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Математика **Математика** **Mathematics**

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ON POTENTIAL THEORY FOR THE GENERALIZED BI-AXIALLY SYMMETRIC ELLIPTIC EQUATION IN THE PLANE

Fundamental solutions of the generalized biaxially symmetric elliptic equation are expressed in terms of the well-known Appel hypergeometric function in two variables, the properties of which are necessary for studying boundary value problems for the above equation. In this paper, using some properties of the Appel hypergeometric function, we prove limit theorems and derive integral equations for the double- and simple-layer potentials and apply the results of the constructed potential theory to the study of the Dirichlet problem for a two-dimensional elliptic equation with two singular coefficients in a domain bounded in the first quarter of the plane.
Key words: Appell hypergeometric function, generalized bi-axially symmetric elliptic equation, potential theory, Green's function, Dirichlet problem.

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Жазықтықтағы жалпыланған екі өске симметриялық эллиптикалық теңдеудің потенциалдық теориясы жайында

Жалпыланған екі өске симметриялық эллиптикалық теңдеудің іргелі шешімдері екі айнымалысы бар Аппелдің гипергеометриялық функциясы арқылы өрнектеледі, олардың қасиеттері жоғарыда келтірілген теңдеу үшін шекті есептерді зерттеу үшін қажет. Бұл жұмыста Аппелдің гипергеометриялық функциясының кейбір қасиеттерін қолдана отырып, біз қос қабатты және жай қабатты потенциалдардың тығыздығы үшін шекті теоремаларды дәлелдейміз және интегралдық теңдеулер аламыз. Құрылған потенциалдар теориясының нәтижелерін жазықтықтың бірінші ширегінде шектелген облыста екі сингулярлы коэффициенті бар екі өлшемді эллиптикалық теңдеу үшін Дирихле есебін зерттеуге қолданамыз.
Түйін сөздер: Аппелдің екі айнымалы гипергеометрияк функциясы, жәй қабатты және қос қабатты потенциалдар, Грин функциясы, іргелі шешім, Дирихле есебі.

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О теории потенциала для обобщенного двуосесимметричного эллиптического уравнения на плоскости

Фундаментальные решения обобщенного двуосесимметричного эллиптического уравнения выражаются через известную гипергеометрическую функцию Аппеля с двумя переменными, свойства которой необходимы для изучения краевых задач для указанного выше уравнения. В данной работе, используя некоторые свойства гипергеометрической функции Аппеля, доказываем предельные теоремы и выводим интегральные уравнения, касающиеся плотности потенциалов двойного и простого слоев. Применим результаты построенной теории потенциала к исследованию задачи Дирихле для двумерного эллиптического уравнения с двумя сингулярными коэффициентами в области, ограниченной в первой четверти плоскости.
Ключевые слова: гипергеометрическая функция Аппеля двух переменных, потенциалы двойного и простого слоев, функция Грина, фундаментальное решение, задача Дирихле.

1 Introduction

Numerous applications of simple- and double-layer potentials, as well as volumetric potentials, occur in fluid mechanics, elastodynamics, electromagnetizm, and acoustics [3]; therefore, the theory of potentials plays an important role in solving boundary value problems for elliptic equations. This, in particular, allows one to reduce the solution of boundary value problems to the solution of integral equations [1, 2].

For the first time S. Gellerstedt [4] constructed a potential theory and applied it to the solution of basic boundary value problems for the model Tricomi equation, i.e. for a two-dimensional elliptic equation with one singular coefficient of the form

$$u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x = 0,\ 0 < 2\alpha < 1,$$

which, later, was developed in the works of F.I.Frankl [5], S.P. Pulkin [6], M.M. Smirnov [7]. This line of research adjoin works [8–10].

The papers [11] and [12] are devoted to investigation of the double- and simple-layer potentials for a three-dimensional singular elliptic equation of the form

$$u_{xx} + u_{yy} + u_{zz} + \frac{2\alpha}{x}u_x = 0,\ 0 < 2\alpha < 1 \tag{1}$$

and solving the mixed problem and the Dirichlet problem for the equation (1) in a domain bounded in the half-space $x > 0$, respectively.

The authors of the papers [13, 14] constructed a potential theory for multidimensional elliptic equation with one singular coefficient

$$\sum_{k=1}^{m} u_{x_k x_k} + \frac{2\alpha}{x_1}u_{x_1} = 0,\ 0 < 2\alpha < 1,\ \ m \geq 2$$

in the domain bounded in a half-space $x_1 > 0$ and with the help of this theory, the solutions of the Dirichlet [13] and Holmgren problems [14] are obtained in forms convenient for further research.

On potential theory for an elliptic equation with two singular coefficients

$$E(u) \equiv u_{xx} + u_{yy} + \frac{2\alpha}{x}u_x + \frac{2\beta}{y}u_y = 0,\ 0 < 2\alpha,\ 2\beta < 1 \tag{2}$$

are devoted to relatively few works. In the works [15–18] the authors studied only the properties of the double-layer potentials for generalized biaxially symmetric elliptic equation (2).

In this paper, for the equation (2), we construct the theory potential and apply it to the solution of the Dirichlet problem in the domain bounded in the first quarter $R_2^{2+} := \{(x,y) : x > 0,\ y > 0\}$ of the xOy-plane.

2 Preliminaries

The Pochhammer symbol $(p)_n$ is defined by the equality

$$(p)_n = p(p+1)...(p+n-1),\ n = 1,2,...;\ (p)_0 \equiv 1. \tag{3}$$

The Gaussian hypergeometric function is defined inside the circle $|z| < 1$ as the sum of the hypergeometric series [19, Ch.2, eq. 2.1(2)]

$$F\left(a,b;c;z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k, \tag{4}$$

and for $|z| \geq 1$ is obtained by an analytic continuation of (4).

For the Gaussian hypergeometric function the summation formula [19, Ch.2, eq. 2.1(14)]

$$F\left(a,b;c;1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\ \ Re(c-a-b) > 0, \tag{5}$$

and Bolts's formula [19, Ch.2, eq. 2.1(22)]

$$F\left(a,b;c;z\right) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right) \tag{6}$$

are valid.

The Appel hypergeometric function of two variables has a form [19, Ch.5, eq. 5.7(7)]

$$F_2\left(a;b_1,b_2;c_1,c_2;x,y\right) \equiv$$
$$\equiv F_2\left[\begin{matrix} a, b_1, b_2; \\ c_1, c_2; \end{matrix}\ x, y\right] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_n} x^m y^n,\ |x| + |y| < 1,$$

where the parameters a, b_1, b_2, c_1, c_2 and variables x, y are arbitrary complex numbers and c_1, $c_2 \neq 0, -1, -2, ...$

We give some elementary relations for F_2 necessary in this study:

$$\begin{aligned}&\frac{\partial^{m+n}}{\partial x^m\partial y^n}F_2\left(a;b_1,b_2;c_1,c_2;x,y\right)=\\&\qquad=\frac{(a)_{m+n}(b_1)_m(b_2)_n}{m!n!(c_1)_m(c_2)_n}F_2\left[\begin{array}{l}a+m+n,b_1+m,b_2+n;\\c_1+m,c_2+n;\end{array}x,y\right],\end{aligned}\tag{7}$$

$$\begin{aligned}&\frac{b_1}{c_1}xF_2\left[\begin{array}{l}a+1,b_1+1,b_2;\\c_1+1,c_2;\end{array}x,y\right]+\frac{b_2}{c_2}yF_2\left[\begin{array}{l}a+1,b_1,b_2+1;\\c_1,c_2+1;\end{array}x,y\right]=\\&\qquad=F_2\left(a+1;b_1,b_2;c_1,c_2;x,y\right)-F_2\left(a;b_1,b_2;c_1,c_2;x,y\right),\end{aligned}\tag{8}$$

$$\begin{aligned}&F_2(a,b_1,b_2;c_1,c_2;x,y)=\\&\quad=(1-x-y)^{-a}F_2\left(a,c_1-b_1,c_2-b_2;c_1,c_2;\frac{x}{x+y-1},\frac{y}{x+y-1}\right).\end{aligned}\tag{9}$$

We note, that every point of the line $x+y=1$ is a logarithmic singularity of the function F_2.

