} = \frac{2}{a}\left[exp\left(\frac{at}{2}\right)C_{i-j}(t) - C_{i-j}(0)\right] - \frac{4}{a^2}\left[exp\left(\frac{at}{2}\right)\frac{dC_{i-j}}{dt} - \frac{dC_{i-j}(0)}{dt}\right], \tag{7}$$

$$I_2^{(1)} = \frac{2}{a}\left[exp\left(\frac{at}{2}\right)C_j(t) - C_j(0)\right] - \frac{4}{a^2}\left[exp\left(\frac{at}{2}\right)\frac{dC_j}{dt} - \frac{dC_j(0)}{dt}\right], \tag{8}$$

$$I_3^{(1)} = \frac{2}{a}\left[exp\left(\frac{at}{2}\right)C_i(t) - C_i(0)\right] - \frac{4}{a^2}\left[exp\left(\frac{at}{2}\right)\frac{dC_i}{dt} - \frac{dC_i(0)}{dt}\right]. \tag{9}$$

As a result, we get

$$\frac{d^2C_i}{dt^2} + a\frac{dC_i}{dt} = \frac{2}{a}\sum_1 \eta_{j,i-j}\left[C_jC_{i-j} - \frac{1}{a}\frac{d}{dt}(C_jC_{i-j})\right] - \frac{4}{a}\sum_2 \eta_{i,j}\left[C_iC_j - \frac{1}{a}\frac{d}{dt}(C_iC_j)\right] -$$
$$-\frac{1}{a}exp(-\frac{at}{2})\sum_1 \eta_{j,i-j}\left[C_j\left(C_{i-j}(0) - \frac{2}{a}\frac{dC_{i-j}(0)}{dt}\right) + C_{i-j}\left(C_j(0) - \frac{2}{a}\frac{dC_j(0)}{dt}\right)\right] +$$
$$+\frac{2}{a}exp(-\frac{at}{2})\sum_2 \eta_{i,j}\left[C_i\left(C_j(0) - \frac{2}{a}\frac{dC_j(0)}{dt}\right) + C_j\left(C_i(0) - \frac{2}{a}\frac{dC_i(0)}{dt}\right)\right] \tag{10}$$

Let's assume $\frac{1}{a} = \tau_*$. Parameter τ_* has a time dimension. So, let T be the characteristic time of the process.

Introducing the small parameter $\varepsilon = \tau_*/T$ we can pass to the dimensionless time $\theta = t/T$ and dimensionless aggregation kernels $\overline{\eta}_{i,j} = T^3\eta_{i,j}$.

Thus equation (10) can be rearranged to the following form:

$$\varepsilon\frac{d^2C_i}{d\theta^2} + \frac{dC_i}{d\theta} = 2\varepsilon^2\sum_1 \overline{\eta}_{j,i-j}\left[C_jC_{i-j} - \varepsilon\frac{d}{d\theta}(C_jC_{i-j})\right] - 4\varepsilon^2\sum_2 \overline{\eta}_{i,j}\left[C_iC_j - \varepsilon\frac{d}{d\theta}(C_iC_j)\right]$$

$$-\varepsilon^2 exp\left(-\frac{\theta}{2\varepsilon}\right)\sum_1 \overline{\eta}_{j,i-j}\left[C_j\left(C_{i-j}(0) - 2\varepsilon\frac{dC_{i-j}(0)}{d\theta}\right) + C_{i-j}\left(C_j(0) - \varepsilon\frac{dC_j(0)}{d\theta}\right)\right] + \tag{11}$$

$$+2\varepsilon^2 exp(-\frac{\theta}{2\varepsilon})\sum_2 \overline{\eta}_{i,j}\left[C_i\left(C_j(0) - 2\varepsilon\frac{dC_j(0)}{d\theta}\right) - C_j\left(C_i(0) - 2\varepsilon\frac{dC_i(0)}{d\theta}\right)\right]$$

Ignoring the fast decreasing terms at time $t >> \tau_*$ we obtain the reducing form of master equation

$$\varepsilon\frac{d^2C_i}{d\theta^2} + \frac{dC_i}{d\theta} = 2\varepsilon^2\sum_1 \overline{\eta}_{j,i-j}\left[C_jC_{i-j} - \varepsilon\frac{d}{d\theta}(C_jC_{i-j})\right] - 4\varepsilon^2\sum_2 \overline{\eta}_{i,j}\left[C_iC_j - \varepsilon\frac{d}{d\theta}(C_iC_j)\right]. \tag{12}$$

Essential difference between solutions of equations (11) and (12) may be observed at the initial period $\Delta\theta_{in}$:

$$\Delta\theta_{in} \sim -\varepsilon ln\varepsilon. \tag{13}$$

Depending on the specific correlation between values of the relaxation time and aggregation kernels we consider three different types of the aggregation process. They are the fast, moderate and slow aggregation:

$$\eta_{i,j} = O(1/\tau_*^2), \qquad \eta_{i,j} = O(1/\tau_*), \qquad \eta_{i,j} = O(1). \tag{14}$$

In any case, singularly perturbed kinetic equations should be obtained.

It's obvious that equation (12) can be reduced to the well-known Smoluchowski equation on the zero-approximation only in the case of fast aggregation.

3 Modified third order model for the synchronic delays case

Let us consider a modification of the Smoluchowski equation taking into account the synchronous time lag of the aggregation of clusters of different orders, which is designed to describe the effect of the characteristic time of aggregate formation on the kinetics of the process [2, 10].

Then the following nonlocal modification of the Smoluchowski equation is proposed for the aggregation process in a polydisperse system [10, 12]:

$$\frac{\partial C_i}{\partial t} = \frac{1}{2}\sum_{j=1}^{i-1}\int dt_1\Phi_{i-j,j}(t,t_1)C_{i-j}(t_1)C_j(t_1) - \sum_{j=1}^{\infty}\int dt_1\Phi_{i,j}(t,t_1)C_i(t_1)C_j(t_1). \tag{15}$$

Model equations for the elements of the coagulation matrix are as follows [10]:

$$\frac{\partial}{\partial}\Phi_{i,j}+\frac{\Phi_{i,j}}{\tau_{i,j}}f^0_{i,j}=0. \tag{16}$$

Then the integro-differential equations take the form:

$$\frac{\partial C_i}{\partial t}=\frac{1}{2}\sum_{j=1}^{i-1}\int dt_1\Phi^0_{i-j,j}exp\left(-\frac{f^0_{i-j,j}}{\tau_{i-j,j}}(t-t_1)\right)C_{i-j}(t_1)C_j(t_1)-$$
$$-\sum_{j=1}^{\infty}\int dt_1\Phi^0_{i,j}(t,t_1)exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t-t_1)\right)C_i(t_1)C_j(t_1). \tag{17}$$

The time derivatives of the integral terms have the form

$$\Phi^0_{i,j}C_i(t)C_j(t)-\frac{f^0_{i,j}}{\tau_{i,j}}\Phi^0_{i,j}\int_0^t dt_1C_i(t_1)C_j(t_1)exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t-t_1)\right). \tag{18}$$

Then the governing equation can be transformed to the form:

$$\frac{d^2C_i}{dt^2}=\frac{1}{2}\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)-\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t)-$$
$$-\frac{1}{2}\sum_{j=1}^{i-1}\frac{f^0_{i-j,j}}{\tau_{i-j,j}}\int dt_1\Phi^0_{i-j,j}exp\left(-\frac{f^0_{i-j,j}}{\tau_{i-j,j}}(t-t_1)\right)C_{i-j}(t_1)C_j(t_1)+ \tag{19}$$
$$+\sum_{j=1}^{\infty}\frac{f^0_{i,j}}{\tau_{i,j}}\int dt_1\Phi^0_{i,j}exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t-t_1)\right)C_i(t_1)C_j(t_1).$$

Taking the time derivative again the following equation can be derived:

$$\frac{d^3C_i}{dt^3}=\frac{d}{dt}\left(\frac{1}{2}\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)-\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t)\right)-\frac{1}{2}\sum_{j=1}^{i-1}\frac{f^0_{i-j,j}}{\tau_{i-j,j}}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)+$$
$$+\frac{1}{2}\sum_{j=1}^{i-1}\left(\frac{f^0_{i-j,j}}{\tau_{i-j,j}}\right)^2\int dt_1\Phi^0_{i-j,j}exp\left(-\frac{f^0_{i-j,j}}{\tau_{i-j,j}}(t-t_1)\right)C_{i-j}(t_1)C_j(t_1)+ \tag{20}$$
$$+\sum_{j=1}^{\infty}\frac{f^0_{i,j}}{\tau_{i,j}}\Phi^0_{i,j}C_i(t_1)C_j(t_1)-\sum_{j=1}^{\infty}\left(\frac{f^0_{i,j}}{\tau_{i,j}}\right)^2\int dt_1\Phi^0_{i,j}exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t-t_1)\right)C_i(t_1)C_j(t_1).$$

Performing a separate averaging over the groups of indices for the terms describing the formation and destruction i- measures, the following system has been obtained

$$\frac{d^3C_i}{dt^3}=\frac{d}{dt}\left(\frac{1}{2}\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)-\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t)\right)-\frac{1}{2}A_1\sum_{j=1}^{i-j}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)+$$

$$+\frac{1}{2}B_1^2\sum_{j=1}^{i-1}\int dt_1\Phi^0_{i-j,j}exp\left(-\frac{f^0_{i-j,j}}{\tau_{i-j,j}}(t-t_1)\right)C_{i-j}(t_1)C_j(t_1)+ \qquad (21)$$

$$+A_2\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t)-B_2^2\sum_{j=1}^{\infty}\int dt_1\Phi^0_{i,j}exp\left(-\frac{f^0_{i,j}}{\tau_{i,j}}(t-t_1)\right)C_i(t_1)C_j(t_1).$$

After transformations, a more compact form of the system has been obtained

$$\frac{d^3C_i}{dt^3}+(B_1+B_2)\frac{d^2C_i}{dt^2}+B_1B_2\frac{dC_i}{dt}=$$

$$=\left(B_1+B_2+\frac{d}{dt}\right)\left(\frac{1}{2}\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)-\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t)\right)- \qquad (22)$$

$$-\frac{1}{2}A_1\sum_{j=1}^{i-1}\Phi^0_{i-j,j}C_{i-j}(t)C_j(t)+A_2\sum_{j=1}^{\infty}\Phi^0_{i,j}C_i(t)C_j(t).$$

A feature of the obtained equation (22) is the presence of solutions describing the propagation of perturbations with a finite velocity in the form of single waves [8].

At the same time, the analysis of the obtained equation shows that for small values of the parameter τ/τ_c, the use of the local form of the Smoluchowski equations with aggregation matrices obeying equations of the form (3) is quite correct, since the correction to the local form has no less than the second order of smallness [10].

4 Conclusion

In this paper a brief introduction to the problem of time nonlocality applying to aggregation process kinetics has been submitted. The main result is that the relaxation kernels approach may be advantageous for deriving master equations with accounting of hierarchy of relaxation times.

Comparing the obtained equations with equations submitted in [8] we notice that account of different time delays for clusters of different orders essentially changes the form of kinetic equations. This circumstance can especially show itself at the initial time when the master equation must be considered in extended form (11). Need for the information about derivatives of clusters production at the initial time manifests, in our opinion, more profound physical content of the submitted model. Of course, this information is out of the competence of the model as such. A separate description of synchronous and asynchronous delays in the formation of clusters of different orders has shown significant differences in the fundamental models of the kinetics of aggregation processes in these situations.

Succeeding analysis of aggregation processes on the base of submitted ideology can be directed to generalizing master equations with allowing for space non-locality too. In our opinion the submitted approach opens up fresh opportunities for detailed study of influence of relaxation times hierarchy on the intensity of aggregation and gelation processes in non-crystalline media containing dispersed solid phase. This opens up new possibilities not only for calculating kinetics, but also for developing new approaches to controlling fine technological processes.

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AUTOMATION OF OBTAINING VINYL ACETATE IN A MICROREACTOR

Due to the lack of domestic manufacturers, the production of a microreactor with a control system for chemical processes will remain relevant. The reason for this is the demand for automated solutions and, in general, automation and research into the optimal conditions for promising chemical processes in companies and in production. Moreover, the microreactor proposed with a chemical process control system will be able to save money and time on the development of effective chemical processes for the production of a wide variety of substances. In addition, the project provides for the use of composite and polymer materials for work, which in turn will reduce the cost of manufactured products, and at the same time increase the competitive attractiveness. The aim is to develop a microreactor and a control system for chemical processes. This goal is achieved by justifying the choice of the direction of research, analysis of existing equipment for carrying out microreactor synthesis, application for their creation lightweight, composite and other technical materials, as well as through the development of technology for creating microreactor equipment using 3D printing, milling and engraving of light metals, composite and polymer materials. The article presents microreactors and the development of a microreactor for the production of vinyl acetate, a detailed description of the methodology for the development of this complex device, including a microreactor, a SCARA type robot and a control unit. Methods for the production of vinyl acetate and the possibility of automating this process with a complex device were also studied.
Key words: automation, microreactor, SCARA type robot, vinyl acetate.

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Винил ацетатын микрореакторда алуды автоматтацизациялау

Отандық өндірушілердің болмауына байланысты химиялық процестерді басқару жүйесі бар микрореактор өндірісі өзекті болып қала береді. Мұның себебі – автоматтандырылған шешімдерге сұраныс және жалпы алғанда, автоматтандыру және кәсіпорындардағы және өндірістегі перспективалы химиялық процестердің оңтайлы шарттарын зерттеу. Сонымен қатар, химиялық процестерді басқару жүйесімен ұсынылған микрореактор әртүрлі заттарды алу үшін тиімді химиялық процестерді әзірлеуге ақша мен уақытты үнемдеуге мүмкіндік береді. Сонымен қатар, жоба жұмыс үшін композиттік және полимерлі материалдарды пайдалануды қарастырады, бұл өз кезегінде өндірілген өнімнің өзіндік құнын төмендетуге, сонымен қатар бәсекеге қабілеттілікті арттыруға мүмкіндік береді. Мақсат – микрореактор мен химиялық процестерді басқару жүйесін жасау. Бұл мақсат зерттеу бағытын таңдауды негіздеу, микрореакторлық синтезді жүргізу үшін қолданыстағы жабдықты талдау, оларды жеңіл, композициялық және басқа да техникалық материалдарды жасауға қолдану, сондай-ақ 3D көмегімен микрореакторлық жабдықты құру технологиясын әзірлеу арқылы қол жеткізіледі. жеңіл металдарды, композициялық және полимерлі материалдарды басып шығару, фрезерлеу және ою.

Мақалада микрореакторлар мен винилацетат өндірісіне арналған микрореактордың дамуы, микрореакторды, SCARA типті роботын және оның басқару блогын қоса алғанда, осы күрделі құрылғыны жасау әдістемесінің толық сипаттамасы талқыланады. Сондай-ақ, винилацетат алу әдістері және осы процесті күрделі құрылғымен автоматтандыру мүмкіндігі зерттелді.
Түйін сөздер: автоматика, микрореактор, типті SCARA робот, винилацетат.

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Автоматизация получения винилацетата в микрореакторе

Из-за отсутствия отечественных производителей производство микрореактора с системой управления химическими процессами останется актуальным. Причина этого - потребность в автоматизированных решениях и в целом автоматизации и исследованиях оптимальных условий для перспективных химических процессов на предприятиях и на производстве. Более того, предложенный микрореактор с системой управления химическим процессом позволит сэкономить деньги и время на разработке эффективных химических процессов для производства самых разных веществ. Кроме того, проектом предусмотрено использование в работе композиционных и полимерных материалов, что, в свою очередь, снизит стоимость производимой продукции и одновременно повысит конкурентоспособность. Целью является разработка микрореактора и системы управления химическими процессами. Данная цель достигается за счет обоснования выбора направления исследований, анализа существующего оборудования для проведения микрореакторного синтеза, применения для их создания легких, композиционных и других технических материалов, а также за счет разработки технологии создания микрореакторного оборудования с использованием 3D. печать, фрезерование и гравировка легких металлов, композитных и полимерных материалов. В статье рассматриваются микрореакторы и разработка микрореактора для получения винилацетата, подробное описание методики разработки данного комплексного прибора, включающего микрореактор, робот типа SCARA и блок управления. Также были изучены методы получения винилацетата и возможность автоматизации данного процесса комплексным прибором.

Ключевые слова: автоматизация, миниатюризация, микрореактор, робот типа SCARA.

1 Introduction

Markets demand products that are smart, feature-rich, communicative, clean, safe, portable, lightweight and self-contained. The production of these products requires miniaturization of components and systems, which can be accomplished using microelectronic technologies developed in recent decades. This allowed not only to reduce the size of components and sensors, but also to increase their density in integrated circuits.

Microsystems are typically less than a millimeter in size. They can be produced using manufacturing technologies related to microelectronics. In this case, it is possible to combine sensors, actuators, and microelectronic components. This technology has contributed to the development of microreactors for controlling chemical reactions.

Leading companies in the world, for example, Lonza, DSM, Sigma Aldrich, Bayer, Astra Zeneca, Novartis, Eli Lilly, GlaxoSmithKline, Pfizer, MSD and research institutes: IMM – Institute of Microtechnology Mainz / Institute of Chemical Technology (Germany), TNO – Government Institute of Applied Research (Holland), MIT – Massachusetts Institute of Technology (USA) is already implementing all these solutions in industry, primarily at the pilot and semi-industrial level, less often at the industrial level [1].

Microreactors have provided a better understanding of the effects of miniaturization on flows and transport phenomena in the chemical industry.

Microreactor technology currently represents a serious alternative to conventional macroscopic production. Microreactors are reactor systems that include structures for the transfer or containment of a gas or liquid, in which at least one size is measured in micrometers and does not exceed 1 mm [2].

For the design of microreactors, various technologies are used: lithography, electroplating, casting, etc. Silicon, quartz, polymers, metals are used as structural materials [3].

Microreactor synthesis of active substances belongs to modern breakthrough technologies of chemical synthesis, which allow the production of complex substances with much lower operating costs. When using micro-reactor technologies, it is possible to provide the following technological advantages: guaranteed process safety, energy efficiency, compliance with regulatory standards, modular design, the possibility of its accelerated scaling, reproducibility of the technological process, compactness and high selectivity [4].

Moreover, microreactor technologies are able to reduce the costs of implementing the synthesis process, since they do not require huge premises and a large staff of specialists. At the same time, the use of a microreactor allows providing all the necessary conditions for the correct course of chemical processes, continuous operation under full automatic control of all parameters [5].

Next, we will consider microreactors of different generations for modeling chemical reactions. Currently, 3 generations of microreactors are known. These are microreactors of the first generation (1G), microreactors of the second generation (2G), as well as microreactors of the third generation (3G).

2 First generation microreactors (1G)

The first generation microreactors made it possible to measure frictional pressure loss. Unfortunately, the asymmetry of the drilled hole in the glass relative to the surface of the well was too great to provide a good seal at the fluid-microreactor junction.

The first generation of microreactors is made from silicon and glass. For this, a 2 mm high drilled glass plate must be coated with etched silicon one millimeter thick (Figure 1) [6].

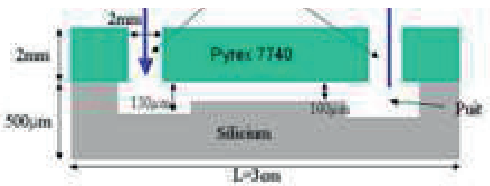

Figure 1: Superposition of silicon and glass plates on top of each other (in the figure, the arrows indicate the entrance and exit).

Next, the second generation microreactors will be presented. (Стекло7740-glass, silicon).