Lemma 1 *[20]. If x and y are positive and $\alpha>0$, $\beta>0$, then*

$$F_2\left(\alpha+\beta,\alpha,\beta;2\alpha,2\beta;x,y\right)\sim-\frac{\Gamma(2\alpha)\Gamma(2\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta)}x^{-\alpha}y^{-\beta}\ln(1-x-y)\tag{10}$$

as $x+y\to 1-0$.

Let $c_1>b_1$, $c_2>b_2$ u $a+b_1+b_2=c_1+c_2$. If $x>0$, $y>0$, then

$$F_2\left(a,b_1,b_2;c_1,c_2;x,y\right)\sim-\frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)}x^{b_1-c_1}y^{b_2-c_2}\ln(1-x-y)\tag{11}$$

as $x+y\to 1-0$.

If $c_1+c_2<a+b_1+b_2$, then

$$\begin{aligned}F_2\left(a,b_1,b_2;c_1,c_2;x,y\right)\sim&\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(a+b_1+b_2-c_1-c_2)}{\Gamma(a)\Gamma(b_1)\Gamma(b_2)}\times\\&\times x^{b_1-c_1}y^{b_2-c_2}(1-x-y)^{c_1+c_2-a-b_1-b_2}.\end{aligned}\tag{12}$$

In addition, the fundamental solutions of the equation (2) are expressed in terms of the Appell hypergeometric function F_2, one of which has the form [21]:

$$\begin{aligned}q\left(x,y;\xi,\eta\right)=&\kappa r^{2\alpha+2\beta-4}x^{1-2\alpha}y^{1-2\beta}\xi^{1-2\alpha}\eta^{1-2\beta}\times\\&\times F_2\left(2-\alpha-\beta,1-\alpha,1-\beta;2-2\alpha,2-2\beta;\sigma_1,\sigma_2\right);\end{aligned}\tag{13}$$

where

$$\sigma_1=1-\frac{r_1^2}{r^2},\ \ \sigma_2=1-\frac{r_2^2}{r^2};\ \ r^2=\left(x-\xi\right)^2+\left(y-\eta\right)^2,$$

$$r_1^2 = (x+\xi)^2 + (y-\eta)^2 \,, \; r_2^2 = (x-\xi)^2 + (y+\eta)^2 \,,$$

$$\kappa = \frac{2^{4-2\alpha-2\beta}}{4\pi} \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(2-\alpha-\beta)}{\Gamma(2-2\alpha)\Gamma(2-2\beta)}.$$

The function $q\,(x, y; \xi, \eta)$ satisfies the equation by the variables (x, y), and by virtue of the formula (10), it has a logarithmic singularity at $r \to 0$ ($x > 0$, $y > 0$) and, therefore, the function $q\,(x, y; \xi, \eta)$ is a fundamental solution to the equation (2).

The fundamental solution given by (13) possesses the following potentially useful property:

$$q\,(x, y; \xi, \eta)|_{x=0} = q\,(x, y; \xi, \eta)|_{y=0} = 0. \tag{14}$$

3 Green's formula

We consider the following identity:

$$\begin{aligned} & x^{2\alpha} y^{2\beta} \left[uE(v) - vE(u) \right] = \\ & \qquad = y^{2\beta} \frac{\partial}{\partial x} \left[x^{2\alpha} \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) \right] + x^{2\alpha} \frac{\partial}{\partial y} \left[y^{2\beta} \left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) \right]. \end{aligned} \tag{15}$$

Integrating both sides of this identity in a domain D, which is located and bounded in the quarter-plane $x > 0$, $y > 0$, and using the Ostrogradsky formula, we obtain

$$\begin{aligned} & \int\!\!\int_D x^{2\alpha} y^{2\beta} \left[uE(v) - vE(u) \right] dxdy = \\ & \qquad = \int_\gamma x^{2\alpha} y^{2\beta} \left[- \left(u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial y} \right) dx + \left(u \frac{\partial v}{\partial x} - v \frac{\partial u}{\partial x} \right) dy \right], \end{aligned} \tag{16}$$

where γ is a contour of D.

The Green's formula (16) is derived under the following assumptions: (a) The functions $u(x, y)$ and $v(x, y)$, and their first-order derivatives, are continuous in the closed domain $\overline{D}$; (b) The second-order partial derivatives are continuous inside the domain D.

The integrals over D, consisting of $E(u)$ and $E(v)$, have a meaning. If $E(u)$ and $E(v)$ are not continuous up to S, then they are improper integrals obtained as limits on any sequence of domains D_n contained inside D when these domains D_n tend to D, so that any point in this D_n will be inside of D, starting with some number n.

If u and v are solutions of equation (2), then we find from formula (16) that

$$\int_\gamma \left(uA_n^{\alpha,\beta}[v] - vA_n^{\alpha,\beta}[u] \right) ds = 0, \tag{17}$$

where $A_n^{\alpha,\beta}[\;\;]$ is the conormal derivative with respect to (x, y):

$$A_n^{\alpha,\beta}[\;\;] \equiv x^{2\alpha} y^{2\beta} \left(\frac{dy}{ds} \frac{\partial}{\partial x} - \frac{dx}{ds} \frac{\partial}{\partial y} \right).$$

Here $\dfrac{dy}{ds}=\cos(n,x)$, $\dfrac{dx}{ds}=-\cos(n,y)$, n is the outer normal to the curve γ.

Assuming that $v\equiv 1$ in (16) and replacing u by u^2, we obtain

$$\int_D x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy=\int_\gamma uA_n^{\alpha,\beta}[u]ds, \tag{18}$$

where $u\,(x,y)$ is the solution of equation (2).

The special case of (17) when $v\equiv 1$ reduces to the following form:

$$\int_\gamma A_n^{\alpha,\beta}[u]ds=0. \tag{19}$$

We note from (19) that the integral of the conormal derivative of the solution of equation (2) along the boundary γ of the domain is equal to zero.

4 A double-layer potential

Let D be a domain bounded by two segments $[0,a]$ of the axes x and y, and a curve Γ with the ends at the points $A(a,0)$ and $B(0,a)$ lying in the quarter-plane $x>0$, $y>0$.

Let the parametric equation of the curve Γ be $x=x(s)$, $y=y(s)$, where s is the length of the arc measured from the point A. With respect to the curve Γ, we will assume that:

(i) the functions $x(s)$ and $y(s)$ have the continuous derivatives $x'(s)$ and $y'(s)$ on the segment $[0,l]$, which do not vanish simultaneously; the derivatives $x''(s)$ and $y''(s)$ satisfy the Holder condition on $[0,l]$, where l is the length of the curve Γ;

(ii) in a neighborhoods of the points A and B on the curve Γ the following conditions are satisfied

$$\left|\frac{dx}{ds}\right|\le C_1y(s),\ \left|\frac{dy}{ds}\right|\le C_2x(s), \tag{20}$$

respectively.

The coordinates of a variable point on the curve Γ will be denoted by (ξ,η).

We now consider the following integral:

$$w(x,y)=\int_0^l \mu\,(s)\,A_\nu^{\alpha,\beta}\left[q\,(\xi,\eta;x,y)\right]ds, \tag{21}$$

where $\mu(s)\in C\left(\overline{\Gamma}\right)$ and $q\,(\xi,\eta;x,y)$ is a fundamental solution of the equation (2) defined by (13). Here

$$A_\nu^{\alpha,\beta}[\ ]=\xi^{2\alpha}\eta^{2\beta}\left[cos(\nu,\xi)\cdot\frac{\partial[\ ]}{\partial\xi}+cos(\nu,\eta)\cdot\frac{\partial[\ ]}{\partial\eta}\right], \tag{22}$$

is the conormal derivative with respect to (ξ,η), ν is outer normal to the curve Γ.