3 Microreactors of the second generation 2G

Second generation microreactors are made from polydimethylsiloxane. The manufacturing process for second generation microreactors includes 3 stages:

- in the first stage, a layer containing channels is obtained by pouring polydimethylsiloxane into a mold (20 g of polymer makes a layer 1.5 mm thick after application to a silicon wafer with a diameter of 4 inches);
- in the second stage, holes with a diameter of 2 mm are drilled in this layer using a punch, at the level of the tanks;
- in the third stage, the sealing plate is made by casting polydimethylsiloxane on a solid silicon substrate without a pattern.

The two plates (stages 1 and 2) are then brought into contact after being exposed to oxygen plasma for 20 seconds at a plasma power of 200 watts (Figure 2).

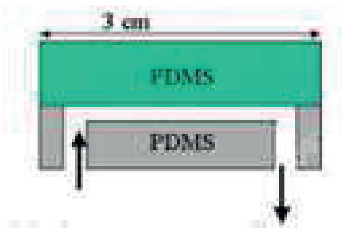

Figure 2: Microreactor in the second generation (PDMS-polydimethylsiloxane).

Next, consider the third generation microreactors.

4 Microreactors of the third generation 3G

Microreactors of the third generation are made of silicon and glass with features of liquid access from the silicon side. These microreactors have exactly the same geometry as the 2G microreactors [7].

Figure 3: Microreactor of the third generation 3G.

Next, we will consider a technique for creating a complex device for vinyl acetate production.

5 Method of creating a complex device for vinyl acetate production

To create a complex device, it is necessary to design and assemble a microreactor, SCARA robot, install a microreactor and SCARA type robot control system, connect with them peristaltic pumps for pumping liquids and gases, a refrigeration device and a thermostat for cooling and temperature control in the microreactor. Further in Figure 4, a diagram of a microreactor and a SCARA type robot in a complex will be presented.

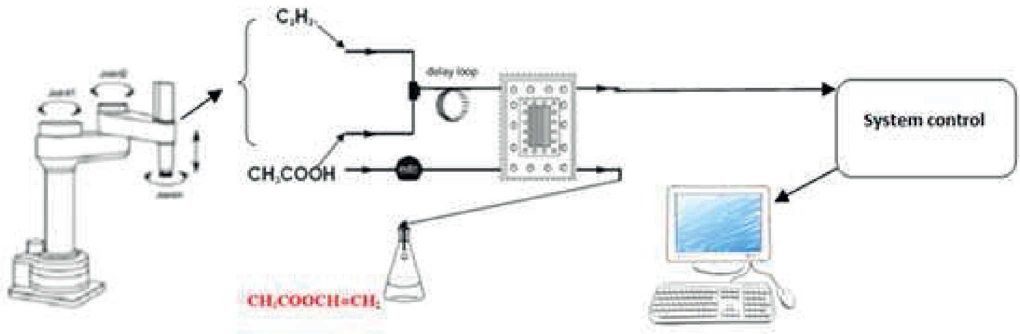

Figure 4: Scheme of a microreactor and a SCARA type robot in a complex.

6 Microreactor design technique

It is planned to create two entrances in the microreactor being designed. Acetylene and acetic acid will flow through these outlets [8]. The material for creating a microreactor is aluminum. Aluminum is a material that is resistant to corrosion and acids. For the manufacture of the microreactor body, polymeric materials will be used, for example, acrylonitrile / butadiene / styrene copolymers and polylactide. For the manufacture of the microreactor, a diagram of its body was created using the Autocad program. This diagram is shown below in Figure 5.

To create a prototype, the following components were used, which are presented in Table 1:

Next, the SCARA type robot will be discussed.

7 SCARA type robot - an integral part of a complex device

3D EXPERIECE Solidworks software was used to design the type SCARA robot. The following is a 3D model of a SCARA type robot (Figure 6).

The robot has 4 degrees of freedom and is driven by four stepper motors. In addition, it includes a servo motor for end-grip control. To create the robot, most of the body parts were designed in Solidworks and will be printed on a 3D printer. The SCARA robot and microreactor will be monitored by a microcontroller.

To control the SCARA type robot, a graphical user interface was created, in which there is control of forward and reverse kinematics. With forward kinematics, each joint of the robot

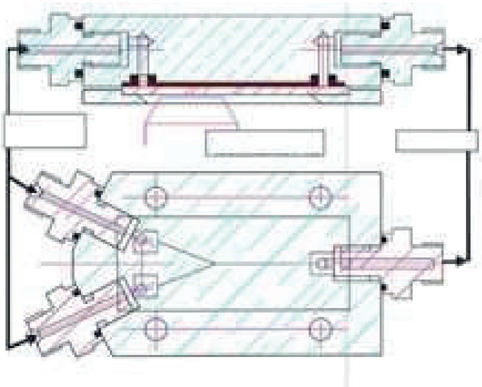

Figure 5: Diagram of the microreactor housing.

Table 1: Name of elements and materials for the microreactor

No	Name of equipment, device, elements and materials of the microreactor	Appointment of equipment, device, inventory	Parameters
1	refrigeration device, thermoelectric in the form of a Peltier plate element	for cooling the microreactor	4-stage multistage refrigeration unit
2	digital temperature controller	for temperature control	12 B, 24 B
3	aluminum heatsink with tight teeth	water cooling system	The dense teeth aluminum radiator has a bottom plate thickness: 4.6 mm and tooth thickness: 1.0 mm and the number of teeth: 27 tablets
4	transparent silicone tube	for bay from a probe	transparent flexible silicone tube size 0.5mm x 1mm non-toxic
5	peristaltic pump	for automatic dosing pump	cylinder automatic titration pump
6	tripod	for collecting probes for sending analyzes	tripod made of material PLA
7	sample collection container	to collect data	container for collecting material

can be manually moved to obtain the desired position [9]. Using the sliders on the left side, you can set the angle of each joint. The final position of the end gripper, the value of its X, Y and Z positions are calculated and printed [10]. On the other hand, using inverse kinematics, you can set the desired position, and the program will automatically calculate the angles for each joint: in order for the robot to get to the desired position [11]. The joint angles and their X, Y and Z values of the end clamp are linked and always present on the screen.

Figure 6: 3D model of the SCARA type robot.

8 Vinyl acetate production method

Vinyl acetate - obtained in a microreactor can serve as a raw material for the production of polyvinyl acetate. Vinyl acetate can be obtained in the process of catalytic vapor-phase synthesis based on the addition of acetylene and acetic acid:

$$2H2 + CH3COOH \rightarrow CH2 = CHOCOCH3$$

Zinc acetate supported on active carbons characterized by the presence of macro- and mesopores can be used as a catalyst [12].

Vinyl acetate can be obtained from acetic cystola and acytelene in two ways. This is a vapor-phase and liquid-phase method. The advantages of the vapor-phase method in comparison with the liquid-phase method are: ease of design, reduced corrosion, increased conversion of both ethylene and acetic acid, and increased selectivity of the process.

8.1 Vapor-phase method of vinyl acetate production

Vapor-phase vinylation is carried out with a large excess of acetylene. The higher the molar ratio of acetylene to acetic acid, the greater the conversion of the acid in one pass through the catalyst. The highest conversion is achieved at a molar ratio of acetylene to acid from 8: 1 to 10. However, due to the difficulty of the subsequent isolation of vinyl acetate from very dilute contact gases, it is necessary to carry out with a much lower excess of acetylene (4: 1). In this case, the degree of conversion in one pass decreases and the amount of unreacted acid increases, which is separated from the contact gases and returned to the process [13].

8.2 Liquid-phase method for producing vinyl acetate

The liquid-phase process for the production of vinyl acetate is carried out at $60 - 65°C$, passing at high speed an excess of acetylene through the reactor, which contains a mixture of glacial acetic acid and acetic anhydride containing dispersed mercury salts. Vinyl acetate, as it is formed, is removed from the reaction zone in the form of vapors entrained in excess acetylene. Vapors of vinyl acetate are condensed and sent to rectification. The acetylene separated from the liquid is returned to the production cycle [14]. Next, we will consider the method of obtaining vinyl acetate.

9 Technique for producing vinyl acetate in a microreactor

To obtain vinyl acetate, you must first obtain acetylene. To do this, take 5-10 grams of calcium carbide and place it in a 250 ml flask. Calcium carbide reacts violently with water. To slow down this reaction, you must use a saturated solution of sodium chloride. We will add a few drops of sodium chloride solution to the funnel [15]. Further addition of the solution is carried out so that a uniform gas flow is established at a rate that allows the formed bubbles to be counted. The evolved gas is acetylene and the second reaction product is calcium hydroxide.

To obtain vinyl acetate in a microreactor, the resulting acetylene must be mixed with acetic acid in a ratio of $3.5 - 5 : 1$. The mixture enters the microreactor and the reaction of the combination of acetylene and acetic acid occurs. The resulting vinyl acetate goes into a container for further analysis in the laboratory.

10 Conclusion

The possibility of using a microreactor and a SCARA type robot considered in the article has shown its applicability as a complex device for the production of vinyl acetate. A detailed description of the methodology for creating this complex device was developed, including a microreactor, SCARA type robot, control unit, pumps, tubes, thermostat and refrigerator. To implement the creation of a complex device in a real environment, it is necessary to assemble several prototypes and analyze the yield of vinyl acetate in the laboratory.

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SECULAR PERTURBATIONS OF TRANSLATIONAL-ROTATIONAL MOTION IN THE NON-STATIONARY THREE-BODY PROBLEM

Modern observational data in astronomy show that real space systems are non-stationary, their masses, sizes, shapes and a few other physical characteristics change over time during evolution. In this connection, the creation of mathematical models of the motion of non-stationary celestial bodies becomes relevant. We consider a non-stationary three-body problem with axisymmetric dynamical structure, shape and variable compression. The Newtonian interaction force is characterized by an approximate expression of the force function accurate to the second harmonic. The masses of bodies change isotropically at different rates. The axes of inertia of the proper coordinate system of non-stationary axisymmetric three bodies coincide with the major axes of inertia of the bodies, and it is assumed that their relative orientations remain unchanged in the process of evolution. Differential equations of translation-rotational motion of three non-stationary axisymmetric bodies with variable masses and dimensions in the relative coordinate system, with the origin in the center of the more massive body, are presented. The analytical expression for the Newtonian force function of the interaction of three bodies with variable masses and dimensions is given. The canonical equations of translational-rotational motion of three bodies in Delaunay-Andoyer analogues are obtained. The equations of secular perturbations of translational-rotational motion of non-stationary axisymmetric three-bodies in the Delaunay-Andoyer analogues of osculating elements have been obtained. The new results obtained can be used to analyze the dynamic evolution of the translation-rotational motion of the three-body problem. The problem is investigated by methods of perturbation theory.
Key words: celestial mechanics, three-body problem, variable mass, secular perturbation, axisymmetric body, translational-rotational motion.

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Бейстационар үш дене есебінің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқулары

Астрономиядағы заманауи бақылау деректері нақты ғарыштық жүйелердің бейстационар екендігін, олардың массалары, өлшемдері, пішіні және басқа да бірқатар физикалық сипаттамалары эволюция барысында өзгеретіндігін көрсетеді. Осыған байланысты бейстационар аспан денелерінің қозғалысының математикалық модельдерін жасау актуалды мәселе болып табылады. Динамикалық құрылымы мен формасы өстік симметриялы және сығылуы ауыспалы бейстационар үш дене есебі қарастырылған. Ньютондық өзара әрекеттесу күші күш функциясының екінші гармоникаға дәл келетін жуық мәнімен өрнегімен сипатталады. Дене массалары әр түрлі жылдамдықта изотропты түрде өзгереді. Бейстационар өстік симметриялы үш дененің өзіндік координаттар жүйесінің өстері денелердің негізгі инерция өстерімен сәйкес келеді және эволюция барысында олардың салыстырмалы бағдары өзгеріссіз қалады деп есептеледі. Массалары мен өлшемдері айнымалы, бейстационар, өстік-симметриялы үш дененің ілгерілемелі-айналмалы қозғалысының дифференциалдық теңдеулері массасы үлкенірек дененің центрінен басталатын салыстырмалы координаталар жүйесінде келтірілген.

Массалары мен өлшемдері айнымалы үш дененің Ньютондық өзара әрекеттесуінің күштік функциялары үшін аналитикалық өрнек келтірілген. Үш дененің ілгерілемелі-айналмалы қозғалысының канондық теңдеулері Делоне-Андуайе оскуляциялаушы элементтерінің аналогтарында алынған. Бейстационар өстік-симметриялы үш дененің ілгерілемелі-айналмалы қозғалысының ғасырлық ұйытқу теңдеулері Делоне-Андуайе оскуляциялаушы элементтерінің аналогтарында алынған. Алынған жаңа нәтижелерді үш дене есебінің ілгерілемелі-айналмалы қозғалысының динамикалық эволюциясын талдауға пайдалануға болады. Мәселе ұйытқу теориясының әдістерімен зерттеледі.
Түйін сөздер: Аспан механикасы, үш дене есебі, айнымалы масса, ғасырлық ұйытқу, өстік симметриялық дене, ілгерілемелі-айналмалы қозғалыс.

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Вековые возмущения поступательно-вращательного движения в нестационарной задаче трех тел

Современные данные наблюдений в астрономии показывают, что реальные космические системы являются нестационарными, их массы, размеры, форма и ряд других физических характеристик изменяются с течением времени в процессе эволюции. В связи с этим, становится актуальным создание математических моделей движения нестационарных небесных тел. Рассматривается нестационарная задача трех тел, обладающих осесимметричным динамическим строением, формой и переменным сжатием. Ньютоновская сила взаимодействия характеризуется приближенным выражением силовой функции с точностью до второй гармоники. Массы тел изменяются изотропно в различных темпах. Оси инерции собственной системы координат нестационарных осесимметричных трех тел совпадают с главными осями инерции тел, и предполагается, что в ходе эволюции их относительная ориентация остаются неизменными. Приведены дифференциальные уравнения поступательно-вращательного движения трех нестационарных осесимметричных тел с переменными массами и размерами в относительной системе координат, с началом в центре более массивного тела. Приведены аналитическое выражение силовой функций ньютоновского взаимодействия трех тел с переменными массами и размерами. Получены канонические уравнения поступательно-вращательного движения трех тел в аналогах оскулирующих элементов Делоне-Андуайе. Получены уравнения вековых возмущений поступательно-вращательного движения нестационарных осесимметричных трех тел в аналогах оскулирующих элементов Делоне-Андуайе. Полученные новые результаты могут быть использованы для анализа динамической эволюции поступательно-вращательного движения задачи трех тел. Задача исследована методами теории возмущений.

Ключевые слова: Небесная механика, задача трех тел, переменная масса, вековое возмущение, осесимметричное тело, поступательно-вращательное движение.

1 Introduction

In classical celestial mechanics, real celestial bodies are considered as material points moving in absolutely empty space under the action of the forces of mutual attraction according to Newton's law of universal gravitation [1]. However, it is not always possible to be satisfied with this first approximation. In other cases, it is impossible to consider real celestial bodies as material points and we have to take into account the influence of their shape by considering them as rigid bodies.

But in reality celestial bodies are not material points (spherically symmetric bodies), but they are not, of course, absolutely rigid bodies either, but always possess a certain degree of plasticity or even are liquid (or gaseous, or dusty) formations [2]. Modern observational data in astronomy show that the real space systems are non-stationary, their masses, sizes, shapes and a number of other physical characteristics change over time during the evolution [5, 12]. In this connection, the creation of mathematical models of the motion of nonstationary celestial bodies becomes actual.

The goal of this work is to obtain differential equations of secular perturbations of the translational-rotational motion of nonstationary axisymmetric three-body dynamic structure, shape, and variable compression. The solution of this problem is associated with rather cumbersome symbolic calculations, which are best performed using computer algebra systems [9, 11].

2 Problem formulation and equations of motion in the relative coordinate system

Let us consider the motion of three non-stationary axisymmetric celestial bodies T_0, T_1, T_2 with variable masses, sizes and variable compression moving in a absolutely empty space under the action of mutual attraction forces according to Newton's law of universal gravitation.

Let the shapes of bodies T_0, T_1, T_2 are different, axisymmetric, and have their own equatorial symmetry plane. Let also assume that the compressions of the bodies with respect to the equatorial plane are variable. The initial locations of the main axes of inertia and the center of inertia in the body of axisymmetric bodies remain unchanged during evolution and are directed along the intersection of the three mutually perpendicular planes.

Let $m_i = m_i(t_0)\nu_i$ be the mass, $l_i = l_i(t_0)\chi_i$ be the characteristic linear dimension, and A_i, B_i, C_i be the second-order principal moments of inertia of the bodies T_i, t_0 be the initial time, ν_i, χ_i $(i = 0, 1, 2)$ be the dimensionless known time functions.

Let us make the following assumptions:

1. Bodies with variable masses $m_i = m_i(t)$ have equatorial symmetry planes and characteristic linear sizes $l_i = l_i(t)$. The second order moments of inertia of the considered bodies are variable

$$A_i = A_i(t), \quad B_i = B_i(t), \quad C_i = C_i(t). \tag{1}$$

2. Bodies are axisymmetric and remain axisymmetric with respect to their own equatorial symmetry planes during evolution

$$A_i(t) = B_i(t) \neq C_i(t) \tag{2}$$

3. The axes of inertia of the proper coordinate system $G_i\widetilde{\xi}_i\widetilde{\eta}_i\widetilde{\zeta}_i$ coincide with the main axes of inertia and this position in the process of evolution is saved.

4. Masses and characteristic sizes of bodies change at different specific rates

$$\frac{\dot m_0(t)}{m_0(t)} \neq \frac{\dot m_1(t)}{m_1(t)} \neq \frac{\dot m_2(t)}{m_2(t)}, \quad \frac{\dot l_0(t)}{l_0(t)} \neq \frac{\dot l_1(t)}{l_1(t)} \neq \frac{\dot l_2(t)}{l_2(t)}. \tag{3}$$

5. Let us assume that the masses of the bodies change isotropically and there are no reactive forces as well as additional rotational moments.

$$\vec{F}_{(reac)} = 0, \quad \vec{M}^{(add)} = 0 \tag{4}$$

6. In the expression for the force function, we restrict the approximation to the second harmonic inclusive.

$$U \approx U^{(0)} + U^{(2)} \tag{5}$$

If the above assumptions are satisfied, the translational motions of bodies T_1 and T_2 in the gravitational field of the "central" body T_0 in the relative coordinate system (Fig. 1) are described by the equations [1-3]:

$$\ddot{x}_i = \frac{1}{\mu_i(t)} \frac{\partial U_{i0}^{(0)}}{\partial x_i} + \frac{\partial V_i}{\partial x_i} \ddot{y}_i = \frac{1}{\mu_i(t)} \frac{\partial U_{i0}^{(0)}}{\partial y_i} + \frac{\partial V_i}{\partial y_i} \ddot{z}_i = \frac{1}{\mu_i(t)} \frac{\partial U_{i0}^{(0)}}{\partial z_i} + \frac{\partial V_i}{\partial z_i} \tag{6}$$

where x_i, y_i, z_i coordinates of the center of mass of the body T_1 and T_2 in the relative coordinate system G_0xyz with origin in the center of the body T_0, $\mu_i(t) = m_0 m_i/(m_i + m_i)$ – reduced masses, V_i – perturbing functions have the form[2]

$$V_i = \frac{1}{\mu_i} U_{i0}^{(2)} + \frac{1}{m_i} U_{ij} + \frac{1}{m_0} \left[x_i \frac{\partial U_{j0}}{\partial x_j} + y_i \frac{\partial U_{j0}}{\partial y_j} + z_i \frac{\partial U_{j0}}{\partial z_j} \right], \quad (i, j = 1, 2), \quad (i \neq j) \tag{7}$$

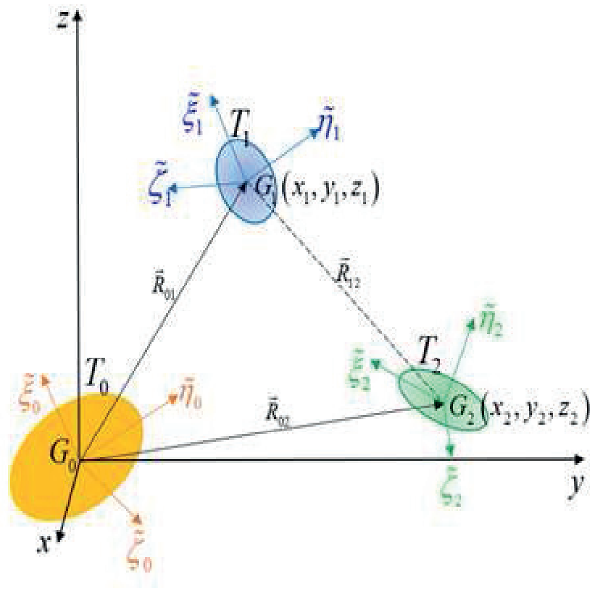

Figure 1: Bodies in a relative coordinate system G_0xyz.