Definition 1 . *We call the integral (21) a double-layer potential with density $\mu\,(s)$.*

In the study of the double layer potential (21), the conormal derivative of the fundamental solution $q(\xi,\eta;x,y)$ plays an important role. Applying successively the formula for the derivative of the Appel hypergeometric function (7) and the adjacent relation (8), taking into account (22), we obtain (for details, see [18]):

$$\begin{aligned}
A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x,y)\right] = &-(2-\alpha-\beta)\kappa\frac{1}{r^{4-2\alpha-2\beta}}x^{1-2\alpha}y^{1-2\beta}\xi^{1-2\alpha}\eta^{1-2\beta}\times \\
&\times F_2\left[\begin{matrix}3-\alpha-\beta,1-\alpha,1-\beta; \\ 2-2\alpha,2-2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]A_\nu^{\alpha,\beta}\left[\ln r^2\right]- \\
&-2(2-\alpha-\beta)\kappa\frac{x^{2-2\alpha}y^{1-2\beta}\xi\eta}{r^{6-2\alpha-2\beta}}F_2\left[\begin{matrix}3-\alpha-\beta,2-\alpha,1-\beta; \\ 3-2\alpha,2-2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\eta(s)}{ds}+ \\
&+2(2-\alpha-\beta)\kappa\frac{x^{1-2\alpha}y^{2-2\beta}\xi\eta}{r^{6-2\alpha-2\beta}}F_2\left[\begin{matrix}3-\alpha-\beta,1-\alpha,2-\beta; \\ 2-2\alpha,3-2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\xi(s)}{ds}+ \\
&+(1-2\alpha)\kappa\frac{x^{1-2\alpha}y^{1-2\beta}\eta}{r^{4-2\alpha-2\beta}}F_2\left[\begin{matrix}2-\alpha-\beta,1-\alpha,1-\beta; \\ 2-2\alpha,2-2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\eta(s)}{ds}- \\
&-(1-2\beta)\kappa\frac{x^{1-2\alpha}y^{1-2\beta}\xi}{r^{4-2\alpha-2\beta}}F_2\left[\begin{matrix}2-\alpha-\beta,1-\alpha,1-\beta; \\ 2-2\alpha,2-2\beta;\end{matrix}\ \sigma_1,\sigma_2\right]\frac{d\xi(s)}{ds}.
\end{aligned} \tag{23}$$

We introduce the notation:

$$w_1(x,y)\equiv\int_0^l A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x,y)\right]ds,$$

Lemma 2 . *The following formula holds true:*

$$w_1(x,y)=\begin{cases} i(x,y)-1, & (x,y)\in D, \\ i(x,y)-\dfrac{1}{2}, & (x,y)\in\Gamma, \\ i(x,y), & (x,y)\notin D\cup\Gamma,\end{cases} \tag{24}$$

where

$$\begin{aligned}
i(x,y)\equiv&(1-2\beta)\kappa x^{1-2\alpha}y^{1-2\beta}\int_0^a\frac{\xi F\left(2-\alpha-\beta,1-\alpha;2-2\alpha;\sigma_{10}\right)}{\left[(x-\xi)^2+y^2\right]^{2-\alpha-\beta}}d\xi+ \\
&+(1-2\alpha)\kappa x^{1-2\alpha}y^{1-2\beta}\int_0^a\frac{\eta F\left(2-\alpha-\beta,1-\beta;2-2\beta;\sigma_{20}\right)}{\left[x^2+(y-\eta)^2\right]^{2-\alpha-\beta}}d\eta,
\end{aligned}$$

$$\sigma_{10}=-\frac{4x\xi}{(x-\xi)^2+y^2},\quad \sigma_{20}=-\frac{4y\eta}{x^2+(y-\eta)^2}.$$

Proof. Lemma 2 was proved in [18].

Lemma 3 . *If $(x,y)\in\Gamma$, then*

$$\left|A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x,y)\right]\right|\le\frac{B_1}{r_1^{2\alpha}r_2^{2\beta}}\left(\ln\frac{r_1r_2}{r_{12}r}+1\right). \tag{25}$$

where B_1 is a constant.

Proof. The estimate (25) follows from the formula (23) and Lemma 1.

Lemma 4 . *If a curve Γ satisfies the conditions (i) and (ii), then the following inequality holds true:*

$$\int_0^l \left| A_\nu^{\alpha,\beta}[q(\xi,\eta;x,y)] \right| ds \le \frac{B_2}{x^\alpha y^\beta},$$

whrer B_2 is a constant.

Proof. Using the formula transformations (9), the conormal derivative $A_\nu^{\alpha,\beta}[q(\xi,\eta;x,y)]$, defined by the formula (23), can be represented as

$$A_\nu^{\alpha,\beta}[q(\xi,\eta;x,y)] = \sum_{i=0}^{4} P_i(s;x,y),$$

where

$$P_0(s;x,y) = -\kappa \frac{(2-\alpha-\beta)r^2}{r_{12}^{6-2\alpha-2\beta}} x^{1-2\alpha} y^{1-2\beta} \xi^{1-2\alpha} \eta^{1-2\beta} \times \\ \times F_2 \left[\begin{matrix} 3-\alpha-\beta, 1-\alpha, 1-\beta; \\ 2-2\alpha, 2-2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \left[\ln r^2 \right],$$

$$P_1(s;x,y) = -2(2-\alpha-\beta)\kappa \times \\ \times \frac{x^{2-2\alpha} y^{1-2\beta} \xi\eta}{r_{12}^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3-\alpha-\beta, 2-\alpha, 1-\beta; \\ 3-2\alpha, 2-2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\eta(s)}{ds},$$

$$P_2(s;x,y) = 2(2-\alpha-\beta)\kappa \times \\ \times \frac{x^{1-2\alpha} y^{2-2\beta} \xi\eta}{r_{12}^{6-2\alpha-2\beta}} F_2 \left[\begin{matrix} 3-\alpha-\beta, 1-\alpha, 2-\beta; \\ 2-2\alpha, 3-2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\xi(s)}{ds},$$

$$P_3(s;x,y) = (1-2\alpha)\kappa \times \\ \times \frac{x^{-2\alpha} y^{1-2\beta} \xi\eta}{r_{12}^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2-\alpha-\beta, 1-\alpha, 1-\beta; \\ 2-2\alpha, 2-2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\eta(s)}{ds},$$

$$P_4(s;x,y) = -(1-2\beta)\kappa \times \\ \times \frac{x^{1-2\alpha} y^{-2\beta} \xi\eta}{r_{12}^{4-2\alpha-2\beta}} F_2 \left[\begin{matrix} 2-\alpha-\beta, 1-\alpha, 1-\beta; \\ 2-2\alpha, 2-2\beta; \end{matrix} \bar{\sigma}_1, \bar{\sigma}_2 \right] \frac{d\xi(s)}{ds},$$

$$r_{12}^2 = (x+\xi)^2 + (y+\eta)^2, \ \ \bar{\sigma}_1 = \frac{4x\xi}{r_{12}^2}, \ \ \bar{\sigma}_2 = \frac{4y\eta}{r_{12}^2}, \ \ 0 \le \bar{\sigma}_1 + \bar{\sigma}_1 \le 1.$$

By virtue of (12), we obtain

$$\begin{aligned}\int_0^l |P_0(s;x,y)|ds \le C_2 \int_0^l \frac{x^{1-2\alpha}y^{1-2\beta}\xi\eta r^2}{r_{12}^{6-2\alpha-2\beta}}\times \\ \times\left(\frac{x\xi}{r_{12}^2}\right)^{\alpha-1}\left(\frac{y\eta}{r_{12}^2}\right)^{\beta-1}\left(\frac{r^2}{r_{12}^2}\right)^{-1}\left|\frac{\partial}{\partial\nu}\left(\ln\frac{1}{r}\right)\right|ds \le \\ \le \frac{C_2}{x^\alpha y^\beta}\int_0^l \xi^\alpha\eta^\beta\left|\frac{\partial}{\partial\nu}\left(\ln\frac{1}{r}\right)\right|ds \le \frac{C_3}{x^\alpha y^\beta}\int_0^l \frac{|cos\vartheta|}{r}ds,\end{aligned} \tag{26}$$

ϑ is an angle between r and outer normal ν to the curve Γ.

From the theory of the logarithmic potential we have

$$\int_0^l \frac{|cos\vartheta|}{r}ds < C_4. \tag{27}$$

Similarly we estimate $P_1(s;x,y)$ and $P_2(s;x,y)$:

$$\int_0^l |P_k(s;x,y)|\,ds \le \frac{D_k}{x^\alpha y^\beta}\ \ (k=1,2). \tag{28}$$

Now we will estimate $P_3(s;x,y)$ and $P_4(s;x,y)$. It is easy to see that

$$\int_{\varepsilon_k}^{l-\varepsilon_k} |P_k(s;x,y)|\,ds \le \frac{D_k}{x^\alpha y^\beta}\ \ (\varepsilon_k>0,\ k=3,4), \tag{29}$$

where D_3 and D_4 are independent of (x,y).