$G_0\tilde{\xi}_0\tilde{\eta}_0\tilde{\zeta}_0$, $G_1\tilde{\xi}_1\tilde{\eta}_1\tilde{\zeta}_1$, $G_2\tilde{\xi}_2\tilde{\eta}_2\tilde{\zeta}_2$ – own coordinate systems.

The expression of the Newtonian force function of the interaction of three non-stationary bodies has the form [1, 2]

$$U = \frac{1}{2}\sum_{i=0}^{2}\sum_{j=0,i\neq j}^{2} U_{ij} \tag{8}$$

U_{ij} – the force function of the mutual attraction of the two bodies T_i and T_j is

$$U_{ij} \approx U_{ij}^{(0)} + U_{ij}^{(2)} \tag{9}$$

$U_{ij}^{(0)}$ – the first term of the force function decomposition is

$$U_{ij}^{(0)} = f\frac{m_i m_j}{R_{ij}} \tag{10}$$

$U_{ij}^{(2)}$ – the second term of the force function decomposition is equal to

$$U_{ij}^{(2)} = f m_i \frac{2A_j + C_j - 3I_j^{(j,i)}}{2R_{ij}^3} + f m_j \frac{2A_i + C_i - 3I_i^{(i,j)}}{2R_{ij}^3} \tag{11}$$

where $R_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$ – the mutual distance between the centers of inertia of the considered bodies, f gravitational constant, $I_i^{(i,j)}$ and$I_j^{(j,i)}$ – moments of inertia of bodies T_i and T_j with relative to vector G_iG_j, connecting the centers of mass of two bodies, is defined by the expression

$$I_i^{(i,j)} = A_i(\alpha_{ij}^2 + \beta_{ij}^2) + C_i\gamma_{ij}^2 \quad I_j^{(i,j)} = A_j(\alpha_{ji}^2 + \beta_{ji}^2) + C_j\gamma_{ji}^2 \tag{12}$$

Where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ – the directional cosines of the vector G_iG_j with the main central axes of inertia of the body T_i. The rotational motions of bodies T_0, T_1, T_2 around their own center of masses in Euler variables are described by the equations [1]

$$\begin{aligned}
&\frac{d}{dt}(A_j p_j) - (A_j - C_j)q_j r_j = \left[\frac{\partial U}{\partial \psi_j} - cos\theta_j \frac{\partial U}{\partial \varphi_j}\right]\frac{sin\varphi_j}{sin\theta_j} + cos\varphi_j \frac{\partial U}{\partial \theta_j},\\
&\frac{d}{dt}(A_j q_j) - (C_j - A_j)p_j r_j = \left[\frac{\partial U}{\partial \psi_j} - cos\theta_j \frac{\partial U}{\partial \varphi_j}\right]\frac{cos\varphi_j}{sin\theta_j} - sin\varphi_j \frac{\partial U}{\partial \theta_j},\\
&\frac{d}{dt}(C_j r_j) = \frac{\partial U}{\partial \varphi_j} \quad j = 0, 1, 2
\end{aligned} \tag{13}$$

where

$$p_j = \dot{\psi}_j \sin\theta_j \sin\varphi_j + \dot{\theta}_j \cos\varphi_j, \quad q_j = \dot{\psi}_j \sin\theta_j \cos\varphi_j - \dot{\theta}_j \sin\varphi_j, \quad r_j = \dot{\psi}_j \cos\theta_j + \varphi_j \tag{14}$$

p_j, q_j, r_j – the projections of the angular velocities of bodies T_j on the axes of their own coordinate systems $G_j\widetilde{\xi}_j\widetilde{\eta}_j\widetilde{\zeta}_j$, $\varphi_j, \psi_j, \theta_j$ – Euler angles [6].

The resulting equations (6) and (13) fully characterize the translational-rotational motion of bodies T_1 and T_2 and the rotational motion of body T_0 in the relative coordinate system G_0xyz in the considered statement.

The equations of perturbed motion in the form of Newton's equations, although they are the most general in the case when the perturbing forces admit a force function, are inconvenient. In this case the equations of perturbed motion in the form of the canonical Hamilton equations are preferable, which have, as in the classical problems, a number of advantages and elegance [5].

In the considered formulation of the problem is very complicated, so we will use the methods of perturbation theory for its investigation [1].

3 Equations of motion in osculating Delaunay-Andoyer elements

For our purposes, the canonical equations of perturbed motion in osculating analogues of Delaunay-Andoyer elements are preferable [1].

Let us consider the analogues of Delaunay-Andoyer elements.

$$L, G, H, l, g, h \quad \text{– Delaunay elements} \tag{15}$$

$$L', G', H', l', g', h' \quad \text{– Andoyerelements (see Fig. 2)} \tag{16}$$

Equations of translational motion of bodies T_1 and T_2 in osculating Delaunay elements have the form [3].

$$\dot{L}_i = \frac{\partial F_i}{\partial l_i}, \quad \dot{G}_i = \frac{\partial F_i}{\partial g_i}, \quad \dot{H}_i = \frac{\partial F_i}{\partial h_i}, \quad \dot{l}_i = -\frac{\partial F_i}{\partial L_i}, \quad \dot{g}_i = -\frac{\partial F_i}{\partial G_i}, \quad \dot{h}_i = -\frac{\partial F_i}{\partial H_i} \tag{17}$$

where

$$F_i = \frac{1}{\sigma_i^2}\frac{\mu_{0i}^2}{2L_i^2} + F_{ipert} \tag{18}$$

$$F_{ipert} = V_i - \frac{1}{2}b_i R_{i0}^2 \tag{19}$$

$$b_i = b_i(t_0) = \frac{\ddot{\sigma}_i}{\sigma_i} = (m_0 + m_i)\frac{d^2}{dt^2}\left(\frac{1}{m_0 + m_i}\right), \quad \sigma_i = \frac{m_0(t_0) + m_i(t_0)}{m_0(t) + m_i(t)}, \quad i = 1, 2 \tag{20}$$

Given the relation $\alpha_{ij}^2+\beta_{ij}^2+\gamma_{ij}^2=1$ and formulas (9) – (12), expression (19) has the form

$$F_{ipert}=\frac{1}{\mu_i}\left(fm_0(C_i-A_i)\left[\frac{1-3\gamma_{i0}^2}{2R_{i0}^3}\right]+fm_i(C_0-A_0)\left[\frac{1-3\gamma_{0i}^2}{R_{0i}^3}\right]\right)+\frac{1}{m_i}\times$$
$$\times\left(fm_im_j\left[\frac{1}{R_{ij}}\right]+fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]+fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{2R_{ji}^3}\right]\right)+\frac{1}{m_0}\times \tag{21}$$
$$\times\left(x_i\frac{\partial}{\partial x_j}+y_i\frac{\partial}{\partial y_j}+z_i\frac{\partial}{\partial z_j}\right)\begin{pmatrix} fm_0m_j\left[\frac{1}{R_{0j}}\right]+fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]+ \\ +fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right] \end{pmatrix}-\frac{1}{2}b_i[R_{i0}^2]$$

The rotational motion of an axisymmetric body ($A=B$) around its center of inertia is described in the analogues of the Andoyerosculating elements. As noted above, the axes of the own coordinate system coincide with the main central axes of inertia of the body.
In the Euler variables, the kinetic energy of rotational motions of non-stationary axisymmetric bodies has the form

$$K_j^{rot}=\frac{1}{2}(A_j(p_j^2+q_j^2)+C_jr_j^2) \tag{22}$$

On the other hand, in the Andoyervariables we get [1]:

$$A_jp_j=\sqrt{G_j'^2-L_j'^2}\sin l_j'B_jq_j=\sqrt{G_j'^2-L_j'^2}\cos l_j'C_jr_j=L_j' \tag{23}$$

Therefore, the expression for kinetic energy (22) in the Andoyervariables can generally be written as

$$K_j^{rot}=\frac{1}{2}(G_j'^2-L_j'^2)\left[\frac{\sin^2 l_j'}{A_j}+\frac{\cos^2 l_j'}{B_j}\right]+\frac{L_j'^2}{2C_j} \tag{24}$$

In the case of an axisymmetric body, expression (24) is greatly simplified

$$K_j^{rot}=\frac{1}{2A_j}(G_j'^2-L_j'^2)+\frac{L_j'^2}{2C_j} \tag{25}$$

Hence, the Hamiltonian of rotational motions of axisymmetric bodies can be written in the form

$$F_j'=\frac{1}{2}(G_j'^2-L_j'^2)\frac{1}{A_j}+\frac{L_j'^2}{2C_j}+F_{jpert}' \tag{26}$$

where

$$F_{jpert}'=U^{(2)},\quad j=0,1,2. \tag{27}$$

$$U^{(2)}=\frac{1}{2}\sum_{i=0,\ i\neq j}^{2}U_{ij}^{(2)} \tag{28}$$

Accordingly, the rotational motions of axisymmetric bodies T_0, T_1, T_2 around their own center of inertia are defined by the equations of perturbed motion in the Andoyerosculating elements of the form [3].

$$\dot{L}'_j = \frac{\partial F'_j}{\partial l'_j}, \quad \dot{G}'_j = \frac{\partial F'_j}{\partial g'_j}, \quad \dot{H}'_j = \frac{\partial F'_j}{\partial h'_j}, \quad \dot{l}'_j = -\frac{\partial F'_j}{\partial L'_j}, \quad \dot{g}'_j = -\frac{\partial F'_j}{\partial G'_j}, \quad \dot{h}'_j = -\frac{\partial F'_j}{\partial H'_j} \tag{29}$$

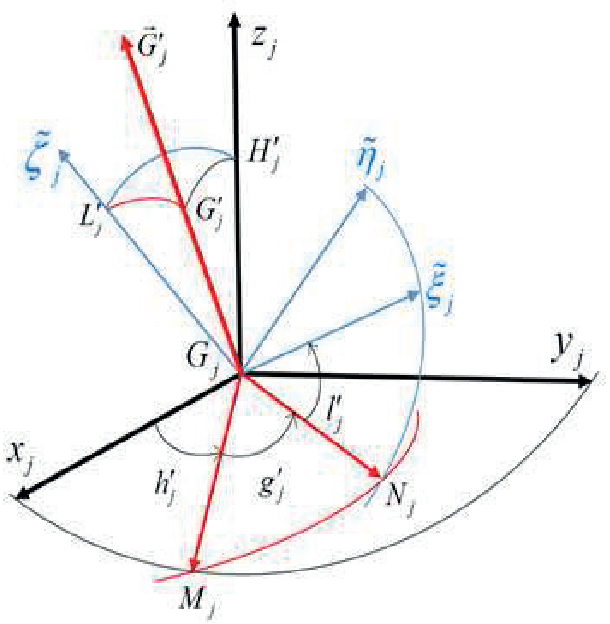

Figure 2: Andoyer variables

The geometric meaning of the analogues of the Andoyer variables are given in [10].

Given the relation $\alpha_{ij}^2 + \beta_{ij}^2 + \gamma_{ij}^2 = 1$ and formulas (9) – (12), expression (27) has the form

$$F'_{jpert} = \frac{1}{2} \sum_{i=0,\ i \neq j}^{2} \left(f m_j (C_i - A_i) \left[\frac{1 - 3\gamma_{ij}^2}{2R_{ij}^3} \right] + f m_i (C_j - A_j) \left[\frac{1 - 3\gamma_{ji}^2}{R_{ji}^3} \right] \right) \tag{30}$$

The values in square brackets in the right-hand side of equation (21) and (30) must be expressed in terms of the Delaunay-Andoyerosculating elements.

4 Differential equations of secular perturbations

Let us consider the nonresonant case. By averaging the right-hand side of equation (17) and (29) over the variables g', l, we obtain the equations for secular perturbations of the translational-rotational motion of bodies T_1 and T_2 and the rotational motion of body T_0 in the problem under consideration [10, 11]. If we denote the secular parts of the perturbing functions F_i, F'_j as $F_{i\,\mathrm{sec}}$, $F'_{j\,\mathrm{sec}}$, then according to the Gaussian scheme we have

$$F_{isec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_i dl_i dg'_i, \quad F'_{jsec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F'_j dl_j dg'_j, \quad i=1,2, \quad j=0,1,2. \tag{31}$$

Accordingly, it is possible to write

$$F'_{jsec} = \frac{1}{2A_j}(G'^2_j)_{sec} + \frac{1}{2}\left(\frac{1}{C_j}-\frac{1}{A_j}\right)(L'^2_j)_{sec} + \frac{1}{2}\sum_{i=0,i\neq j}^{2}\left(\begin{array}{c} fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \\ +fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right]_{sec}\end{array}\right) \tag{32}$$

$$\begin{aligned} F_{isec} = {} & \frac{\mu_{0i}^2}{2\sigma_i^2}\left(\frac{1}{L_i^2}\right)_{sec} + \frac{(m_0+m_i)}{m_0 m_i}\left(fm_0(C_i-A_i)\left[\frac{1-3\gamma_{i0}^2}{2R_{i0}^3}\right]_{sec} + fm_i(C_0-A_0)\times\right. \\ & \left.\times\left[\frac{1-3\gamma_{0i}^2}{R_{0i}^3}\right]_{sec}\right) + \frac{1}{m_i}\left(fm_i m_j\left[\frac{1}{R_{ij}}\right]_{sec} + fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \right. \\ & \left. + fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{2R_{ji}^3}\right]_{sec}\right) + \frac{1}{m_0}\left(x_i\frac{\partial}{\partial x_j} + y_i\frac{\partial}{\partial y_j} + z_i\frac{\partial}{\partial z_j}\right)\times \\ & \times\left(\begin{array}{l} fm_0 m_j\left[\frac{1}{R_{0j}}\right]_{sec} + fm_j(C_i-A_i)\left[\frac{1-3\gamma_{ij}^2}{2R_{ij}^3}\right]_{sec} + \\ \\ +fm_i(C_j-A_j)\left[\frac{1-3\gamma_{ji}^2}{R_{ji}^3}\right]_{sec}\end{array}\right) - \frac{1}{2}b_i[R_{i0}^2]_{sec} \end{aligned} \tag{33}$$

5 Conclusion

The translational-rotational motion of three non-stationary axisymmetric mutually gravitating bodies according to Newton's law is studied by perturbation theory methods. Equations for secular perturbations are obtained. Further it is planned to express the values in square brackets in the right part of equations (32) and (33) through the Delaunaye-Andoyer osculating elements.

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3-бөлім	Раздел 3	Section 3
Информатика	Информатика	Computer Science

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A COMPARATIVE ANALYSIS OF MODERN TRENDS IN CARSHARING: WITH REFERENCE TO KAZAKHSTAN

This article presents the results of a systematic literature review on modern trends in the implementation of car-sharing and conducts a comparative analysis of Kazakhstani carsharing solutions to some of the world's most popular alternatives. The findings reveal some socio-economic and cultural barriers for adopting and promoting carsharing services by studying the results of recent literature. It also considers business models and features of various car-sharing services and suggests a P2P(peer-to-peer) model as the main model of the proposed carsharing system, a system architecture of which is presented as well. The main contribution of this article to the research topic is analyzing existing carsharing solutions both in Kazakhstan and abroad, identifying the gaps in the development of carsharing in the country, suggesting the preferred business model, system architecture and finding directions for future research.

Key words: carsharing, sustainability, carbon emissions, shared mobility, sharing economy.

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КАРШЕРИНГТЕГІ ЗАМАНАУИ ТРЕНДТЕРДІ САЛЫСТЫРМАЛЫ ТАЛДАУ: ҚАЗАҚСТАНҒА ШОЛУ

Бұл мақалада каршерингті жүзеге асырудың ағымдағы тенденциялары туралы әдебиеттерге жүйелі шолу жасалады және қазақстандық каршеринг шешімдерінің кейбір әлемдегі ең танымал баламаларымен салыстырмалы талдауы қарастырылады. Ғылыми әдебиеттерді зерделеу арқылы алынған мәліметтер каршеринг қызметін енгізу мен ілгерілетудегі кейбір әлеуметтік-экономикалық және мәдени кедергілерді анықтады. Сондай-ақ мақалада айтылған автомобильдерді ортақ пайдаланудың әртүрлі үлгілері талданады. Бұл мақаланың берілген тақырыпты зерттеудегі негізгі үлесі Қазақстандағы каршеринг сервистерін дамытудағы олқылықтарды және болашақ зерттеулердің бағыттарын анықтау болып табылады.

Түйін сөздер: каршеринг, тұрақты даму, көміртегі айналымы, ортақ мобилділік, ортақ қолдану экономикасы.

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СРАВНИТЕЛЬНЫЙ АНАЛИЗ СОВРЕМЕННЫХ ТРЕНДОВ В КАРШЕРИНГЕ: СО ССЫЛКОЙ НА КАЗАХСТАН

В данной статье представлены результаты систематического обзора литературы о современных тенденциях в реализации каршеринга и проводится сравнительный анализ решений казахстанского каршеринга с некоторыми из самых популярных в мире альтернатив. Полученные путем изучения научной литературы данные выявили некоторые социально-экономические и культурные барьеры для внедрения и продвижения услуг каршеринга. Также статья анализирует различные бизнес-модели и особенности каршеринга и предлагает модель P2P(peer-to-peer) как наиболее эффективную для предложенной системы каршеринга, архитектура которой также предлагается. Основной вклад данной статьи заключается в выявлении пробелов в развитии каршеринга в Казахстане, путем сравнительного анализа существующих решений по каршерингу с зарубежными аналогами, предложение архитектуры и модель для системы каршеринга в Казахстане и определение направлений для будущих исследований.

Ключевые слова: каршеринг, устойчивое развитие, выбросы углерода, совместная мобильность, экономика совместного использования.

1 Introduction

Unprecedent development of automobile industry supported by population growth is bringing new environmental challenges to the society. These challenges are seen more evenly in large cities due to ever-increasing urbanization and human mobility. Nowadays urban population has limitless opportunities of overcoming large distances not only in a short period of time, but also with high level of comfort by means of different mobility solutions. This level of mobility is rather a requirement of time than a choice. Thus, to make cities more sustainable world's developed economies are now concerned about implementing the concept of Smart Cities, which, according to the literature, includes following elements: smart government, smart people, smart economy, smart transportation, environment and living"(Razmjoo and et al., 2021). Carsharing as a type of shared mobility is gaining more and more popularity and is one of the key solutions to a smart transportation and consequently to Smart cities. According to consulting agencies carsharing suggests strong new opportunities for automakers, suppliers, and many more mobility players. Consumer survey conducted by McKinsey and Company in 2017 indicate continued growth potential for shared mobility, stating that by 2030 "...of those currently using non-taxi ride-hailing services, 63 percent expect to increase their usage "a lot"in the next two years, and even more (67 percent) say they will do the same concerning car sharing"(McKinsey and Company, 2017). However, the inuence of carsharing in urban mobility is a subject of research. According to corresponding study on this topic conducted in the city of Madrid in 2018, "...it is essential to ensure that the arrival of new carsharing modes lead to more sustainable cities and complements public transport"(Ampudia-Renuncio, Guirao, Molina, 2018). Although the effect of carsharing on economies and smart transportation in general is a subject of further research, authors state that "...car- sharing is predicted to have a transformative effect on the industry. Vehicle electrication and automation have the potential to reduce GHG emissions and reduce private-vehicle ownership in favor of shared automated vehicles"(Shaheen*, Cohen, Farrar, 2019). Thus, as part of smart transportation understanding motivation, socio-economic factors and identifying areas for development of carsharing is important to implement the concept of Smart cities. This article will discuss some of the current trends in this topic referencing to the development of carsharing in Kazakhstan.

2 Methodology and design

The focus of this article is to understand the modern trends in carsharing by looking at different literature. First, definitions of carsharing were examined and differences between some of them were described. A diagram illustrating the types and models of carsharing is created in section 3 with a short description of each of the models described. The next focus of the literature review was in understanding the motivation towards adopting carsharing, socio-economic and cultural factors that affect one's choice in opting for a personal car ownership. A comparative analysis of the few popular carsharing services were examined both internationally and in Kazakhstan.

3 Carsharing definition, models, and types

Recent literature uses several terms associated with sharing mobility such as carsharing, carpooling, ride sharing etc. Although some literature uses these terms interchangeably, there is a considerable difference between the first two: (1) carsharing - is the use of cars provided by an individual, a community or a specialized company and (2) carpooling - is the joint and organized use of a car by several individuals to travel (Bulteau, Feuillet, & Dantan, 2019). Transportation research board also refers to carsharing as a service that provides members with access to a fleet of vehicles on an hourly basis (Transportation research board, 2005). Another literature refers to carsharing as "... a form of person-to-person lending or collaborative consumption in a sharing economy, where existing car owners rent their vehicles to other people for short periods of time"(Saurabh & et.al., 2021). Despite the differences, some literature considers carsharing and carpooling as "... a common trend that is transforming the automotive and transportation industries"(European Economic Comission UN , 2020). This article clearly distinguishes between traditional carsharing, where it is considered rather a timely service provided by individuals or companies and a shared use of cars as the means of transport to reach the common destination. Thus, further research in this article is only relevant to traditional carsharing. There are several business models and types of carsharing discussed in the literature. Following figure illustrates existing business models with reference to the type of carsharing provided.

Source*: Adopted from Goncalves, G.,Santos, D. (2016). One-Way Carsharing Systems: Real-Time Optimization of Staff Movements and Operations. UNIVERSIDADE DE LISBOA INSTITUTO SUPERIOR TECNICO*

Depending on the type carsharing allows: (1) one-way and (2) round-trip services. If free-floating carsharing and peer-to-peer carsharing allows only one-way travels, station based carsharing can allow only round-trip travels as vehicles in this case must be returned to the

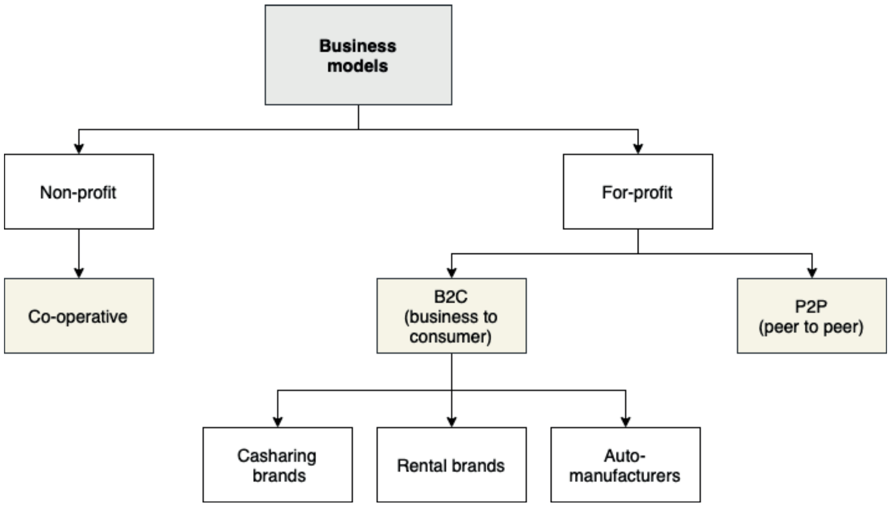

Figure 1: Carsharing models and types

car owner (peer-to-peer) or a station (stationary). Additionally, we distinguish three models: (1) peer-to-peer - is a type of carsharing that offers a shared use of vehicles that belong to an individual to a specific user community. In this type of carsharing, players usually provide a platform to handle the transaction, offer insurance, and equip cars with devices to ensure access; (2) free-floating - relatively new type of car sharing, where customers pick up and return the vehicle in any location within a certain area. This type of carsharing, therefore, is usually practiced as a replacement of a taxi experience within a certain city for short distance travels, whereas (3) stationary carsharing allows users pick and return the vehicle only in the fixed (usually the same) location (Deloitte, 2020).

4 Motivation, socio-economic, ecological, and cultural aspects

Although carsharing has gained large popularity in most of the developed courtierscountries there are still non-adopters. Understanding the motivation behind the choice for carsharing services is one of the first steps. Thus, "... gaining insights into the characteristics and motives of adopters and non-adopters of carsharing is important for policymakers and businesses alike to support the wider diffusion of carsharing"(Munzel, Piscicelli, Boon, Frenken, 2019). Regardless of the number of owners of a personal car, more and more inhabitants of the world are beginning to give preference to car sharing services. According to the United Nations Economic Commission for Europe (UNECE), "... this is already a reality in large cities, where, due to excessive traffic congestion, high cost of parking or their lack, the private car as a general phenomenon begins to give way to new forms of mobility"(European Economic Comission UN , 2020). Reducing carbon emissions is one of the most rational reasons for countries to adopt and promote carsharing technologies and services. According to the European Transport and Environment Community "... Transport emissions have increased by a quarter since 1990 and are continuing to rise with 2017 oil consumption in the EU increasing at its fastest pace since 2001.1 Unless transport emissions are brought under control national 2030 climate goals will be missed. To meet the 2050 Paris climate commitments cars and vans must be entirely decarbonized. This requires ending sales of cars with an internal combustion engine by 2035. Such a transformation requires wholesale changes, not only to vehicles but also how they are owned, taxed, and driven"(European Federation for Transport and Environment, 2018).

Despite ecological benefits some municipalities do not seem to give preference for organizing infrastructure that promote carsharing. For example, in Tokyo "...it has been stated that ride-hailing increases motorized traffic and causes traffic congestion, and those for these reasons, there are areas where ride-hailing is not approved by the municipality of the city"(Ikezoe, Kiriyama, Fujimura, 2021). Based on the results of this study it seems like motivation towards choosing personal car ownership over carsharing is still present even in such high-tech city like Tokyo. The study identified sociological and cultural factors that keep citizens of Tokyo using personal cars. Thus, qualitative indicators that determined the preference for personal car ownership were "emotional factor "convenience factor"and opting for "private space". Although this study has a limitation of observing only 23 cases in Tokyo it clearly illustrates that promoting carsharing services cannot be successful without understanding socio-economic, cultural, and mental aspects that may justify one's choice for the type of mobility. Thus, the results showed that the strength of the emotional factor as a utility of owning a car was more than twice that of the convenience factor. Changing mobility behavior of potential users of carsharing services is stated to be a challenge in another study. Were authors claim that "...one of the main challenges remains to attract mainstream consumers to adopt carsharing services and change their mobility behavior"(Munzel, Piscicelli, Boon, & Frenken, 2019). Some financial implications can also act as a barrier to adopt and promote carsharing. The automobile industry is one of the priority drivers of the economy for several countries. Data show that 6.7% of the total number of jobs in Europe are in the automobile industry. It includes following sectors: (1) direct manufacturing, (2) indirect manufacturing, (3) personal cars, (4) transportation and recent (5) construction. Literature calls such economies "dependent"on automobile industry. It was therefore a reason for some giants of automobile industry like General Motors in U.S. to exit the carsharing market. In case of GM the carsharing service Maven was shut down in 2016.

Nevertheless, some literature states, that "...carsharing provides the potential to reduce the costs of vehicle travel to the individual as well as the society"(B. Caufield, 2021). According to PwC 8% of all US adults have participated in some form of car-sharing economy, with 1% serving as service providers for this new model by carpooling or renting their cars for an hour, day, or week (European Economic Comission UN , 2020). The number of registered carsharing users accounted to more than 2 million people around the world in 2020 (European Economic Comission UN , 2020). In comparison with the increasing number of personal car owners, this figure is insignificant and, therefore, the development of carsharing services require more attention in modern society.

5 Carsharing in Kazakhstan: a comparative analysis

Compared to European countries, Kazakhstan is a relatively new adopter of such sharing mobility services. Thus, there is a limited literature about the current state or development of carsharing in Kazakhstan. To contribute to the body of knowledge and understand the state of carsharing in the country a comparative analysis is conducted. It focuses on understanding the strength and weaknesses of existing carsharing solutions and compares them to European analogs. Table 1 presents the results of the analysis:

Table 1. A comparative analysis of carsharing solutions: Kazakhstan vs international

N	Service provider	Area of Service	Type of carsharing	Model of operation	Website	App
International						
1	Turo	Worldwide	One-way, round-trip	Peer-to-peer	yes	yes
2	ZipCar	Worldwide	One-way, round-trip	B2C, B2B	yes	yes
3	Getaround	USA	one-way, round-trip	B2C, peer-to-peer	yes	yes
4	Enterprise Carshare	USA, Europe, Latin America	One-way, round-trip	B2C, B2B	yes	yes
5	HyreCar	USA	One-way, round-trip, Business-related rentals	B2C, B2B	yes	yes
6	Car2go	Worldwide	One-way, round-trip	B2C, peer-to-peer	yes	yes
Kazakhstan						
1	Anytime	Almaty	round-trip, short-term rentals	B2C, B2B, free-floating	yes	yes
2	Rentacars	Almaty, Nur-Sultan	round-trip, long-term rental	B2C, station-based	yes	no
3	Avis Kazakhstan	Almaty	round-trip	B2C, station-based	yes	no
4	Almacar	Almaty	round-trip	B2C, station-based	yes	no
5	Imageauto	Almaty	round-trip	B2C, station-based	yes	no
6	Pegas Auto	Almaty and Almaty region	round-trip	B2C, station-based	yes	no

Note: Service providers from 2 to 6 are not carsharing services as such but are rather car rental companies.

As it can be seen competition in Kazakhstani carsharing market is not as high as it is in Europe or U.S. Correspondingly, there is only one active carsharing service provider in Kazakhstan (anytime) to this moment (table 1). Unlike car rental services, anytime is a fully operating carsharing company which offers both web and mobile solutions to its customers which in turn makes it the only competitive carsharing service provider in Kazakhstan. Like ZipCar, Turo and other international operators anytime provides a full technological support to its customers by providing both web-based and mobile solutions to its customers. Another advantage of this service is the level of mobility that it supports by supporting a free-floating model, where customers can pick and drop the vehicle at any convenient station within available zones. However, compared to its international analogs it still has some limitations. If, for instance, international carsharing operators like Turo, ZipCar and Car2Go are available in most parts of European and US cities, anytime is a local based service for Almaty only and, respectively, is not available in any other cities of Kazakhstan. An additional limitation of anytime is that, compared to international analogs, providing a valid driving license only is not enough to use the service. It additionally requires a proof of National ID card, which makes it unavailable to use for non-Kazakhstani citizens. Foreign citizens in turn are more likely to need the carsharing service as mostly they do not own a personal car. Consequently, expanding the areas of service by including another two big cities of Kazakhstan like Nur-Sultan and Shymkent, simplifying the registration process by requiring only a valid driving license, and by which making it available for use to foreign citizens could be an advantage for this platform.

Limitations persist when looking at car rental services, that have been included in this study as analogs of carsharing in Kazakhstan.While popular international analogs offer a wider variety of mobility options by providing both one-way carsharing where a vehicle can be dropped at a convenient location, a round-trip carsharing, when the car is returned to the initial station and free-floating carsharing, where a user has an option of returning the vehicle to any station within available zones all observed car rental services are station-based. This hinders opportunities for carsharing or car rental use as the purpose of carsharing initially is in increasing the possibilities for human mobility by making it more convenient to opt for rental or shared mobility services than owning a personal car. Hence the status of car mobility services in Kazakhstan at the moment does not satisfy the requirements for advanced human mobility as compared to international analogs.

Additionally in Kazakhstan (anytime) free-floating zones are only available within the city while international carsharing operators offer inter-city free-floating options. When looking at the convenience of the use and availability of technology solutions Kazakhstan falls behind by having only one service provider (anytime) with a mobile solution for its' customers. The rest of the observations (car rental companies) only support a web-based solution. Another limitation that can be seen from the table is that all six observed carsharing (and car rental) solutions in Kazakhstan are based in a certain location (mostly in Almaty and Nur-sultan).

Although there is a good deal of car rental services in Kazakhstan most of them operate only around big cities such as Almaty and Nur-Sultan. Some of the rental companies are included in this analysis. Thus, one of the car rental companies (rentacars) offer their services in two large cities (Almaty and Nur-Sultan), and another one (pegas auto) expand their services a little further by including Almaty region as well. Above mentioned car rental services are similar to station-based carsharing solutions, with considerable differences: (1) companies offering car rental services often do not provide a full range of technical solutions and usually limit with websites only; (2) they operate during working hours only, consequently, do not provide a 24/7 customer support. There are also other companies, that have not been included in this analysis like autoprokat.kz, which offer a peer-to-peer based solution for the customers who want to rent their car. In such cases a car rental company acts as a platform where customers can offer their cars for rental in a way traditional car rental works. Renting a personal car to such companies offer customers a possibility of renting their cars for long periods. However, this has certain limitation too. For instance, renting a personal car to companies in a discussed manner limits one's ability to control and manage their cars. Management and control over the use of the car in such cases totally belong to a company. One of the perspective companies was "Doscar club startup based in Almaty, which also offered car rental services, but the service has not been updated since July 2019. It had around 5000 users offering some advantages to its customers like a non-key access to cars, per/min and per/hour options of rental, comfortable rates on rental and cost reduction by including the petrol and carwash price in the service cost. According to the data available there were approximately 20 cars available in Almaty. Although this carsharing solution has many advantages over some other alternate services there are some disadvantages like limited number of available vehicles, and no up-to-date data on the current state of the service.

6 Peer-to-Peer app design

Having considered the business models and features of various car-sharing services, P2P was chosen as the main model of the proposed carsharing system. The P2P business model will allow to deploy the service without the need to purchase a fleet of cars and their further maintenance and rent parking spaces. In this section, a high-level structural model of the system modules is presented both from the side of the user and from the side of the host (returning the car) (Figures 1 and 2). The modules are sorted by priority, how critical is their inclusion in the minimum viable product (MVP).
The following diagrams show the use cases that each actor can do.

7 System architecture design of proposed system

Figure 4 demonstrates system design of proposed system. The main actors like Customer, Staff, Host are included in the system design architecture. Also, the back-end server, database, load balancer can be seen as a part of the system design. The Machine Learning (ML) server and middleware API which connects the backend server with ML server is also an essential part of the system.

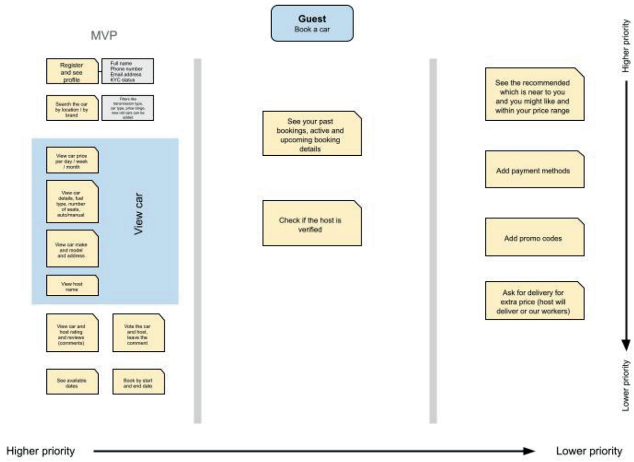

Figure 2: Use cases that Guest(customer) can perform

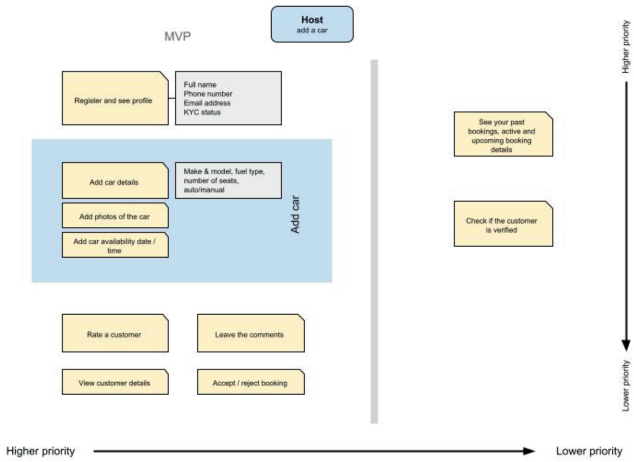

Figure 3: Use cases that Host can perform

8 Discussion and future research

Despite the difference in the state of carsharing between international players and Kazakhstan carsharing services are more likely to gain larger recognition in the coming years. Since the need for human mobility is growing and governments are looking for more sustainable solutions to human mobility the future of carsharing market is largely positive. However, there is a lot more that must be done in terms of creating a supporting urban infrastructure for carsharing solutions, including the creation of normative documents regulating these services in Kazakhstan than internationally. According to some of the world's reputable consulting agencies technology will revolutionize and disrupt the car sharing market.

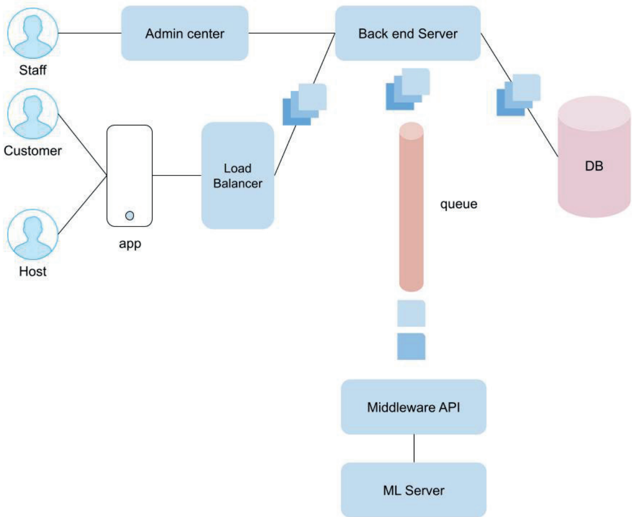

Figure 4: System architecture

Different smartphone solutions are expected to improve the use of carsharing services. Hence, Deloitte states that "...the future will further transform from automobiles to automobility, which will unleash new levels of convenience and efficiency for all road users by promoting the relationship between cars, infrastructure, and users"(Deloitte, 2020). A recent case study on Netherlands have stated that "...the increasing diffusion of carsharing in recent years is supported by technological innovations like smart keys, GPS services, as well as the widespread use of smartphones"(Munzel, Piscicelli, Boon, Frenken, 2019).