Integrals $\int_0^{\varepsilon_k} |P_k(s;x,y)|\,ds$ and $\int_{l-\varepsilon_k}^{l} |P_k(s;x,y)|\,ds$ are estimated similarly. Let us estimate the first of them for $k=3$. Using the estimate (11), taking into account the first of the conditions (20), we get

$$\int_0^{\varepsilon_3} |P_3(s;x,y)|\,ds \le \frac{E_1}{x^\alpha y^\beta}\int_0^{\varepsilon_3}\ln\left[\frac{r}{r_{12}}\right]ds \le \frac{E_2}{x^\alpha y^\beta}. \tag{30}$$

Thus, the obtained estimates (26) - (30) imply the validity of the Lemma 4.

Theorem 1 . *The following limit formulas hold true for a double-layer potential (21):*

$$\begin{aligned} w_i(s) = -\frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s,t)dt, \\ w_e(s) = \frac{1}{2}\mu(s) + \int_0^l \mu(t)K(s,t)dt, \end{aligned} \tag{31}$$

where

$$K(s,t) = A_\nu^{\alpha,\beta}\left[q\left(\xi(t),\eta(t);x(s),y(s)\right)\right].$$

$A_n^{\alpha,\beta}\left[w\left(x,y\right)\right]_i$ *and* $A_n^{\alpha,\beta}\left[w\left(x,y\right)\right]_e$ *are limiting values of the double-layer potential (21) at the point* $t\in\Gamma$ *from the inside and the outside, respectively.*

Proof. Theorem 1 follows from the Lemmas 2 and 4.

5 The simple-layer potential

In this section, we consider the following integral:

$$v(x,y) = \int_0^l \rho(t)q(\xi,\eta;x,y)dt, \tag{32}$$

where the density $\rho(t) \in C\left(\overline{\Gamma}\right)$ and $q(\xi,\eta;x,y)$ is given in (13). We call the integral (32) a simple-layer potential with density $\rho(t)$.

The simple-layer potential (32) is defined throughout the quarter-plane $x > 0$, $y > 0$ and is a continuous function when passing through the curve Γ. Obviously, a simple-layer potential is a regular solution of equation (2) in any domain lying in the quarter-plane $x > 0$, $y > 0$. It is easy to see that, as the point (x,y) tends to ∞ , a simple-layer potential $v(x,y)$ tends to 0. Indeed, we let the point (x,y) be on the quarter-circle given by C_R: $x^2+y^2 = R^2\,(x > 0,\ y > 0)$. Then, by virtue of (13), we have

$$|v(x,y)| \le \int_0^l |\rho(t)||q(\xi,\eta;x,y)|dt \le \frac{M}{R^2}, \tag{33}$$

where M is a constant.$(R \ge R_0)$.

We take an arbitrary point $N(x(x),y(s))$ on the curve Γ and draw a normal at this point. By considering on this normal any point $M(x,y)$, not lying on the curve Γ, we find the conormal derivative of the simple-layer potential (32):

$$A_n^{\alpha,\beta}\left[v(x,y)\right] = \int_0^l \rho(t)A_n^{\alpha\beta}\left[q(\xi,\eta;x,y)\right]dt, \tag{34}$$

where

$$A_n^{\alpha,\beta}[\ ] = x^{2\alpha}y^{2\beta}\left(cos(n,x)\cdot\frac{\partial}{\partial x} + cos(n,y)\cdot\frac{\partial}{\partial y}\right).$$

The integral in (34) exists also in the case when the point $M(x,y)$ coincides with the point N, which we mentioned above.

Theorem 2 . *The following limit formulas hold true for a simple-layer potential (32):*

$$\begin{aligned} A_n^{\alpha,\beta}\left[v(x,y)\right]_i &= \frac{1}{2}\rho(s) + \int_0^l \rho(t)K(t,s)dt, \\ A_n^{\alpha,\beta}\left[v(x,y)\right]_e &= -\frac{1}{2}\rho(s) + \int_0^l \rho(t)K(t,s)dt, \end{aligned} \tag{35}$$

where

$$K(t,s) = A_n^{\alpha,\beta}\left[q(\xi(t),\eta(t);x(s),y(s))\right].$$

$A_n^{\alpha,\beta}\left[v(x,y)\right]_i$ *and* $A_n^{\alpha,\beta}\left[v(x,y)\right]_e$ *are limiting values of the normal derivative of simple-layer potential (32) at the point* $t \in \Gamma$ *from the inside and the outside, respectively.*

Proof. Theorem 2 is proved in the same way as theorem 1.

Making use of these formulas, the jump in the normal derivative of the simple-layer potential follows immediately:

$$A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]_i - A_n^{\alpha\beta}\left[v\left(x,y\right)\right]_e = \rho(x,y). \tag{36}$$

For future researches on the subject of the present investigation, it will be useful to note that when the point (x, y) tends to ∞, the following inequality

$$\left|A_n^{\alpha,\beta}\left[v\left(x,y\right)\right]\right| \leq \frac{M}{R^{4-2\alpha-2\beta}}, \tag{37}$$

is valid, M is a constant $(R \geq R_0)$.

In exactly the same way as in the derivation of (18), it is not difficult to show that Green's formulas are applicable to the simple-layer potential (32) as follows:

$$\int\int_D x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dxdy = \int_\Gamma vA_n^{\alpha,\beta}\left[v\right]_i ds, \tag{38}$$

$$\int\int_{D'} x^{2\alpha}y^{2\beta}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]dxdy = -\int_\Gamma vA_n^{\alpha,\beta}\left[v\right]_e ds. \tag{39}$$

Hereinafter $D' = R_2^{2+} \setminus \overline{D}$ is the unbounded domain at $x > 0$, $y > 0$.

6 Integral Equations For Denseness

Formulas (31) and (35) can be written as the following integral equations for densities:

$$\mu(s) - \lambda\int_0^l K(s,t)\mu(t)dt = f(s), \tag{40}$$

$$\rho(s) - \lambda\int_0^l K(t,s)\rho(t)dt = g(s), \tag{41}$$

where

$$\lambda = 2, \quad f(s) = -2w_i(s), \ g(s) = -2A_n^{\alpha,\beta}\left[v\right]_e,$$

$$\lambda = -2, \quad f(s) = 2w_e(s), \ g(s) = 2A_n^{\alpha,\beta}\left[v\right]_i.$$

Equations (40) and (41) are mutually conjugated and, by Lemma 3, Fredholm theory is applicable to them. We show that $\lambda = 2$ is not an eigenvalue of the kernel $K\left(s, t\right)$. This assertion is equivalent to the fact that the homogeneous integral equation

$$\rho(s) - 2\int_0^l K(t, s)\rho(t)dt = 0, \tag{42}$$

has no non-trivial solutions.

Let $\tilde{\rho}\left(t\right)$ be a continuous non-trivial solution of the equation (42). The simple-layer potential with density $\tilde{\rho}\left(t\right)$ gives us a function $\tilde{v}\left(x, y\right)$, which is a solution of the equation (2) in the domains D and D'. By virtue of the equation (42), the limiting values of the normal derivative of $A_n^{\alpha,\beta}\left[\tilde{v}\right]_e$ are zero. The formula (39) is applicable to the simple-layer potential $\tilde{v}(x, y)$, from which it follows that $\tilde{v}(x, y) = \mathit{const}$ in domain D'. At infinity, a simple layer potential is zero, and consequently $\tilde{v}(x, y) \equiv 0$ in D', and also on the curve Γ. Applying now (38), we find that $\tilde{v}(x, y) \equiv 0$ is valid also inside the domain D. But then $A_n^{\alpha,\beta}\left[\tilde{v}\right]_i = 0$, and by virtue of formula (36) we obtain $\tilde{\rho}\left(t\right) \equiv 0$. Thus, clearly, the homogeneous equation (42) has only the trivial solution; consequently, $\lambda = 2$ is not an eigenvalue of the kernel $K\left(s; t\right)$.

7 The Uniqueness of the Solution of Dirichlet Problem

We apply the obtained results of potential theory to the solving the boundary value problem for the equation (2) in the domain D.