However, the degree to which citizens are likely to opt for carsharing usage highly depends on how supportive the government policies are and if the sufficient infrastructure was created. Thus, for most of the countries a carsharing is still a niche phenomenon. A study of 58 households in Norway states that "...policies should combine Electric Vehicles (EVs) and car-sharing, e.g., in Oslo, the focus of promoting EVs should include shared EVs, and in Rotterdam, improved charging infrastructure would be effective"(M.C.Svennevika, Dijkb, Arnfalkc, 2020).

The studies show that successful projects in carsharing around the world, including those initiated exclusively through private initiative, have been supported and facilitated by government agencies and built on a solid regulatory framework. Another key success factor is a well-chosen sustainable business model and the availability of investment opportunities (European Economic Comission UN , 2020). This is supported by another study that suggests that to promote the carsharing culture "...regulation should focus on shaping favorable conditions for a connected multi-modal transportation system instead of specific regulations for each carsharing business model"(Munzel, Piscicelli, Boon, Frenken, 2019).

Although Kazakhstan's carsharing market is developing there are many limitations observed as a result of a comparative analysis. Thus, different factors might affect the adoption and development of carsharing in Kazakhstan:

- in comparison with European countries, the distances between cities in Kazakhstan are larger and even if carsharing develops, it is more likely to occur in large cities (Nur-Sultan, Almaty, Shymkent);

- the regulation of carsharing by normative documents remain limited, so a clear understanding of how carsharing service providers operate has not been formed. In this regard, some data is provided by the UNECE report (European Economic Comission UN , 2020);

- in comparison with numerous studies carried out in some of the world's biggest cities like Tokyo, Paris etc. we can observe the lack of empirical data for Kazakhstan on the motivation, limitations,and potential of carsharing. This limits the development of this type of service.

These outcomes might be limited with the scope of the study since for this article only few cases were observed from both local and Kazakhstani carsharing providers. However, it certainly illustrates the visible difference in the status quo of carsharing between early adopters and Kazakhstan. There is little data available on what is the motivation towards personal car ownership versus carsharing, as well as socio-demographic and cultural factors that have never been accessed in the literature for a national context. This can be a direction towards further research on this topic. As a suggestion focus group interviews could be conducted with all the beneficiary of carsharing services, including potential customers and government parties. This can give a bigger picture of the future of carsharing in Kazakhstan and suggest ideas for improvement of the current status quo. Finally, as studies suggest "...understanding the effects of carsharing when combined with other existing travel modes is an important pre- requisite for decision-makers for them to be able to positively utilize the benefits of carsharing services"(Matowski, Pribyl, Pecherova, 2021). Therefore, future empirical studies must be conducted on behavioral patterns of carsharing adopters and non-adopters in Kazakhstan, including the research of above-mentioned limitations.

9 Acknowledgements

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INTEGRATION OF THE LABVIEW 2019 DEVELOPMENT TOOL IN WORKING WITH MICROCONTROLLERS

When connecting various sensors to microcontrollers, there are problems of the data received from them arise: analysis, data visualization, transmission and storage on remote storage media, etc This usually requires additional connection to microcontrollers of modules with the necessary functions. The way to use applications developed using the LabVIEW tool in the microcontroller operation scheme is one of the simple and reliable solutions in such cases. The study proposes to analyze the developed connection diagram, the wireless data transmission algorithm and their processing, using the example of connecting the GY21 sensor module, the ESP32 NodeMCU hardware platform and the application developed in the LabVIEW 2019 environment. The main technical characteristics of these modules are also given. Software products created using this LabVIEW software package can be supplemented with code fragments developed in other traditional programming languages, such as C / C ++, Pascal, Basic, FORTRAN. Conversely, you can use modules developed in LabVIEW in projects created in other programming systems. Thus, LabVIEW allows you to develop almost any application that interacts with any kind of hardware supported by the PC operating system. Using the technology of virtual instruments, a developer can turn a standard personal computer and a set of arbitrary control and measuring equipment into a multifunctional measuring and computing complex that allows remote control and monitoring via the Internet.
Key words: LabVIEW, ESP32 NodeMCU, sensor module GY21, ARDUINO IDE, Wireless data transmission.

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Микроконтроллерлермен жұмыс жасау кезінде LabVIEW 2019 әзірлеу құралын біріктіру

Датчиктердің барлық түрлерін микроконтроллерлерге қосқан кезде, олардан алынған деректерді одан әрі өңдеу міндеттері туындайды: талдау, деректерді визуализациялау, қашықтағы ақпарат тасығыштарда беру және сақтау және т.б. Бұл әдетте микроконтроллерлерге қажетті функциялары бар модульдерге қосымша қосылуды қажет етеді. Микроконтроллерлердің жұмыс сұлбасында LabVIEW құралын қолдана отырып жасалған қосымшаларды пайдалану әдісі мұндай жағдайларда қарапайым және сенімді шешімдердің бірі болып табылады. Зерттеуде GY21 датчик модулін, ESP32 NodeMCU аппараттық платформасын және LabVIEW 2019 ортасында жасалған қосымшаны қосу мысалында жасалынған байланыс сұлбасын, деректерді сымсыз беру және өңдеу алгоритмін талдауды ұсынылады. Сондай-ақ, осы модульдердің негізгі техникалық сипаттамалары келтірілген. Осы LabVIEW бағдарламалық кешенін қолдана отырып жасалған бағдарламалық өнімдер C/C++, Pascal, Basic, FORTRAN сияқты басқа дәстүрлі бағдарламалау тілдерінде жасалған код үзінділерімен толықтырылуы мүмкін. Керісінше, LabVIEW-де жасалған модульдерді басқа бағдарламалау жүйелерінде жасалған жобаларда қолдануға болады.
Осылайша, LabVIEW дербес компьютердің операциялық жүйесі қолдайтын кез-келген аппараттық құралдармен өзара әрекеттесетін кез-келген қосымшаны жасауға мүмкіндік береді. Виртуалды аспаптар технологиясын қолдана отырып, әзірлеуші стандартты дербес компьютерді және ерікті бақылау-өлшеу жабдықтарының жиынтығын Internet арқылы қашықтан басқаруға және бақылауға мүмкіндік беретін көп функциялы өлшеу-есептеу кешеніне айналдыра алады.
Түйін сөздер: LabVIEW, ESP32 NodeMCU, GY21 датчик модулі, ARDUINO IDE, деректерді сымсыз жіберу.

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Интеграция инструмента разработки LabVIEW 2019 в работе с микроконтроллерами

При подключении всевозможных датчиков к микроконтроллерам возникают задачи дальнейшей их обработки получаемых от них данных: анализа, визуализации данных, передачи и хранения на удаленных носителях информации и т.д. Для этого как правило требуется дополнительное подключение к микроконтроллерам модулей с необходимыми функциями. Способ использования приложений, разработанных с помощью инструмента LabVIEW в схеме работы микроконтроллеров, является в таких случаях одним из простых и надежных решений. В исследовании разбирается разработанная схема подключения, алгоритм беспроводной передачи данных и их обработки, на примере подключения модуля датчика GY21, аппаратной платформы ESP32 NodeMCU и приложения разработанного в среде LabVIEW 2019. Также приводятся основные технические характеристики указанных модулей. Программные продукты, созданные с использованием данного программного комплекса LabVIEW, могут быть дополнены фрагментами кода, разработанными на других традиционных языках программирования, например C/C++, Pascal, Basic, FORTRAN. И наоборот можно использовать модули, разработанные в LabVIEW в проектах, создаваемых в других системах программирования. Таким образом, LabVIEW позволяет разрабатывать практически любые приложения, взаимодействующие с любыми видами аппаратных средств, поддерживаемых операционной системой ПК. Используя технологию виртуальных приборов, разработчик может превратить стандартный персональный компьютер и набор произвольного контрольно-измерительного оборудования в многофункциональный измерительно-вычислительный комплекс, допускающий удаленное управление и наблюдение через Internet.

Ключевые слова: LabVIEW, ESP32 NodeMCU, модуль датчика GY21, ARDUINO IDE, беспроводная передача данных.

1 Introduction

LabVIEW (Laboratory Virtual Instrumentation Engineering Workbench) is a specialized, flexible programming environment used to create unique utilities and applications for measuring instruments. LabVIEW is a data collection, analysis and processing system that includes powerful tools for visualizing results. It is also used to control technical objects and technological processes [1].

Currently, LabVIEW is widely used in the following areas [2]:

– Automotive industry;
– Telecommunications;
– Aerospace industry;
– Semiconductor industry;
– Development and production of electronics;
– Extractive industry;
– Shipbuilding industry;
– Biomedicine;

ESP32 NodeMCU is a highly integrated, combined (Wi-Fi + Bluetooth) chip designed for solutions that require the lowest power consumption figures, shown in (Fig. 1).

ESP32 NodeMCU supports the entire stack of Wi-Fi 802.11n and BT4.2 protocols, providing this functionality via SPI / SDIO or I^2C / UART [3].

ESP32 NodeMCU Module Specifications:

CPU:	Xtensa Dual-Core 32-bit LX6, 160 MHz или 240 MHz (до 600 DMIPS);
Memory:	520 KByte SRAM, 448 KByte ROM;
Flash:	1, 2, 4 . . . 64 Mb;
Wireless:	Wi-Fi: 802.11b/g/n/e/i, до 150 Mbps с HT40;
Bluetooth:	v4.2 BR/EDR и BLE;
Manufacturer:	Espressif Systems;

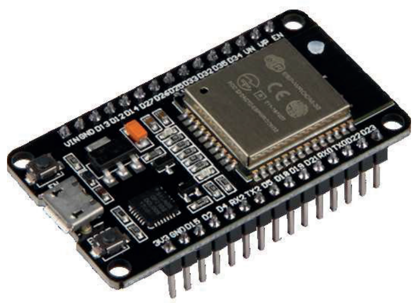


Figure 1: Module ESP32 NodeMCU

The GY21 humidity and temperature sensor module based on the SHT21 sensor is a high-precision module for measuring temperature and humidity, shown in (Fig.2). It has a very low error in its class: when measuring temperature, it is 0.4%, and humidity is 2%.

Due to such a low error and universal I2C interface, this module is suitable for temperature and humidity measurement in industrial areas [4].

GY-21 Specifications:

– Sensor:	SHT21
– Interface:	I2C
– Sensor operating voltage range:	$1.9 - 3.6V$
– Module supply voltage:	$5V$
– Current consumption in measuring mode:	$300\mu A$
– Standby current consumption:	$0.15\mu A$
– Humidity: operating range:	0 to 100%
– Measurement accuracy:	$\pm 3\%(max), 0 - 80\%$
– Temperature: operating range:	$-40to + 125°C$
– Measurement accuracy:	$\pm 0.4°C(max), -10to85°C$

– Factory calibration
– Built-in battery discharge detector (sets the flag if the supply voltage drops below 2.25 V)
– Built-in heater for self-diagnosis of sensors

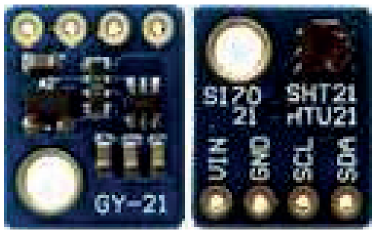


Figure 2: Sensor module GY21 (SHT21)

2 Material and methods

The algorithm of actions of our circuit is as follows: Sensor module GY21 (SHT21) connected to the ESP32 NodeMCU Module will transmit temperature and humidity parameters to it according to the program executed on the ESP32 NodeMCU Module. To do this, we will develop the necessary program (sketch for uploading) in the **Arduino IDE** [5]. The received data, already available in the RAM of the Module ESP32 NodeMCU, upon request from the application developed in LabVIEW, will be read and processed on the computer via the TCP IP protocol wirelessly.

Our experiment starts with assembling a circuit diagram, which were designed in the program Fritzing [6]. The connection diagram of the GY21 humidity and temperature sensor module to the ESP32 NODMCU v.3 hardware platform is shown in (Fig. 3).

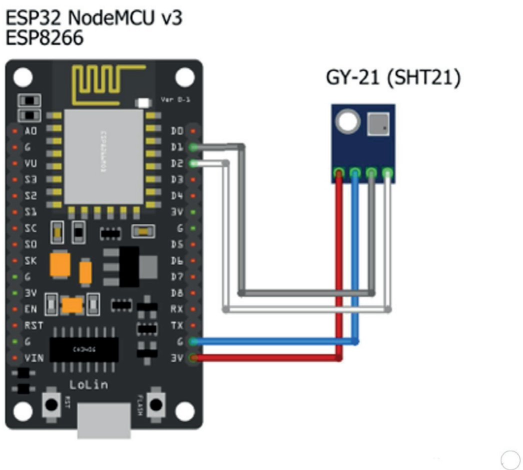


Figure 3: Connection diagram

The next step is to write a sketch in the Arduino IDE program.

Arduino IDE is an integrated development environment for Windows, MacOS and Linux, developed in C and C ++, designed to create and download programs on Arduino-compatible boards, as well as on boards from other manufacturers.

The ESP8266 WiFi.h library was used to work with the ESP32 hardware platform interface:

– The library "Sodaq_SHT2x.h" was also connected to work with the selected module of the humidity and temperature sensor GY21;

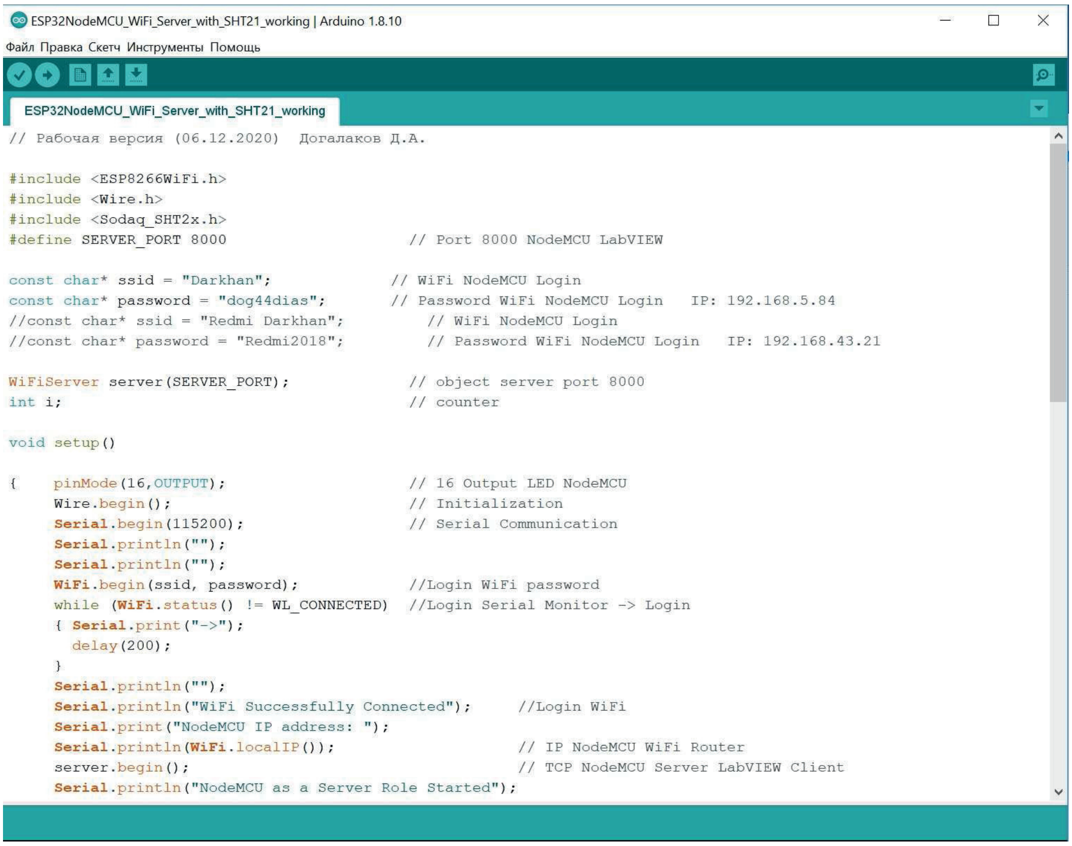
– Protocol for data transfer between ESP32 NodeMCU and the main control program: TCP IP;

– Number of data transmission port: 8000;

– Size of data transmission packet: 12 bytes;

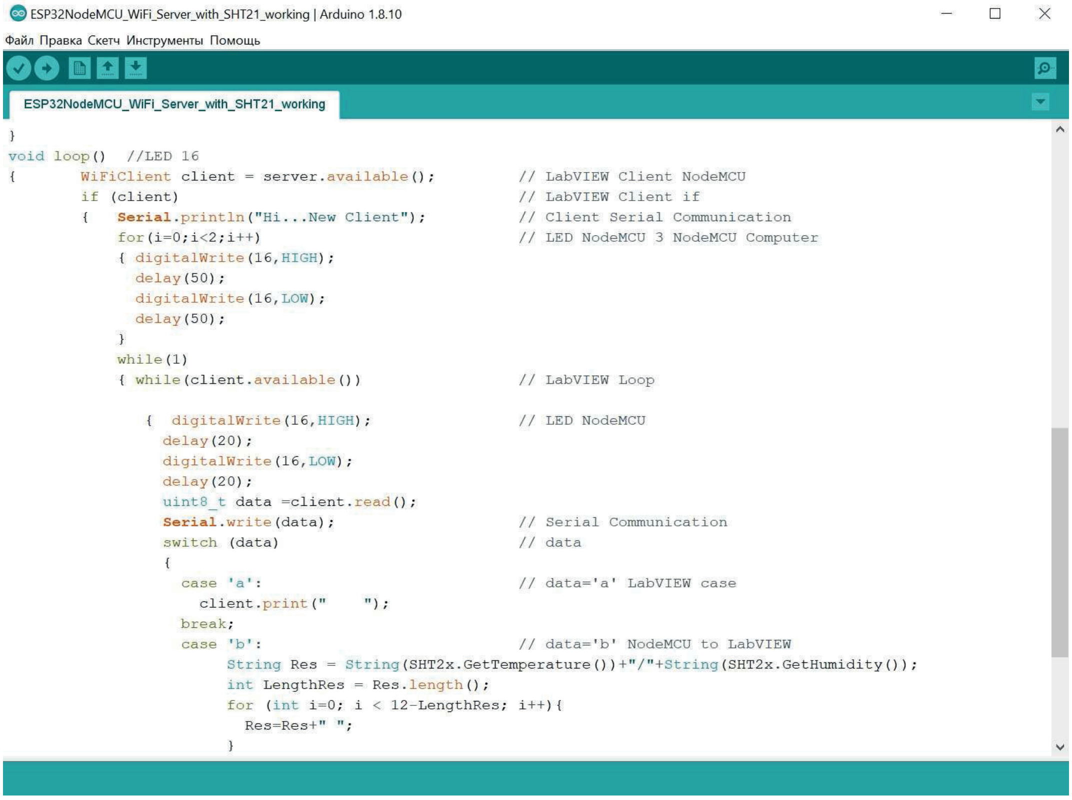
When forming a data packet of temperature and humidity readings before sending to the control program, its size (in bytes) may not be constant due to the dimension of the readings themselves, for example, the temperature may be $T = -12°C$ or $0°C$ or $25°C$, respectively for temperature readings we must to allocate 3 or 1 or 2 bytes. To do this, we set the maximum packet size for send 12 bytes, taking into account the humidity readings and other possible parameters in the future, and in the case of missing bytes from the data, we fill our packet with characters ("non-breaking space"code Alt + 255) up to 12 bytes.