We consider the Dirichlet problem for equation (2) in the domain D defined in Section 4. We assume that the curve Γ satisfies conditions (i) and (ii) in Section 4.

Dirichlet problem. Find a regular solution $u(x, y)$ of equation (2) in the domain D that is continuous in the closed domain $\overline{D}$ and satisfies the following boundary conditions:

$$u|_{\Gamma} = \varphi(s)\ (0 \le s \le l), \tag{43}$$

$$\lim_{x\to 0} u(x, y) = \tau_1(y), \quad \lim_{y\to 0} u(x, y) = \tau_2(x)\ (0 \le x, y \le a), \tag{44}$$

where $\varphi(s)$ is given continuous function in $0 \le s \le l$; $\tau_1(y)$ and $\tau_2(x)$ are continuous functions at $0 \le x,\ y \le a$; $\tau_1(0) = \tau_2(0)$, $\tau_1(a) = \varphi(l)$, $\tau_2(a) = \varphi(0)$.

Theorem 3 . *If the Dirichlet problem has a regular solution, then it is unique.*

Proof. Consider the domain $D_{\varepsilon,\delta_1,\delta_2} \subset D$, bounded by the curve Γ_ε, parallel to the curve Γ, and line segments $x = \delta_1 > \varepsilon$ and $y = \delta_2 > \varepsilon$.

Integrating both sides of the identity (15) along the domain D_ε and using the Gauss-Ostrogradsky formula, we obtain

$$\int\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}\left[uE(v) - vE(u)\right]dxdy =$$
$$= \int_{S_{\varepsilon,\delta_1,\delta_2}} \left(uA_n^{\alpha,\beta}[v] - vA_n^{\alpha,\beta}[u]\right)dS_{\varepsilon,\delta_1,\delta_2},$$

where $S_{\varepsilon,\delta_1,\delta_2}$ is a contour of the domain $D_{\varepsilon,\delta_1,\delta_2}$.

One can easily check that the following equality holds:

$$\int\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}uE(u)dxdy = \int\int_{D_{\varepsilon,\delta_1,\delta_2}} x^{2\alpha}y^{2\beta}\left[u_x^2+u_y^2\right]dxdy-$$
$$-\int\int_{D_{\varepsilon,\delta_1,\delta_2}}\left[y^{2\beta}\frac{\partial}{\partial x}\left(x^{2\alpha}uu_x\right)+x^{2\alpha}\frac{\partial}{\partial y}\left(y^{2\beta}uu_y\right)\right]dxdy.$$

Application of the Ostrogradsky formula to this equality after $\delta_1\to 0$, $\delta_2\to 0$ and $\varepsilon\to 0$ yields

$$\int\int_D x^{2\alpha}y^{2\beta}\left[u_x^2+u_y^2\right]dxdy = -\int_\Gamma \varphi(s)A_n^{\alpha,\beta}[u]ds+$$
$$+\int_0^a x^{2\alpha}\frac{\partial u}{\partial x}\bigg|_{x=0}\cdot y^{2\beta}\tau_1(y)dy+\int_0^a y^{2\beta}\frac{\partial u}{\partial y}\bigg|_{y=0}\cdot x^{2\alpha}\tau_2(x)dx. \tag{45}$$

If we consider the homogeneous Dirichlet problem, then we find from (45):

$$\int\int_D x^{2\alpha}y^{2\beta}\left[u_x^2+u_y^2\right]dxdy = 0.$$

Hence, it follows that $u(x,y)=0$ in $\overline{D}$.

8 Green's Function Revisited

To solve this problem, we use the Green's function method. First, we construct the Green's function for solving the Dirichlet problem for an equation in a domain which is bounded by an arbitrary curve and two mutually perpendicular line segments. We then show that, in view of the Green's function, the solution of the Dirichlet problem in a quadrant takes a simpler form as described below.

Definition 2 . *We refer to $G(x,y;x_0,y_0)$ as Green's function of the Dirichlet problem, if it satisfies following conditions:*

1) The function $G(x,y;x_0,y_0)$ is a regular solution of equation (2) in the domain D, expect at the point (x_0,y_0), which is any fixed point of D.

2) The function $G(x,y;x_0,y_0)$ satisfies the boundary conditions given by

$$G(x,y;x_0,y_0)|_\Gamma = 0,\quad G(x,y;x_0,y_0)|_{x=0}=0,\quad G(x,y;x_0,y_0)|_{y=0}=0; \tag{46}$$

3)The function $G(x,y;x_0,y_0)$ can be represented as follows:

$$G(x,y;x_0,y_0) = q(x,y;x_0,y_0)+v(x,y;x_0,y_0) \tag{47}$$

where $q\left(x,y;x_0,y_0\right)$ is a fundamental solution of the equation (2), defined in the domain D, and the function $v(x,y;x_0,y_0)$ is a regular solution of the equation (2) in the domain D.

The construction of the Green's function $G(x,y;x_0,y_0)$ reduces to finding its regular part $v(x,y;x_0,y_0)$ which, by virtue of (14), (46) and (47), must satisfy the following boundary conditions:

$$v(x,y;x_0,y_0)|_\Gamma = -q(x,y;x_0,y_0)|_\Gamma\,, \tag{48}$$

$$v(x,y;x_0,y_0)|_{x=0} = 0, \quad v(x,y;x_0,y_0)|_{y=0} = 0.$$

We now look for the function $v(x,y;x_0,y_0)$ in the form of a double-layer potential given by

$$v\,(x,y;x_0,y_0) = \int_0^l \mu\,(t;x_0,y_0)\,A_\nu^{\alpha,\beta}[q\,(\xi,\eta;x,y)]dt. \tag{49}$$

By taking into account the equality (31) and the boundary condition (48), we obtain the integral equation for the density $\mu\,(t;x_0,y_0)$ as follows:

$$\mu\,(s;x_0,y_0) - 2\int_0^l K(s,t)\mu\,(t;x_0,y_0)\,dt = 2q\,(x(s),y(s);x_0,y_0)\,. \tag{50}$$

The right-hand side of (50) is a continuous function of s (the point (x_0,y_0) lies inside D). In Section 6, it was proved that $\lambda = 2$ is not an eigenvalue of the kernel $K(s,t)$ and, consequently, the Equation (50) is solvable and its continuous solution can be written in the following form:

$$\mu\,(s;x_0,y_0) = 2q\,(x(s),y(s);x_0,y_0) + 4\int_0^l R(s,t;2)q\,(\xi,\eta;x_0,y_0)\,dt, \tag{51}$$

where $R(s,t;2)$ is the resolvent of kernel $K(s,t)$; $(x(s),y(s)) \in \Gamma$. Thus, upon substituting from (51) into (49), we obtain

$$\begin{aligned} v\,(x,y;x_0,y_0) = 2\int_0^l q\,(\xi,\eta;x_0,y_0)\,A_\nu^{\alpha,\beta}[q\,(\xi,\eta;x,y)]dt+ \\ +4\int_0^l\int_0^l A_\nu^{\alpha,\beta}[q(\xi,\eta;x,y)]R_0(t,s;2)q\,(x(s),y(s);x_0,y_0)\,dtds. \end{aligned} \tag{52}$$

We now define the function $g(x,y)$ as follows:

$$g(x,y) = \begin{cases} v(x,y;x_0,y_0), & (x,y) \in D, \\ -q(x,y;x_0,y_0), & (x,y) \in D'. \end{cases} \tag{53}$$

The function $g(x,y)$ is a regular solution of equation (2) both inside the domain D and inside D' and equal to zero at infinity. Because the point (x_0,y_0) lies inside D, therefore, in D', the function $g(x,y)$ has derivatives of any order in all variables that are continuous up to Γ. We can consider $g(x,y)$ in D' as a solution of Equation (2) satisfying the boundary conditions given by

$$A_n^{\alpha,\beta}\left[g(x,y)\right]\big|_\Gamma = -A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right],$$

$$g(x,y)|_{x=0} = 0, \;\; g(x,y)|_{y=0} = 0.$$

We represent this solution in the form of a simple-layer potential as follows:

$$g(x,y) = \int_0^l \rho(t;x_0,y_0)q(\xi,\eta;x,y)dt, \;\; (x,y) \in D' \tag{54}$$

with an unknown density $\rho(t;x_0,y_0)$.

Using the formula (35), we obtain the following integral equation for the density $\rho(s;x_0,y_0)$:

$$\rho(s;x_0,y_0) - 2\int_0^l K(t,s)\rho(t;x_0,y_0)dt = 2A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right]. \tag{55}$$

Equation (55) is conjugated with the equation (50). Its right-hand side is a continuous function of s. Thus, clearly, the equation (55) has the following continuous solution:

$$\begin{aligned} \rho(s;x_0,y_0) = 2A_n^{\alpha,\beta}\left[q(x(s),y(s);x_0,y_0)\right] + \\ + 4\int_0^l R(t,s;2)A_\nu^{\alpha,\beta}\left[q(\xi,\eta;x_0,y_0)\right]dt. \end{aligned} \tag{56}$$

The values of a simple-layer potential $g(x,y)$ on the curve Γ are equal to $-q(x,y;x_0,y_0)$, that is, just as the functions $v(x,y;x_0,y_0)$ and on the axes x and y their partial derivatives with respect to y and x multiplied, respectively, by $y^{2\beta}$ and $x^{2\alpha}$ are equal to zero. Hence, by virtue of the uniqueness theorem for the Dirichlet problem, it follows that the formula (54) for the function $g(x,y)$ defined by (53) holds throughout in the quarter-plane $x \geq 0$, $y \geq 0$, that is,

$$v(x,y;x_0,y_0) = \int_0^l \rho(t;x_0,y_0)q(\xi,\eta;x,y)dt, \;\; (x,y) \in D. \tag{57}$$

Thus, the regular part $v(x,y;x_0,y_0)$ of Green's function is representable in the form of a simple-layer potential.

Applying the formula (35) to (57), we obtain

$$2A_n^{\alpha,\beta}\left[v\left(x(s),y(s);x_0,y_0\right)\right]_i = \rho(s;x_0,y_0) + 2\int_0^l K(t,s)\rho(t;x_0,y_0)dt,$$

But, according to (55), we have

$$2A_n^{\alpha,\beta}\left[q\left(x(s),y(s);x_0,y_0\right)\right]_i = \rho(s;x_0,y_0) - 2\int_0^l K(t,s)\rho(t;x_0,y_0)dt.$$

Summing the last two equalities by term-by-term and taking equation (47) into account, we find that

$$A_n^{\alpha,\beta}\left[G\left(x(s),y(s);x_0,y_0\right)\right] = \rho(s;x_0,y_0). \tag{58}$$

Consequently, formula (57) can be written in the following form:

$$v(x,y;x_0,y_0) = \int_0^l A_\nu^{\alpha,\beta} \left[G\left(\xi,\eta;x_0,y_0\right) \right] q(\xi,\eta;x,y)dt.$$

Multiplying both sides of (56) by $q\left(x(s),y(s);x,y\right)$, integrating by s over the curve Γ from 0 to l and, by virtue of (51) and (49), we obtain

$$v(x_0,y_0;x,y) = \int_0^l \rho(t;x_0,y_0) q(\xi,\eta;x,y)dt.$$

Comparing this last equation with the formula (57), we have

$$v(x,y;x_0,y_0) = v(x_0,y_0;x,y). \tag{59}$$

if the points (x,y) and (x_0,y_0) are inside the domain D.

Lemma 5 . *If points (x,y) and (x_0,y_0) are inside domain D, then Green's function $G(x,y;x_0,y_0)$ is symmetric about those points.*

Proof. The proof of Lemma 5 follows from the representation (47) of Green's function and the equality (59).

For a quarter circle D_0 bounded by two segments $[0,a]$ of the axes x and y and a quarter circle given by $x^2+y^2=a^2$ $(x \geq 0,\ y \geq 0)$, the Green's function of the Dirichlet problem has the following form

$$G_0(x,y;x_0,y_0) = q(x,y;x_0,y_0) - \left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x,y;\bar{x}_0,\bar{y}_0), \tag{60}$$

where

$$R^2 = x_0^2 + y_0^2,\ \ \bar{x}_0 = \frac{a^2}{R^2}x_0,\ \ \bar{y}_0 = \frac{a^2}{R^2}y_0.$$

We now show that the function given by

$$v_0(x,y;x_0,y_0) = -\left(\frac{a}{R}\right)^{2\alpha+2\beta} q(x,y;\bar{x}_0,\bar{y}_0)$$

can be represented in the following form:

$$v_0(x,y;x_0,y_0) = -\int_0^l \rho(s;x,y) v_0(x(s),y(s);x_0,y_0)ds, \tag{61}$$

where $\rho(s;x,y)$ is a solution of equation (57).

Indeed, by letting an arbitrary point (x_0,y_0) be inside the domain D, we consider the function given by

$$u(x,y;x_0,y_0) = -\int_0^l \rho(s;x,y) v_0(x(s),y(s);x_0,y_0)ds.$$

As a function of (x, y), the function $u\left(x, y; x_0, y_0\right)$ satisfies equation (2), because this equation is satisfied by the function $\rho(s; x, y)$. Substituting the expression (56) for $\rho(s; x, y)$, we obtain

$$u(x, y; x_0, y_0) = -\int_0^l \psi(s; x_0, y_0) A_n^{\alpha,\beta}\left[q(x(s), y(s); x, y)\right] ds, \tag{62}$$

where

$$\psi(s; x_0, y_0) = 2v_0\left(x(s), y(s); x_0, y_0\right) + 4\int_0^l R(s, t; 2) v_0\left(\xi, \eta; x_0, y_0\right) dt,$$

that is, $\psi(s; x_0, y_0)$ is a solution of the integral equation

$$\psi(s; x_0, y_0) - 2\int_0^l K(s, t)\psi(t; x_0, y_0) dt = 2v_0\left(x(s), y(s); x_0, y_0\right). \tag{63}$$

Applying formula (31) to the double-layer potential (62), we obtain

$$u_i\left(x(s), y(s); x_0, y_0\right) = \frac{1}{2}\psi(s; x_0, y_0) - \int_0^l K(s, t)\psi(t; x_0, y_0) dt,$$

whence, by virtue of (63) we get

$$u_i\left(x(s), y(s); x_0, y_0\right) = v_0\left(x(s), y(s); x_0, y_0\right), \ (x(s), y(s)) \in \Gamma.$$

It is easy to see that

$$u\left(x, y; x_0, y_0\right)\big|_{x=0} = 0, \quad v_0\left(x, y; x_0, y_0\right)\big|_{x=0} = 0,$$

$$u\left(x, y; x_0, y_0\right)\big|_{y=0} = 0, \quad v_0\left(x, y; x_0, y_0\right)\big|_{y=0} = 0.$$

Thus, clearly, the functions $u\left(x, y; x_0, y_0\right)$ and $v_0\left(x, y; x_0, y_0\right)$ satisfy the same equation (2) and the same boundary conditions. Also, by virtue of the uniqueness of the solution of the Dirichlet problem, the equality

$$u\left(x, y; x_0, y_0\right) \equiv v_0\left(x, y; x_0, y_0\right).$$

is satisfied.

Now, subtracting the expression (60) from (47), we obtain

$$\begin{aligned} H\left(x, y; x_0, y_0\right) &= G\left(x, y; x_0, y_0\right) - G_0\left(x, y; x_0, y_0\right) = \\ &= v\left(x, y; x_0, y_0\right) - v_0\left(x, y; x_0, y_0\right) \end{aligned}$$

or, by virtue of(57), (59), (60) and (61), we obtain

$$H(x, y; x_0, y_0) = \int_0^l \rho(t; x, y) G_0(\xi, \eta; x_0, y_0) dt. \tag{64}$$

Solving the Dirichlet Problem for Equation (2)

Theorem 4 . *The following function*

$$
\begin{aligned}
u\left(x_0, y_0\right) = & \int_0^a y^{2\beta}\left(x^{2\alpha} \frac{\partial G\left(x, y ; x_0, y_0\right)}{\partial x}\right)\Bigg|_{x=0} \tau_1(y) d y+ \\
& +\int_0^a x^{2\alpha}\left(y^{2\beta} \frac{\partial G\left(x, y ; x_0, y_0\right)}{\partial y}\right)\Bigg|_{y=0} \tau_2(x) d x- \\
& -\int_0^l A_\nu^{\alpha, \beta}\left[G\left(\xi, \eta ; x_0, y_0\right)\right] \varphi(s) d s \\
= & I_1(x_0, y_0)+I_2(x_0, y_0)+I_3(x_0, y_0),
\end{aligned}
\tag{65}
$$

where $\varphi(s)$ is given continuous function in $0 \leq s \leq l$; $\tau_1(y)$ and $\tau_2(x)$ are given continuous functions in $0 \leq x,\, y \leq a$ with $\tau_1(0)=\tau_2(0)$, $\tau_1(a)=\varphi(l)$, $\tau_2(a)=\varphi(0)$, is the solution of the Dirichlet problem for equation (2) in the domain D.