Below in (Fig. 4) and (Fig. 5) there is a listing (code) of the program loaded into the memory of the ESP32 NODMCU v.3 module:


ESP32NodeMCU_WiFi_Server_with_SHT21_working | Arduino 1.8.10
Файл Правка Скетч Инструменты Помощь
ESP32NodeMCU_WiFi_Server_with_SHT21_working

```
// Рабочая версия (06.12.2020)  Догалаков Д.А.

#include <ESP8266WiFi.h>
#include <Wire.h>
#include <Sodaq_SHT2x.h>
#define SERVER_PORT 8000                    // Port 8000 NodeMCU LabVIEW

const char* ssid = "Darkhan";              // WiFi NodeMCU Login
const char* password = "dog44dias";        // Password WiFi NodeMCU Login   IP: 192.168.5.84
//const char* ssid = "Redmi Darkhan";          // WiFi NodeMCU Login
//const char* password = "Redmi2018";          // Password WiFi NodeMCU Login   IP: 192.168.43.21

WiFiServer server(SERVER_PORT);             // object server port 8000
int i;                                      // counter

void setup()

{     pinMode(16,OUTPUT);                   // 16 Output LED NodeMCU
      Wire.begin();                         // Initialization
      Serial.begin(115200);                 // Serial Communication
      Serial.println("");
      Serial.println("");
      WiFi.begin(ssid, password);           //Login WiFi password
      while (WiFi.status() != WL_CONNECTED) //Login Serial Monitor -> Login
      { Serial.print("->");
        delay(200);
      }
      Serial.println("");
      Serial.println("WiFi Successfully Connected");     //Login WiFi
      Serial.print("NodeMCU IP address: ");
      Serial.println(WiFi.localIP());                    // IP NodeMCU WiFi Router
      server.begin();                                    // TCP NodeMCU Server LabVIEW Client
      Serial.println("NodeMCU as a Server Role Started");
```

Figure 4: Listing of the program for loading into the ESP32 NODMCU v.3 module (beginning)



ESP32NodeMCU_WiFi_Server_with_SHT21_working | Arduino 1.8.10

Файл Правка Скетч Инструменты Помощь

ESP32NodeMCU_WiFi_Server_with_SHT21_working

```
}
void loop()  //LED 16
{       WiFiClient client = server.available();          // LabVIEW Client NodeMCU
        if (client)                                        // LabVIEW Client if
        {   Serial.println("Hi...New Client");             // Client Serial Communication
            for(i=0;i<2;i++)                               // LED NodeMCU 3 NodeMCU Computer
            { digitalWrite(16,HIGH);
              delay(50);
              digitalWrite(16,LOW);
              delay(50);
            }
            while(1)
            { while(client.available())                    // LabVIEW Loop

                {  digitalWrite(16,HIGH);                  // LED NodeMCU
                 delay(20);
                 digitalWrite(16,LOW);
                 delay(20);
                 uint8_t data =client.read();
                 Serial.write(data);                       // Serial Communication
                 switch (data)                             // data
                 {
                   case 'a':                               // data='a' LabVIEW case
                     client.print("    ");
                   break;
                   case 'b':                               // data='b' NodeMCU to LabVIEW
                        String Res = String(SHT2x.GetTemperature())+"/"+String(SHT2x.GetHumidity());
                        int LengthRes = Res.length();
                        for (int i=0; i < 12-LengthRes; i++){
                          Res=Res+" ";
                        }
```

Figure 5: Listing of the program for loading into the ESP32 NODMCU v.3 module (continuation)

Further, in the development environment LABVIEW 2019 in the "Block diagram" we create a general block diagram of the operation of our control program, (Fig. 6).

Part 1 describes the instructions and settings required to connect to the ESP32 NODMCU using the TCP IP protocol. To scan the network and determine the IP address assigned by the WI-FI network to the ESP32 NODMCU, the free Hercules SETUP utility was used [7].

Part 2 executes command "b" – sending a request for data transfer to the ESP32 NODMCU module.

Part 3 and part of **Part 7** output the temperature and humidity readings to special separate areas of the program interface in the form of graphs.

Part 4 reads the system time and generates a new line according to the specified format.

Part 5 and **Part 6** describes an algorithm for dividing the received data packet of 12 bytes into two parts, to separate the temperature and humidity readings from it separately.

Part 7 describes the procedure for writing the finally formed data line to a text file at the specified path.

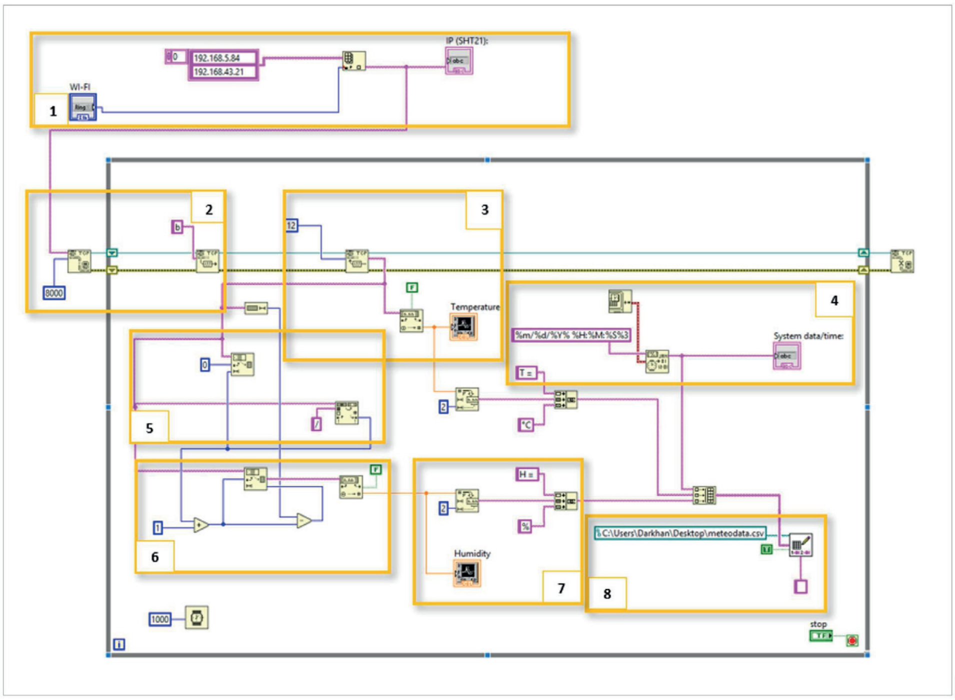


Figure 6: Algorithm of the control program in the LABVIEW 2019 environment

The control program interface developed in the LABVIEW environment is shown in (Fig. 7).

The panel displays: selectable shared wireless network (WI-FI), the IP address received by the ESP32 NODMCU from the network and the graphics of the read data;

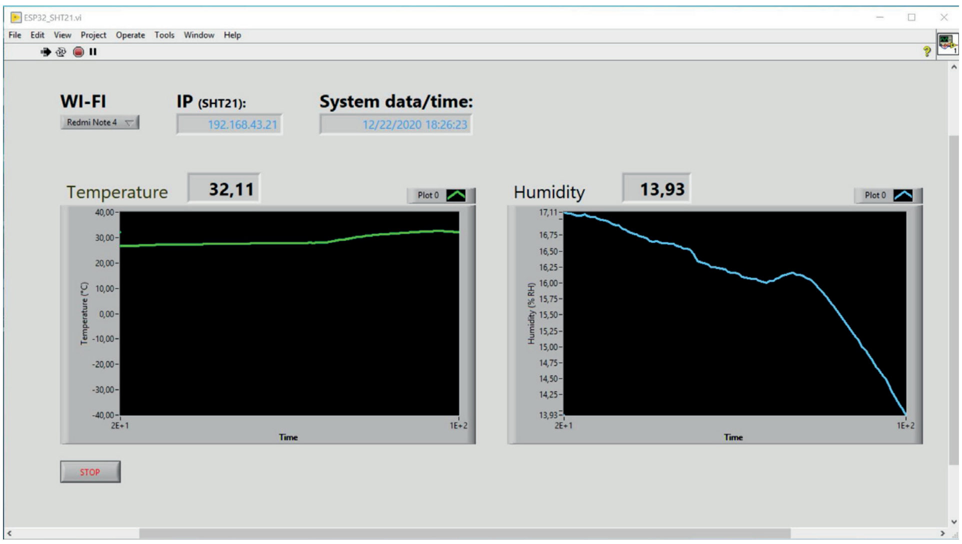


Figure 7: The interface of the control program in the LABVIEW 2019 environment

Read and transmitted temperature and humidity data from the ESP32 NodeMCU to the main control program on the PC. Writing data to a text file with a time interval of 1 second is shown in (Fig. 8).

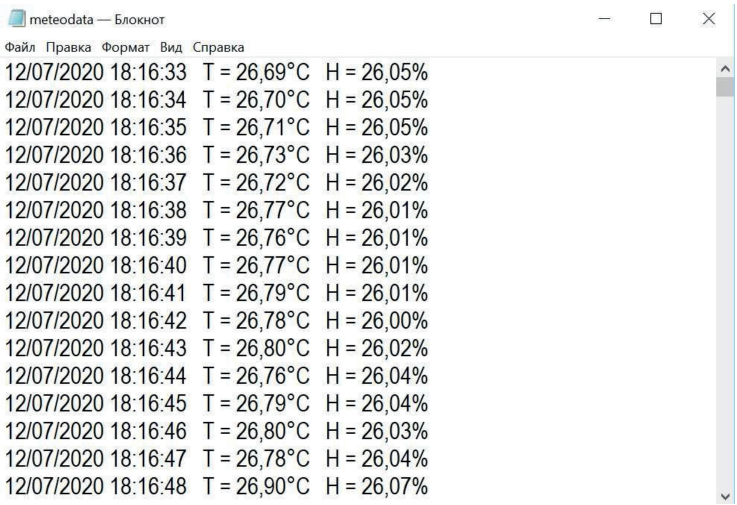


Figure 8: Received data written to text file

You can also use other GPIOs on the ESP32 NodeMCU board to turn on or off peripheral equipment such as relays [8].

3 Results and discussion

The proposed scheme of collaboration between the LABVIEW environment and microcontrollers for data transmission and processing allows you to exclude the purchase of additional output and recording devices. At the same time, the volume of data received can be significantly increased, and their consolidation and visualization in applications using other additional tools of the LABVIEW environment becomes convenient and practical. The selected mechanism for using the LABVIEW environment in integration with microcontrollers is also planned to be used in other tasks related to technical vision for controlling relays and electric drives on electric vehicles [9]. One of the examples of such solutions is the using NI LabVIEW software and the NI Vision Development Module to develop a system to monitor the Paris RER. The key element of the infrastructure of the transport system are rails. After the installation of the controls, their position may change depending on the conditions of the environment, for example, the temperature. This integrated system measures these rail position changes [10].

4 Conclusion

This article presented the joint operation of the LabVIEW 2019 software package and one of the ESP32 NodeMCU hardware platforms with the GY21 module connected to it. The main technical characteristics of these devices are described, as well as commands for connecting and transferring data between them. The great opportunities inherent in the graphical programming and execution environment of LabVIEW 2019 programs (various libraries, integration and data exchange components, visual components) generally show and provide unlimited opportunities for software developers, both for research tasks (data collection and analysis), and for automation of various technological processes. This practical example we have analyzed confirms this.

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POST-EDITING FOR THE KAZAKH LANGUAGE USING OPENNMT

The modern world and our immediate future depend on applied intelligent systems, as new technologies develop every day. One of the tasks of intelligent systems is machine (automated) translation from one natural language to another. Machine translation (MT) allows people to communicate regardless of language differences, as it removes the language barrier and opens up new languages for communication. Machine translation is a new technology, a special step in human development. This type of translation can help when you need to quickly understand what your interlocutor wrote or said in a letter.
The work of online translators used to translate into Kazakh and vice versa. Translation errors are identified, general advantages and disadvantages of online machine translation systems in Kazakh are given. A model for the development of a post-editing machine translation system for the Kazakh language is presented.
OpenNMT (Open Neural Machine Translation) is an open source system for neural machine translation and neural sequence training. To learn languages in OpenNMT, you need parallel corpuses for language pairs. The advantage of OpenNMT is that it can be applied to all languages and can handle large corpora. Experimental data were obtained for the English-Kazakh language pair. Experimental data were obtained for the English-Kazakh language pair.
Key words: Opennmt, neural machine translation, turkic languages.

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OpenNMT көмегімен қазақ тіліне постредактрлеу

Қазіргі әлем және біздің жақын болашағымыз қолданбалы интеллектуалды жүйелерге байланысты, өйткені жаңа технологиялар күн сайын дамып келеді. Интеллектуалды жүйелердің міндеттерінің бірі - бір табиғи тілден екіншісіне машиналық (автоматтандырылған) аударманы қолданып аудару. Машиналық аударма (тілдік аударма) адамдарға тілдік айырмашылықтарға қарамастан байланыс жасауға мүмкіндік береді, өйткені ол тілдік тосқауылды жойып, қарым-қатынас үшін жаңа тілдерді ашады. Машиналық аударма - бұл жаңа технология, адам дамуындағы ерекше қадам. Аударманың бұл түрі сізге әңгімелесушінің хатта не жазғанын немесе не айтқанын тез түсіну қажет болғанда көмектесе алады.
Онлайн аудармашылардың жұмысы бұрын қазақшаға және керісінше аударылатын. Аударма қателері анықталды, онлайн-машиналық аударма жүйесінің қазақ тіліндегі жалпы артықшылықтары мен кемшіліктері келтірілді. Қазақ тіліне арналған өңдеуден кейінгі машиналық аударма жүйесін әзірлеу моделі ұсынылған.
OpenNMT (Open Neural Machine Translation) – нейрон машинасын аудару және нейрон реттілігін оқытуға арналған ашық бастапқы жүйе. OpenNMT-де тілдерді үйрену үшін сізге тілдік жұптарға параллель корпустар қажет. OpenNMT-тің артықшылығы - ол барлық тілдерге қолданыла алады және үлкен корпустарды басқара алады. Эксперименттік мәліметтер ағылшын-қазақ тілі жұбы үшін алынды.
Түйін сөздер: Opennmt, нейрон машиналық аударма, түркі тілдері.

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Постредактирование для казахского языка с использованием openNMT

Современный мир и наше ближайшее будущее зависят от прикладных интеллектуальных систем, так как новые технологии развиваются с каждым днем. Одной из задач интеллектуальных систем является машинный (автоматизированный) перевод с одного естественного языка на другой. Машинный перевод (МП) позволяет людям общаться независимо от различия языков, поскольку это устраняет языковой барьер и открывает новые языки общения. Машинный перевод - это новая технология, особый шаг в развитии человека. Этот тип перевода может помочь, когда нужно быстро понять, что ваш собеседник написал или сказал в письме.
Работа онлайн-переводчиков, используемых для перевода на казахский язык и обратно. Выявлены ошибки перевода, даны общие преимущества и недостатки онлайн систем машинного перевода на казахском языке. Представлена модель разработки системы пост-редактирования машинного перевода для казахского языка.
OpenNMT (Open Neural Machine Translation) – это система с открытым исходным кодом для нейронного машинного перевода и обучения нейронной последовательности. Для обучения языки в OpenNMT нужны параллельные корпуса для языковых пар. Преимуществом OpenNMT является применение ко всем языкам и может работать с большими корпусами.В статье рассматривается обучения тюркские языки в OpenNMT. Было получено экспериментальные данные для англо-казахского языковой пары.

Ключевые слова: Opennmt, нейронный машинный перевод, тюркские языки.

1 Introduction

Türkic languages, a language family, spread over the territory from Turkey in the west to Xinjiang in the east and from the coast of the East Siberian Sea in the north to Khorasan in the south. The speakers of these languages live compactly in the CIS countries (Azerbaijanis – in Azerbaijan, Turkmens – in Turkmenistan, Kazakhs – in Kazakhstan, Kyrgyz – in Kyrgyzstan, Uzbeks – in Uzbekistan; Kumyks, Karachais, Balkars, Chuvashs, Tatars, Bashkirs, Nogais, Yakuts, Tuvans, Khakass, Mountain Altai – in Russia; Gagauz – in the Transnistrian Republic) and beyond its borders – in Turkey (Turks) and China (Uighurs). Currently, the total number of speakers of the Turkic languages is about 120 million.

The Türkic languages are similar in structure and meaning. This can be seen in the following table:

Table 1. Words of Turkic languages

Ancient Turkic	Turkish	Turkmen	Turkmen	In Kazakh	Kyrgyz	In uzbek	In Uyghur	Tyvan
Ana	ana/anne	ene	ana	ана /ana/	Ene	Ona	Ana	Ава
Burun	Burun	burun	borın	мұрын /murın/	murun	Burun	burun	Думчук
Qol	Kol	qol	qul	қол /qol/	Qol	qo‘l	kol	Хол
Yol	Yol	ýol	yul	жол /jol/	Jol	yo‘l	yol	орук (чол)
Semiz	Semiz	semiz	simez	семіз /semiz/	semiz	Semiz	semiz	Семис

But languages are different from other languages, they have special characteristics:

- proximity of the lexical structure;
- the law of harmony;
- agglutination – a series of affixes;
- lack of a category;
- lack of auxiliary words (prepositions);
- special word order.

Therefore, when translating, the Turkic languages give out morphological, lexical, and semantic errors.

To evaluate errors, we will use well-known translation machines, such as: Google, Yandex:

Table 2. Online transfers completed in February 2021 y.

Source text for translation into Türkic language	Name of machine translation systems and translation results		disadvantages
	Yandex	Google	
Таныш булыгыз, бу минем гаилэм (Tatar language)	Meet my family	Meet me, this is my family	Google Translate pays attention to punctuation. If there is an exclamation mark at the end of a sentence, this is correctly translated as "meet", otherwise it is incorrectly translated as "meet".
Сизни тэбриклэшкэ иҗазэт бериң (Uigur language)	No translation	Let me congratulate you	There is no Uyghur translation in Yandex. It accepts both Tatar language.
Мені жерге қаратпа (Kazakh language)	don't put me on the ground	don't put me on the ground	Translate phraseological units into a straight line
Бу тугрида гап хам булиши мумкин емас (Uzbek language)	It's all in tugrida and can not be found	This is out of the question	Yandex translation could not translate the word "tugrida" and completely lost the meaning of the sentence
Birsey icmek istiyorum (Turkish language)	I want to drink birsey	I want to drink something	In the translation of Yandex, the word "something" replaced by the word "birsey".

This article discusses training parallel corpus in Opennmt. The advantage of Opennmt is its universal applicability to different languages, including the Turkic languages.

We also get the results of computational experiments for the Kazakh language.

The rest of the paper is organised as follows:

– Section 2 provides an overview of previous work carried out in this area.

– Section 3 presents Opennmt for Kazakh-English, English-Kazakh language pairs.

– Section 4 presents experimental NMT results for Kazakh-English, English-Kazakh language pairs.

– Section 5 presents conclusions and suggests directions for future work.

2 Previous scientific work

In our country, in Kazakhstan, MT (machine translation) of the Kazakh language has been developing since 2000. Professor U. A. Tukeyev was one of the first to study machine translation. He managed to create a scientific school that is actively engaged in research in the field of MT. Among domestic students, one can note the study of models and methods of semantics of machine translation from Russian into Kazakh language [1], a statistical model of the alignment of English-Kazakh words using the machine translation algorithm [2].

To improve the morphologies of the Turkic languages, vocabulary training on a parallel corpus was used [3-6]. Learning on the parallel corpus of the English-German language pair in Opennmt, studied in foreign works. In this work, 50k vocabulary was learned for each pair and it was shown that in Opennmt Bleu was 17.60 [7]. Opennmt showed a better result than in the Nematus Bleu system by 0.5. Also, the architecture and applicability of Opennmt in other areas were considered.

3 Opennmt for Turkic languages

To build a neural network, the project uses the capabilities of the Torch deep machine learning library [8].

Opennmt in the model contains the following parameters [9-10]:

- encoder_type: transformer;
- decoder_type: transformer;
- position_encoding: true;
- enc_layers: 6;
- dec_layers: 6;
- heads: 8;
- rnn_size: 512;
- word_vec_size: 512;
- transformer_ff: 2048;
- dropout_steps: [0];
- dropout: [0.1];

- attention_dropout: [0.1];
- train_steps: 100000.

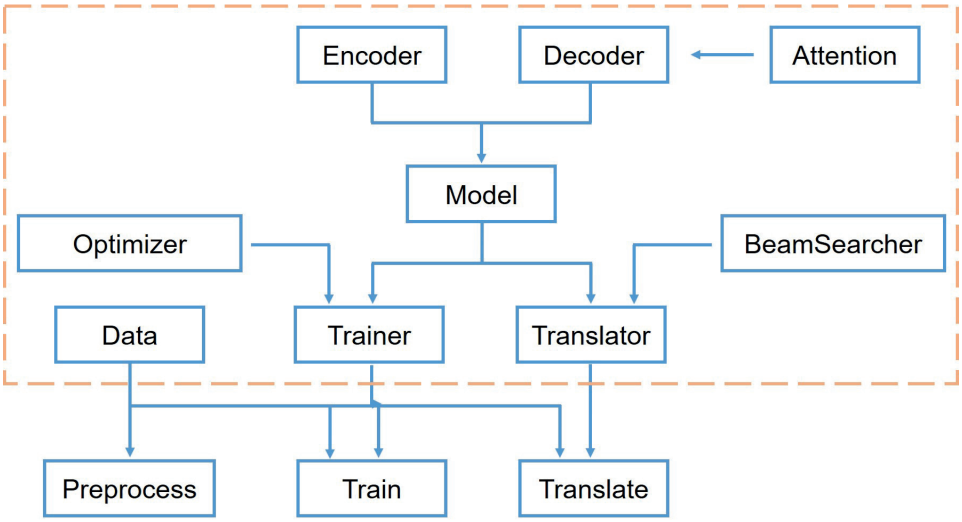

Figure 1: Building an Opennmt Model

4 Results

A Kazakh-English language pair was used for training. 109,772 sentences were used in the corpus. These proposals were taken from the website [11]: Akorda, Primeminister, mfa.gov.kz, economy.gov.kz, strategy2050.kz. Of these, it was taken for train 80000, test 20000, validation 9772.It took 36 hours to train at Opennmt.

Table 3. Obtained result in Opennmt for the English-Kazakh language pair

Language pair	Speed tok/sec	BLEU
Kazakh-English	4185	20.56
English-Kazakh	4185	20.05

As you can see in the table BLEU is less compared to other languages as for English-German, English-French pairs. Because the structure of the language of the Turkic languages is different from these languages. More parallel data is required to improve this metric.

5 Conclusion

This article covered learning parallel corpuses in Opennmt. In the experiment, it was used for the English-Kazakh language pair. To improve the translation, you need to add sentences to the corpus for the English-Kazakh language pair. In the future, corpora will be developed for English-Turkic language pairs and training according to this system.

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4-бөлім	Раздел 4	Section 4
Қолданбалы математика	**Прикладная математика**	**Applied Mathematics**

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INFLUENCE OF THERMAL EFFECTS TO POLLUTANT DISPERSION IN IDEALIZED STREET CANYON: NUMERICAL STUDY

In this work, we numerically investigate the process of atmospheric air pollution at various values of the road temperature in idealized urban canyons. To solve this problem, the Reynolds-averaged Navier-Stokes equations (RANS) were used. Closing this system of equations required the use of various turbulent models. The verification of the mathematical model and the numerical algorithm was carried out using a test problem. The results obtained using various turbulent models were compared with experimental data and calculated data of other authors. The main problem considered in this paper is characterized as follows: estimation of emissions of pollutants between buildings using different types of hedge barriers (continuous and intermittent) at different temperatures. The results have shown that the presence of hedge barriers along the roads significantly reduces the concentration of harmful substances in the air. The use of a grass barriers with a total height of $0.1H$ leads to a decrease in the concentration level to a section $X = 0.05H$ by more than 1.5 times compared with the case of a complete absence of protective barriers. In addition, the temperature conditions (in this case, $T_H = 305K$) also reduce the concentration value by almost 2 times. Increasing the temperature at the side of the road using the barrier reduces the spread and deposition of pollutants.
Key words: Air pollution, RANS, mathematical model, hedge barriers, concentration, temperature.

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Идеалданған көше каньонындағы ластаушы заттардың дисперсиясына жылу эффектісінің әсері: Сандық зерттеу

Бұл жұмыста ауа температурасының әртүрлі мәндерінде жолдың идеализацияланған қалалық каньондарындағы ауаның ластану процесі сандық зерттелген. Бұл мәселені шешу үшін Рейнольдс орташа Навье-Стокс теңдеулері (RANS) қолданылды. Бұл теңдеулер жүйесін жабу әртүрлі турбулентті модельдерді қолдануды қажет етті. Математикалық модель мен сандық алгоритмді тексеру тест тапсырмасы жүргізілді. Әр түрлі турбулентті модельдерді қолдану арқылы алынған нәтижелер эксперименттік мәліметтермен және басқа авторлардың есептеулерімен салыстырылды. Бұл жұмыста қарастырылған негізгі міндет келесідей сипатталады: әр түрлі температура мәндерінде әр түрлі шөп кедергілерін (үздіксіз және үзіліссіз) қолдана отырып, ғимараттар арасындағы ластаушы заттардың шығарындыларын бағалау. Зерттеу нәтижелері жолдар бойында кедергілердің болуы ауадағы зиянды заттардың концентрациясын едәуір төмендететінін көрсетті.

Жалпы биіктігі 10 см болатын кедергілерді қолдану Қорғаныс кедергілерінің толық болмау жағдайымен салыстырғанда шоғырлану деңгейінің $X = 5$см қимасына дейін 1,5 есе төмендеуіне әкеледі. Сонымен қатар, температура жағдайлары (бұл жағдайда $T_H = 305K$) концентрация мәнін 2 есе азайтады. Тосқауылдың көмегімен жол жиегіндегі температураның жоғарылауы қатерлі заттардың таралуын және тұндырылуын азайтады.
Түйін сөздер: Ауаның ластануы, RANS, математикалық модель, шөптік кедергілер, концентрация, температура.

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Влияние тепловых эффектов на рассеивание загрязняющих веществ в идеализированном уличном каньоне: численное исследование

В настоящей работе численно исследован процесс загрязнения атмосферного воздуха в идеализированных городских каньонах дороги при различных значениях температуры. Для решения этой задачи использовались усредненные по Рейнольдсу уравнения Навье-Стокса (RANS). Закрытие этой системы уравнений потребовало использования различных турбулентных моделей. Верификация математической модели и численного алгоритма проводилась с использованием тестовой задачи. Результаты, полученные с использованием различных турбулентных моделей, были сопоставлены с экспериментальными данными и расчетными данными других авторов. Основная задача, рассматриваемая в данной работе, характеризуется следующим образом: оценка выбросов загрязняющих веществ между зданиями с использованием различных типов травяных барьеров (непрерывных и прерывистых) при различных значениях температуры. Результаты исследований показали, что наличие живой изгороди вдоль дорог значительно снижает концентрацию вредных веществ в воздухе. Использование живой изгороди общей высотой 10 см приводит к снижению уровня концентрации до сечения $X = 5$см более чем в 1,5 раза по сравнению со случаем полного отсутствия защитных барьеров. Кроме того, Температурные условия (в данном случае $T_H = 305K$) также снижают значение концентрации почти в 2 раза. Повышение температуры на обочине дороги с помощью барьера уменьшает распространение и отложение злокачественных веществ.

Ключевые слова: Загрязнение атмосферного воздуха, RANS, математическая модель, травяные барьеры, концентрация, температура.

1 Introduction

In large cities, air quality has been an urgent problem in recent decades, as poor air quality can worsen people's health, and the main reason is constant traffic. Most people working in large cities are more likely to go to medical institutions with complaints of problems with the respiratory system. Consequently, people with birth defects and pathologies are at risk – in this case, in addition to death, cardiovascular disorders, cancer of other serious diseases are possible [1, 2, 3]. Therefore, the search for ways to reduce the percentage of pollution along the roads is an urgent problem of mankind today. The constant increase in the number of vehicles in large cities remains the main cause of air pollution [5]. Moreover, the level of pollutants in densely populated cities increases especially strongly during certain periods [6]. To do this, there are a number of methods for reducing air pollution, such as alternative fuels and electric vehicles (EV), and solid barriers can be used to neutralize harmful substances in the air. In addition, hedges reduce noise levels by being a natural signal source, giving the city aesthetic appeal and characterizing its ecosystem services.

Various types of barriers are often used to improve air quality in urban canyons [7, 8, 9]. In addition, subsequent studies assessed the impact of protective barriers on air quality along roads [10, 11, 12, 13]. Trees [14, 15], hedges [16], green roofs and facades [17] can be protective barriers, solid barriers are also used as low boundary walls [18, 19] and noise barriers [20, 21]. The presence of parked vehicles along the road is considered to be another important factor for improving air pollution control methods, which, in turn, works like a barrier and significantly reduce the concentration of pollutants in the air [22]. Herbal barriers play a role in solving such problems; city streets reduce dispersion of pollutants [23, 24].

Vegetation barriers also reduce roadside air pollution by affecting local turbulence and the natural dispersion of pollutants generated during driving [25]. In the paper [26] indicates the best roadside safety barriers to reduce air pollution along roads in urban environments. The main purpose of using protective grasses and artisanal barriers is to mitigate the impact of pollutants by quantifying spatial changes in different pavement configurations. To determine the effectiveness of the barriers to the dispersion of pollutants into the atmosphere, CFD is used, where the barrier from natural wind flow plays the main role. The results show that obstacles increase turbulence and wind speed, and can also reduce the concentration of exhaust gases in the urban environment.

The proposed study assessed the effect of a protective barrier on the level of air pollution in an urban environment [27], taking into account the effect of temperature on the carriageway. Complex urban structures with improved infrastructure, especially during seasonal periods, such as winter seasons accumulate more heat, while in summer, on the contrary, more energy is spent for cooling than rural areas [28, 29]. All these factors affect the heat exchange of the air flow in the urban environment and lead to the urban heat island effect [30, 31]. It turned out that the value of the concentration depends not only on the total amount of malignant emissions, but also on the temperature of the walls, road and source, an analysis is required that has the following parameters: building geometry, type of pollutants and barriers, wind flow conditions, barrier porosity and temperature, which have a significant impact on urban air quality. Thus, to improve air quality, the effectiveness of two various cases of road safety barriers was evaluated various temperatures.

The main goal of this paper is to study the effect of various temperatures when using different types of protective barriers as a possible mitigation for different pavement layouts that are commonly found in the real world. The most dangerous threat to urban canyons comes from stagnant situations, that is, when a surface inversion is accompanied by a weak wind flow. All physical and mathematical assumptions were validated by computer simulations.

2 Materials and methods

2.1 Mathematical model

The system of RANS, which is simulated by the ANSYS Fluent, and used to construct a mathematical model of the flow of liquid and gas. Various RANS turbulent models were used. The implementation of the test problems was based on the existing experimental data of well-known authors. The 3D cases are presented as test problems, for which a non-stationary state model was formed to analyze the gas flow in the $L * B * H$ urban street zone.

Ethylene (C2H4) was used as a pollutant for this problem [32]. The computational model of atmospheric air pollution in the idealized urban canyon road for various temperature values based on the RANS equation, the equation for the transfer of pollutants and the energy equation.

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{1}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \frac{\partial(-\rho \overline{u_j' u_i'})}{\partial x_j} - \frac{\partial P}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right) + f \tag{2}$$

$$\frac{\partial C}{\partial t} + \frac{\partial u_j C}{\partial x_j} = \frac{\partial(-\overline{u_j' C'})}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\gamma \frac{\partial C}{\partial x_j}\right) \tag{3}$$

$$\frac{\partial T}{\partial t} + \frac{\partial u_j T}{\partial x_j} = \frac{\partial(-\overline{u_j' T'})}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\chi \frac{\partial T}{\partial x_j}\right) \tag{4}$$

where μ is the dynamic viscosity, u_i is the velocity; P is the pressure; ρ is the density, T is the temperature, γ is the molecular diffusion coefficient, χ is the thermal diffusivity, $f = \rho g(T - T_0)$ where g is the specific force of gravity, $\overline{u_j' u_i'}$ and are Reynolds averaged velocity stresses and turbulent heat flows, respectively.

2.2 Numerical scheme

A numerical model for the presented problem was simulated by using the SIMPLE method (Semi-Implicit Method for Pressure-Linked Equations) [33-36]. This method is applied in many studies or investigations to solve many problems of hydrodynamics and heat transfer and served to create a whole class of numerical methods. All variables that were used in this simulation are completely dimensional size.

2.3 Model validation

Based on experimental studies [32], a test problem was implemented to verify the chosen mathematical model. The domain of the test problem is shown in Figure 1. The air flow rate under isothermal conditions is $V_{inlet} = 1m/s$. The tracer gas ejection rate was $3.0L/min$ using a mass flow controller. The source of the emission of the pollutant was located at the bottom center of the object under consideration and the emission velocity at the source was $V_{source} = 0.01923m/s$. The emitted contaminant was used identical to the experiment. The source line measures 0.01×0.26 meters. Similarly, to the test problem, for dimensioning, the constant $H = 1m$ will be introduced (the length of the area where the main pollution occurs). To determine the best holding abilities, several solid barriers were built with different heights $H_b = (0.05m, 0.1m, 0.2m, 0.3m$ and $0.4m)$.

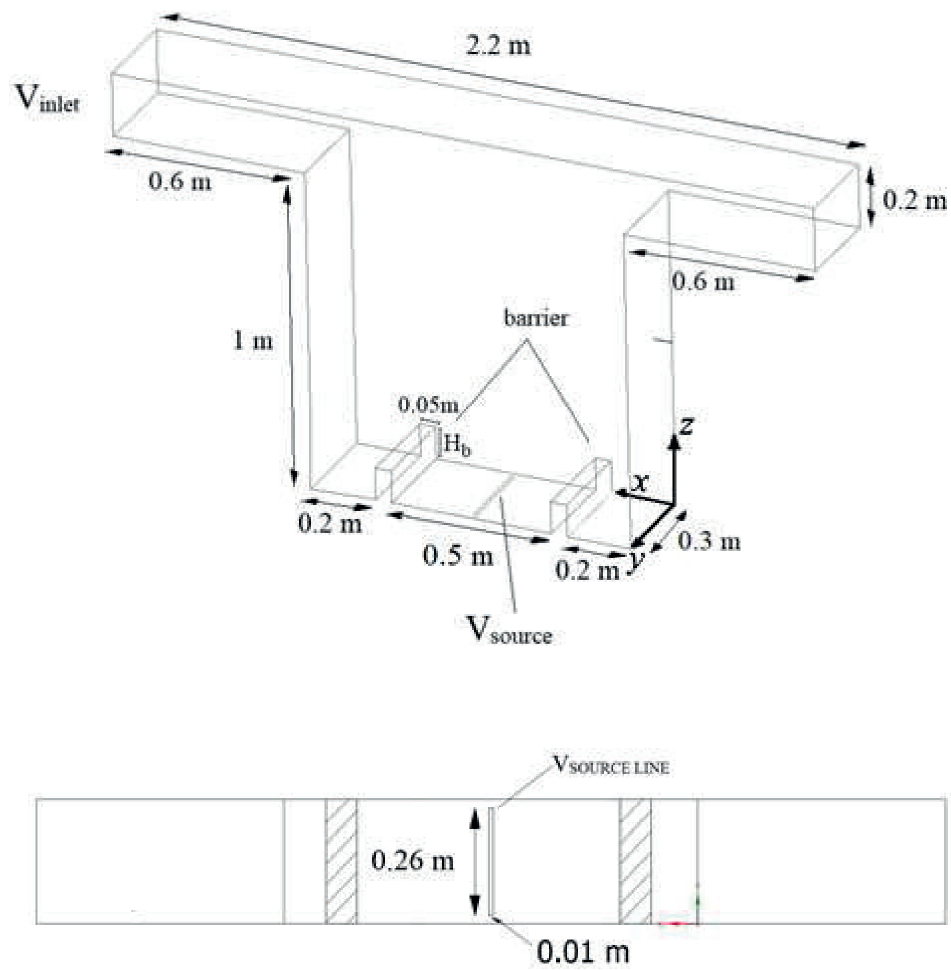


Figure 1: Test case geometry view of the cavity with herbal continuous barriers

As a computational grid, it was used a grid with a refinement to the center of the canyon, where the source of the ejection is located. To obtain more accurate results, high-quality grids were used in the area where velocity and kinetic energy are measured and neglected in fine grids in areas where vortex formation is not observed. As mentioned earlier, when performing the test problem, an unstructured mesh was used with a clustered to the buildings (side walls) – $0.005m$, to the surface of the earth – $0.0025m$, to the source of pollutants – $0.001m$, to the upper wall (sky) – $0.0048m$. The total number of elements and the general view of the computational grid are presented in Table 1 and Figure 2.

To simulate the test problem, the following boundary conditions were adopted, which is shown in Figure 3. The total calculation time for speed is $1800s$ with $dt = 10s$ increments.

Numerical results were compared with measurement and computational values in three control lines. For the mean velocity – exactly in the middle of the considered region in the line $x = 0.5H$, and for concentrations in three lines – $x = 0.95H$, $x = 0.5H$, $x = 0.05H$.

As seen in Figure 4, several turbulence models were used for predicting mean velocity and

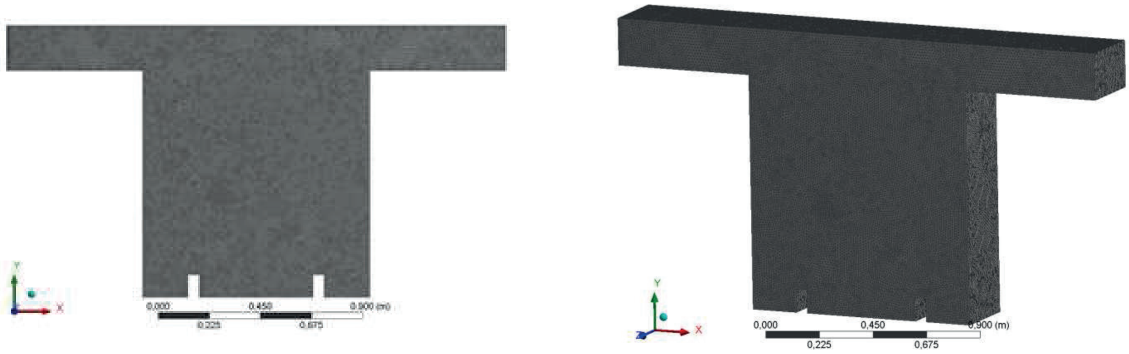


Figure 2: Computational grid

Table 1: Total number of elements and nodes for test cases

Variants of Test cases	**Elements**	**Nodes**
Without barrier	3491125	654944
Barrier 0.05 m	3033302	569837
Barrier 0.1 m	3103267	582834
Barrier 0.2 m	3242149	609064
Barrier 0.3 m	3379810	634887
Barrier 0.4 m	3519383	661186

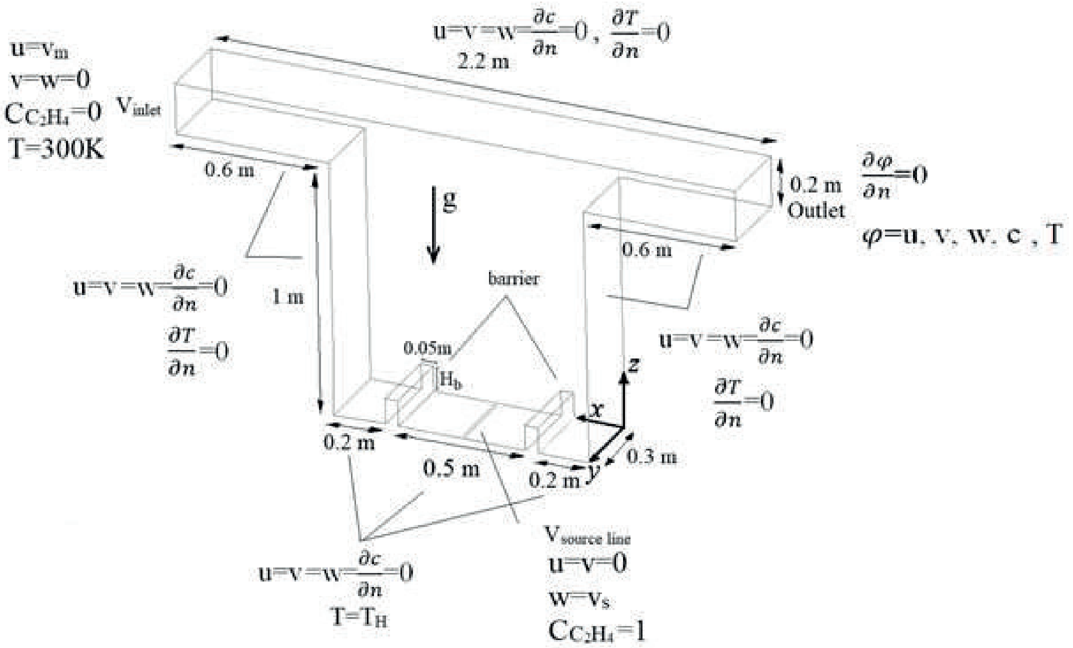


Figure 3: The schematic view of the cavity with herbal continuous barriers

concentrations. From the obtained results the $k-\varepsilon$ RNG turbulence model shows the values that are closest to reality for this problem. Thus, further this turbulence model will be used in order to obtain computational values as close to reality as possible.