Proof. Let (x_0, y_0) be a point inside the domain D. Consider the domain $D_{\varepsilon,\delta_1,\delta_2} \subset D$ bounded by the curve Γ_ε, which is parallel to the curve Γ, and the line segments $x=\delta_1>\varepsilon$ and $y=\delta_2>\varepsilon$.

We choose ε, δ_1 and δ_2 to be so small that the point (x_0, y_0) is inside $D_{\varepsilon,\delta_1,\delta_2}$. We cut out from the domain $D_{\varepsilon,\delta_1,\delta_2}$ a circle of small radius ρ with center at the point (x_0, y_0), and we denote the remainder part of $D_{\varepsilon,\delta_1,\delta_2}$ by $D^\rho_{\varepsilon,\delta}$, in which the Green's function $G(x, y; x_0, y_0)$ is a regular solution of equation (2).

Let $u(x, y)$ be a regular solution of the equation (2) in the domain D that satisfies the boundary conditions (43) and (44). Applying the formula (17), we obtain

$$
\begin{aligned}
& \int_{C_\rho}\left(G A_n^{\alpha, \beta}[u]-u A_n^{\alpha, \beta}[G]\right) d s=\int_{\delta_2}^{y_1} x^{2\alpha} y^{2\beta}\left(u \frac{\partial G}{\partial x}-G \frac{\partial u}{\partial x}\right)\Bigg|_{x=\delta_1} d y+ \\
& +\int_{\delta_1}^{x_1} x^{2\alpha} y^{2\beta}\left(u \frac{\partial G}{\partial y}-G \frac{\partial u}{\partial y}\right)\Bigg|_{y=\delta_2} d x+\int_{\Gamma_\varepsilon}\left(G A_n^{\alpha, \beta}[u]-u A_n^{\alpha, \beta}[G]\right) d s,
\end{aligned}
$$

x_1 and y_1 are an abscissa and ordinate of the intersection points of the curve Γ_ε with the straight lines $y=\delta_2$ and $x=\delta_1$, respectively, and C_ρ is a circumference of the cut circle.

Proceeding to the limit as $\rho \to 0$ and then as $\varepsilon \to 0$, $\delta_1 \to 0$ and $\delta_2 \to 0$, we obtain the formula (65).

We show that the formula (65) gives a solution of the Holmgren problem.

It is easy to see that the first integral $I_1\left(x_0, y_0\right)$ in the formula (65) is a solution of the equation (2) and is regular in the domain D, continuous in $\overline{D}$.

We use the following notation:

$$
\begin{aligned}
\vartheta(x_0, y_0)= & \int_0^a y^{2\beta}\left(x^{2\alpha} \frac{\partial q\left(x, y ; x_0, y_0\right)}{\partial x}\right)\Bigg|_{x=0} \tau_1(y) d y=(1-2\alpha) \kappa \times \\
& \times x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \frac{y F\left(\beta-\alpha, 1-\beta ; 2-2\beta ; \dfrac{4 y y_0}{x_0^2+\left(y+y_0\right)^2}\right)}{\left[x_0^2+\left(y-y_0\right)^2\right]^{1-\alpha}\left[x_0^2+\left(y+y_0\right)^2\right]^{1-\beta}} \tau_1(y) d y .
\end{aligned}
\tag{66}
$$

Here, $\vartheta(x_0, y_0)$ is a continuous function in $\overline{D}$. In view of (66) and (52) and the symmetry of the function $v(x, y; x_0, y_0)$, the integral $I_1(x_0, y_0)$ can be represented in the following form:

$$
\begin{aligned}
I_1(x_0,y_0) =& \vartheta(x_0,y_0) + 2\int_0^l \vartheta(\xi,\eta)\, A_\nu^{\alpha,\beta}[q(\xi,\eta;x_0,y_0)]dt+ \\
&+4\int_0^l\int_0^l R(t,s;2)\,\vartheta(x(s),y(s))A_\nu^{\alpha,\beta}[q(\xi,\eta;x_0,y_0)]dtds.
\end{aligned}
\tag{67}
$$

The last two integrals in the formula (67) are double-layer potentials. Taking into account the formula (31) and the integral equation for the resolvent $R(s,t;2)$ from formula (67), we obtain

$$
I_1(x_0,y_0)|_\Gamma = 0,
$$

It is easy to see that

$$
\lim_{x_0\to 0} u(x,y) = \tau_1(y_0) \quad (0 \le y_0 \le a).
$$

In fact, by virtue of (57) and the symmetry of the function $v(x,y;x_0,y_0)$, the above integral can also be written in the following form:

$$
\begin{aligned}
I_1(x_0,y_0) =& \int_0^a \tau_1(y) q(0,y;x_0,y_0)dy+ \\
&+\int_0^a \tau_1(y)dy\int_0^l \rho(t;0,y)q(\xi,\eta;x_0,y_0)dt.
\end{aligned}
$$

Following the work [7], it is easy to show that

$$
\lim_{x_0\to 0}\int_0^a \tau_1(y)q(0,y;x_0,y_0)dy = \tau(y_0) \quad (0\le y_0\le a)
$$

and

$$
\lim_{x_0\to 0}\int_0^a \tau_1(y)dy\int_0^l \rho(t;0,y)q(\xi,\eta;x_0,y_0)dt = 0 \quad (0\le y_0\le a),
$$

because

$$
q(\xi,\eta;x_0,y_0) = 0
$$

when $x_0 = 0,\ 0\le y_0\le a$.

By virtue of the last from the conditions (46), we have

$$
\lim_{y_0\to 0} u(x,y) = 0 \quad (0\le x_0\le a).
$$

Similarly, we get

$$
I_2(x_0,y_0)|_\Gamma = 0;\ \ \lim_{x_0\to 0} I_2(x_0,y_0) = 0,\ \ \lim_{y_0\to 0} I_2(x_0,y_0) = \tau_2(x_0).
$$

We consider the third integral $I_3(x_0, y_0)$ in the formula (65), which, by virtue of (58) and (56), can be written in the following form:

$$I_3(x_0, y_0) = -\int_0^l \varphi(s)\rho(s; x_0, y_0)ds = -\int_0^l \theta(t)A_\nu^{\alpha,\beta}\left[q(\xi, \eta; x_0, y_0)\right]dt,$$

where

$$\theta(t) = 2\varphi(t) + 4\int_0^l R(t, s; 2)\varphi(s)ds,$$

that is, the function $\theta(s)$ is a solution of the integral equation

$$\theta(s) - 2\int_0^l K(s, t)\theta(t)dt = 2\varphi(s). \tag{68}$$

Because $\theta(s)$ is a continuous function, $I_3(x_0, y_0)$ is a solution of Equation (2), regular in the domain D, that is continuous in $\overline{D}$, which, by virtue of (31) and (68), satisfies following condition:

$$I_3\left(x_0, y_0\right)\big|_\Gamma = \varphi(s).$$

It is now easy to see that

$$\lim_{x_0\to 0} I_3\left(x_0, y_0\right) = 0 \ \ (0 \le y_0 \le a), \quad \lim_{y_0\to 0} I_3\left(x_0, y_0\right) = 0 \ \ (0 \le x_0 \le a).$$

Theorem 4 is proved.