The presence of barriers with various heights significantly affects the spread of pollutants. An increase in the barrier height leads to an increase in the concentration near the source ($x = 0.5H$); however, on the opposite side ($x = 0.95H$) it noticeably decreases. As can be seen from the obtained results, a continuous type of barrier with a height of $0.05H$ and $0.1H$, the spread of the pollutant proceeds clockwise and is distributed evenly over the entire specified area. At heights of $0.2H$, $0.3H$, and $0.4H$, the nature of the movement of the pollutant changes: the direction of propagation changes in the opposite direction (counterclockwise) until reaching the height of the barrier, then the nature of the movement changes again to clockwise movement. Barriers show the ability to retain a pollutant within the source area (between barriers).

The results show that a barrier with a height of $0.1H$ has a higher contaminant retention capacity compared to a barrier height of $0.05H$. However, low height barriers (e.g. $0.05H$) offer less air flow obstacle than higher barriers (e.g. $\geq= 0.1H$) due to the unobstructed path of removal of the substance emitted from the source in the middle of the region under study. Since a barrier with a height of $0.1H$ showed the best results, which show less concentration value, it was chosen to study the effect of temperatures on the behavior of the pollutant.

Figure 6 shows the effect of different temperatures for a barrier with a height of $0.1H$. The results obtained show that an increase in temperature near the building and roads surface in the urban settings leads to an increase in pollutants in the study area. In addition, the speed of movement of the pollutant differs depending on. Temperatures $301K$ and $303K$ and without temperature show the general nature of the flow movement, and at a temperature of $305K$ one can see how the distribution pattern of pollutants changes in the three control lines.

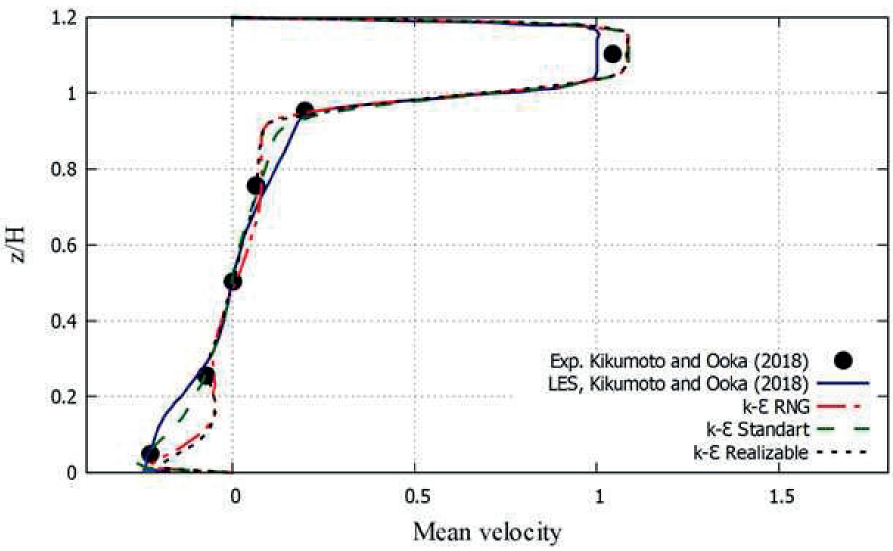
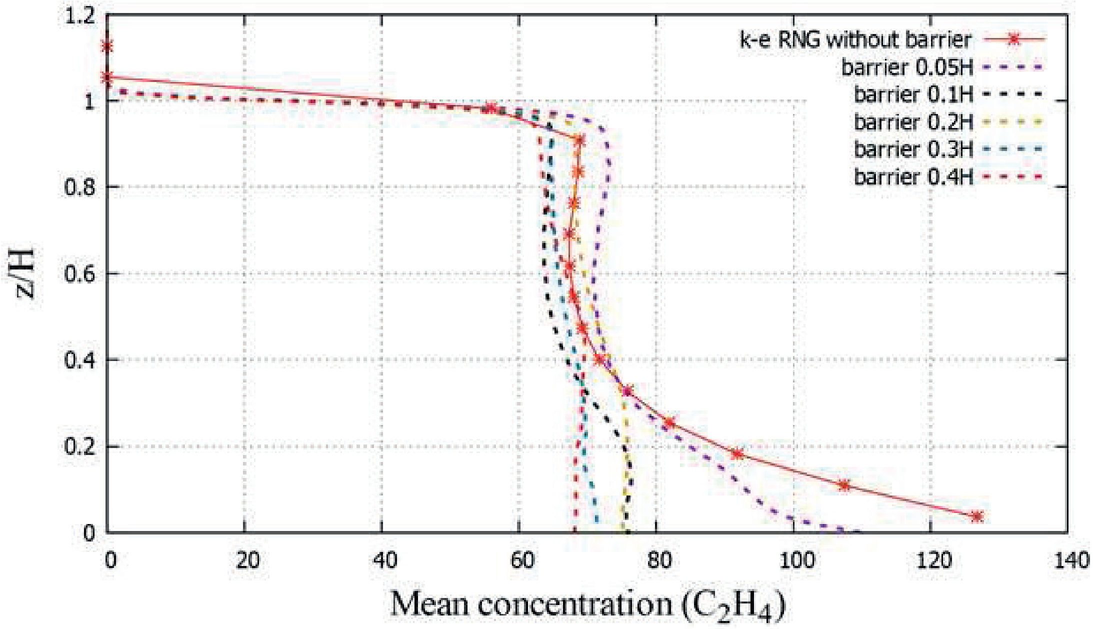
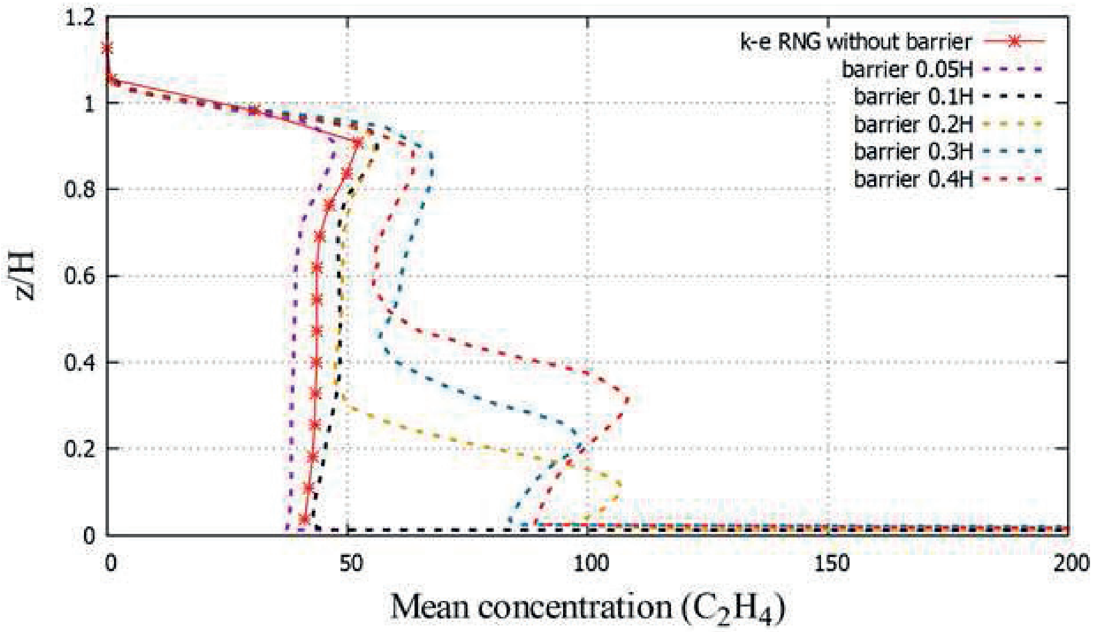


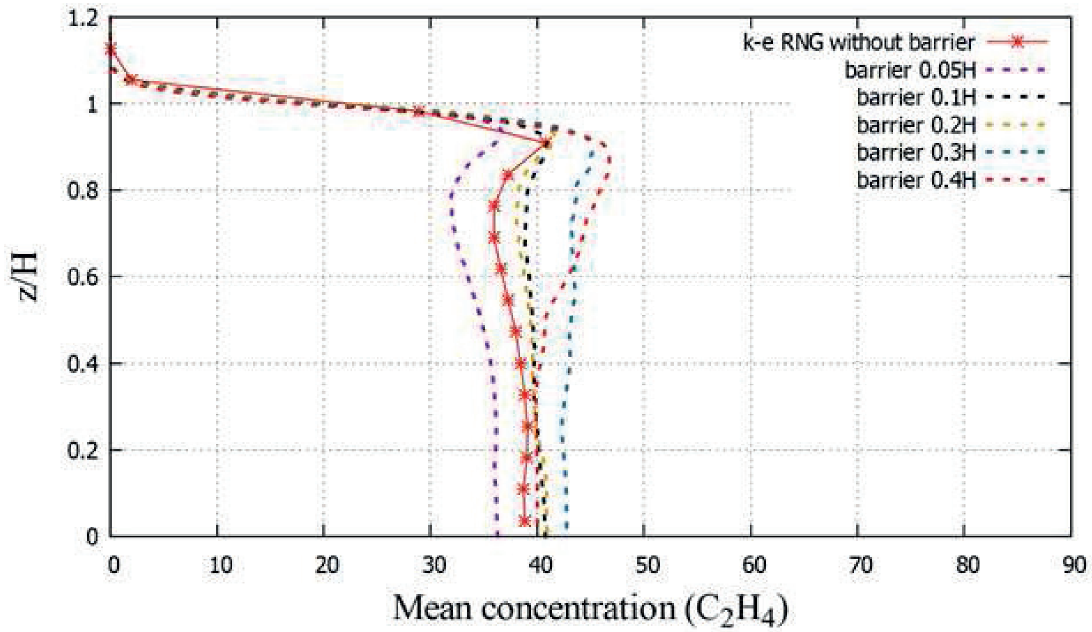
Figure 4: The mean velocity (U) profile at the $x = 0.5m$ and $y = 0.15m$ for various turbulence model



a) Line $x = 0.05H$

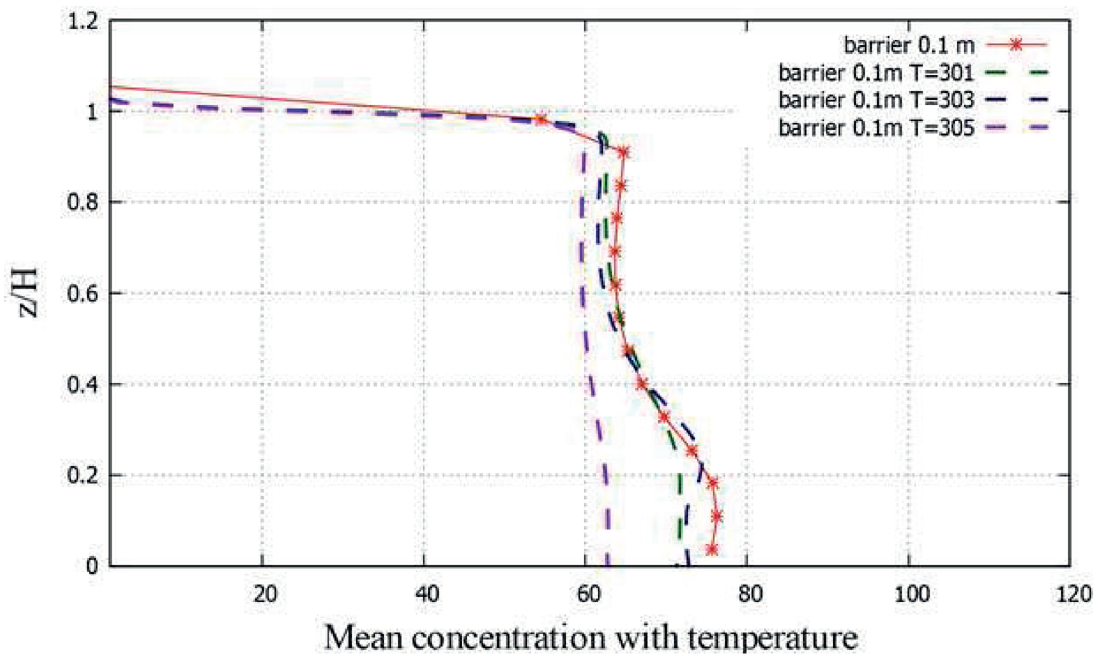


b) Line $x = 0.5H$

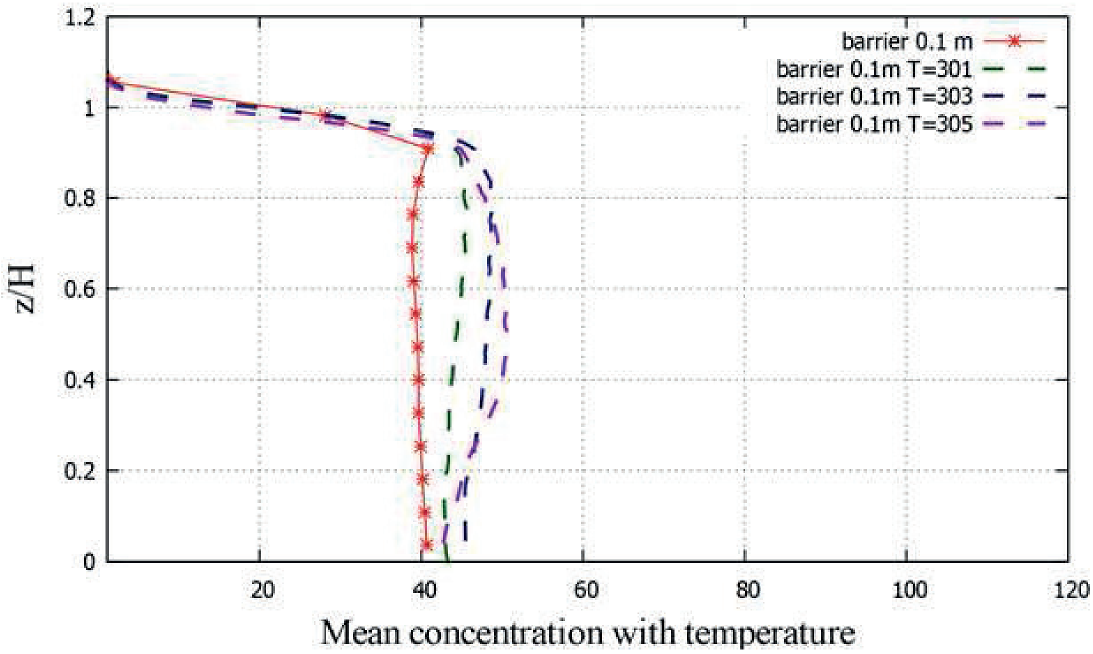


c) Line $x = 0.95H$

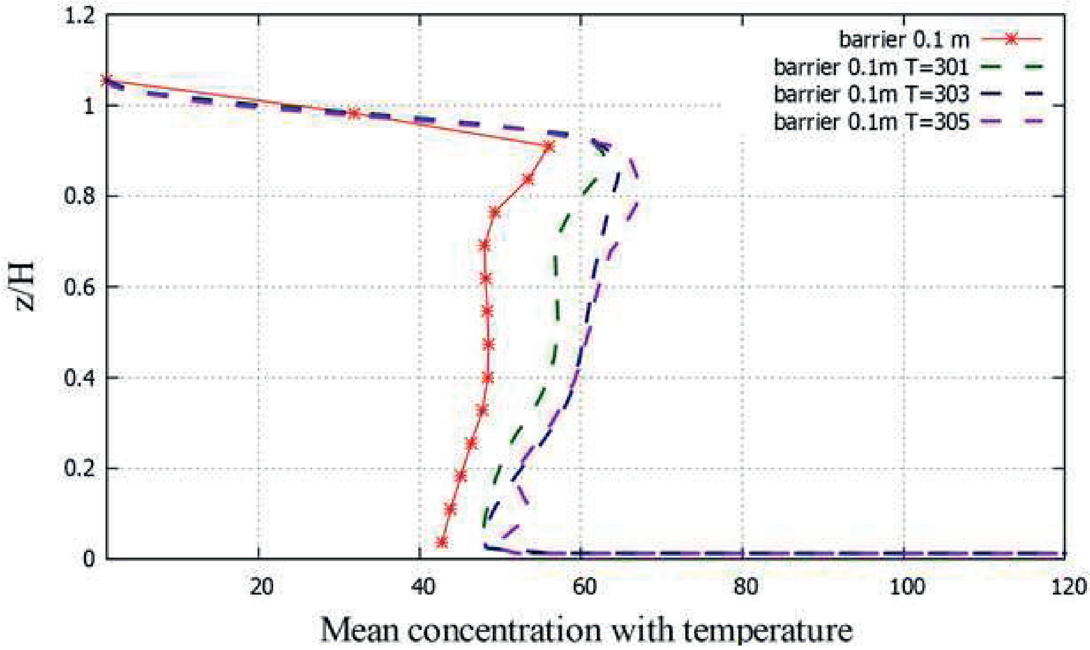
Figure 5: The mean concentration profiles without temperature



a) Line $x = 0.05H$



b) Line $x = 0.5H$



c) Line $x = 0.95H$

Figure 6: The mean concentration profiles with temperature

3 Conclusion

In the present study, several 3D models of a street canyon were built and analyzed. After carefully studying the influence of various mathematical models on the pollutants dispersion rate, a model was chosen that showed the best results when compared with experimental values in [32]. With the help of the chosen $k - \varepsilon$ RNG turbulence model, all subsequent problems were numerically solved: an urban canyon without any internal obstacle – a barrier; a canyon with a solid barrier on either side of the pollution source. The height of $0.1H$ was chosen as the most optimal height of the barrier, as a height with the properties of simultaneous retention of pollutants from the pedestrian zone and the properties of satisfactory ventilation of the whole region. This height was applied for a continuous type of barrier when analyzing the effect of temperature on the nature of changes in the pollutant flux in a given study area.

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