By using formulas (64) and (60), solution (65) of the Dirichlet problem given by (43) and (44) for Equation (2) can be written in the following form:

$$\begin{aligned} &u\left(x_0, y_0\right) = \\ &= \int_0^a \tau_1(y)y^{2\beta}\cdot x^{2\alpha}\frac{\partial}{\partial x}\left[G_0\left(x, y; x_0, y_0\right) + H\left(x, y; x_0, y_0\right)\right]\Big|_{x=0} dy + \\ &+ \int_0^a \tau_2(x)x^{2\alpha}\cdot y^{2\beta}\frac{\partial}{\partial y}\left[G_0\left(x, y; x_0, y_0\right) + H\left(x, y; x_0, y_0\right)\right]\Big|_{y=0} dx - \\ &- \int_0^l \varphi(s)\left\{A_\nu^{\alpha,\beta}\left[G_0\left(\xi, \eta; x_0, y_0\right)\right] + A_\nu^{\alpha,\beta}\left[H\left(\xi, \eta; x_0, y_0\right)\right]\right\}ds, \end{aligned} \tag{69}$$

where

$$H\left(x, y; x_0, y_0\right) = \int_0^l \rho_0\left(t; x_0, y_0\right)G_0\left(\xi, \eta; x, y\right)dt.$$

We remark that solution (69) of the Dirichlet problem is more convenient for further investigations.

In the case of a quarter circle D_0, the function $H(x,y;x_0,y_0) \equiv 0$ and solution (69) assumes a simpler form as follows:

$$
\begin{aligned}
&u(x_0,y_0) = \\
&= (1-2\alpha)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \tau_1(y) y \left[\frac{\tilde{F}_1\left(-\frac{4yy_0}{X_1^2}\right)}{X_1^{4-2\alpha-2\beta}} - \frac{\tilde{F}_1\left(-\frac{4yy_0}{Y_1^2}\right)}{Y_1^{4-2\alpha-2\beta}} \right] dy + \\
&+ (1-2\beta)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^a \tau_2(x) x \left[\frac{\tilde{F}_2\left(-\frac{4xx_0}{X_2^2}\right)}{X_2^{4-2\alpha-2\beta}} - \frac{\tilde{F}_2\left(-\frac{4xx_0}{Y_2^2}\right)}{Y_2^{4-2\alpha-2\beta}} \right] dx - \qquad (70) \\
&+ 2(2-\alpha-\beta)\kappa x_0^{1-2\alpha} y_0^{1-2\beta} \int_0^l \varphi(s)\xi(s)\eta(s) \frac{R^2-a^2}{r_{12}^{6-2\alpha-2\beta}} \times \\
&\times F_2\left(3-\alpha-\beta, 1-\alpha, 1-\beta; 2-2\alpha, 2-2\beta; \frac{r_1^2-r^2}{r_{12}^2}, \frac{r_2^2-r^2}{r_{12}^2}\right) ds,
\end{aligned}
$$

where

$$
\tilde{F}_1(z) = F(2-\alpha-\beta, 1-\beta; 2-2\beta; z), \;\; \tilde{F}_2(z) = F(2-\alpha-\beta, 1-\alpha; 2-2\alpha; z);
$$

$$
\begin{aligned}
&R^2 = x_0^2 + y_0^2, \;\; a^2 = \xi^2 + \eta^2; \;\; r^2 = (\xi - x_0)^2 + (\eta - y_0)^2, \\
&r_1^2 = (\xi + x_0)^2 + (\eta - y_0)^2, \;\; r_2^2 = (\xi - x_0)^2 + (\eta + y_0)^2;
\end{aligned}
$$

$$
\begin{aligned}
&X_1^2 = x_0^2 + (y - y_0)^2, \;\; Y_1^2 = \left(a - \frac{yy_0}{a}\right)^2 + \frac{y^2}{a^2} x_0^2; \\
&X_2^2 = (x - x_0)^2 + y_0^2, \;\; Y_2^2 = \left(a - \frac{xx_0}{a}\right)^2 + \frac{x^2}{a^2} y_0^2.
\end{aligned}
$$

The resulting explicit integral representations (69) and (70) play an important role in the study of problems for equation of the mixed type (that is, elliptic-hyperbolic or elliptic-parabolic types): they make it easy to derive the basic functional relationship between the traces of the sought solution and of its derivative on the line of degeneration from the elliptic part of the mixed domain.

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MAXIMAL REGULARITY ESTIMATE FOR A DIFFERENTIAL EQUATION WITH OSCILLATING COEFFICIENTS

The paper considers a second-order differential equation with unbounded coefficients. Sufficient summability conditions with the weight of the solution and its derivatives up to second order are obtained. The equation studied is singular as it is defined in an infinite domain, and its coefficients may be unbounded. Its main feature is the rapid growth of the coefficient at of the first derivative of the solution required, therefore the well-developed theory of the Sturm-Liouville equations is not applicable. The equation studied and its multidimensional generalizations arise in the modeling of the Brownian motion of particles, in problems of biology and financial mathematics. Their well-known representatives are the Ornstein-Uhlenbeck and Fokker-Planck-Kolmogorov equations, which have been actively studied since the first half of the twentieth century. On the other hand, projection methods are well known in applications (e.g., Fourier or Laplace transforms), which reduce partial differential equations with coefficients depending on one variable to ordinary differential equations. Therefore, the present study is important for partial derivative equations with unbounded coefficients. In contrast to previous works, the senior and intermediate coefficients of the equation studied can be strongly fluctuating. In the proof of the main theorems, the authors use their earlier result on the correct solvability of the mentioned equation.

Key words: second order differential equation, linear differential equation, differential equation in an unbounded domain, maximal regularity, oscillating coefficients.

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Тербелмелі коэффициентті бір дифференциалдық теңдеу үшін максималды регулярлық бағасы

Жұмыста коэффициенттері шенелмеген екінші ретті дифференциалдық теңдеу қарастырылған. Шешім мен оның екінші ретке дейінгі туындыларының салмақпен қосындылануы үшін жеткілікті шарттар алынады. Зерттелген теңдеу сингулярлы, өйткені ол шексіз облыста берілген, ал оның коэффициенттері шенелмеген болуы мүмкін. Оның басты ерекшелігі - шешімнің бірінші ретті туындысы алдындағы коэффициенттің жылдам өсуінде жатыр, соның әсерінен Штурм-Лиувилль теңдеулерінің дамыған теориясын қолдану мүмкін емес. Зерттелген теңдеу мен оның көп өлшемді жалпылаулары бөлшектердің броундық қозғалысын модельдеу кезінде, биология және қаржылық математика мәселелерінде туындайды. Олардың белгілі өкілдері - XX ғасырдың бірінші жартысынан бастап белсенді түрде зерттеліп келе жатқан Орнштейн-Уленбек және Фоккер-Планк-Колмогоров теңдеулері. Екінші жағынан, проекциялық әдістерді қолданып (мысалы, Фурье немесе Лаплас түрлендірулерін) коэффициенттері бір айнымалыға тәуелді дербес туындылардағы теңдеулерді қарапайым дифференциалдық теңдеулерге алып келуге болады. Сондықтан, бұл зерттеудің коэффициенттері шенелмеген дербес туындылардағы теңдеулер үшін маңызы бар. Зерттеліп отырған теңдеудің осыған дейін қарастырылғандардан айырмашылығы - оның жоғарғы және аралық коэффициенттері жылдам тербелуі мүмкін. Негізгі теоремаларды дәлелдеу кезінде авторлар өздерінің осы теңдеудің дұрыс шешілуіне қатысты алдыңғы нәтижелерін пайдаланған.

Түйін сөздер: екінші ретті дифференциалдық теңдеу, сызықты дифференциалдық теңдеу, шенелмеген облыстағы дифференциалдық теңдеу, максималды регулярлық, тербелмелі коэффициенттер.

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Оценка максимальной регулярности для дифференциального уравнения с колеблющимися коэффициентами

В работе рассматривается дифференциальное уравнение второго порядка с неограниченными коэффициентами. Получены достаточные условия суммируемости с весом решения и его производных вплоть до второго порядка. Изучаемое уравнение является сингулярным, так как оно задано в бесконечной области, а его коэффициенты могут быть не ограниченными. Главной его особенностью является быстрый рост коэффициента при первой производной искомого решения, из-за чего не применима хорошо развитая теория уравнений Штурма-Лиувилля. Исследуемое уравнение и его многомерные обобщения возникают в моделировании броуновского движения частиц, в задачах биологии и финансовой математики. Их известными представителями являются уравнения Орнштейна-Уленбека и Фоккера-Планка –Колмогорова, которые активно изучаются начиная с первой половины двадцатого века. С другой стороны, в приложениях хорошо известны проекционные методы (например, преобразования Фурье или Лапласа), которые сводят уравнения в частных производных с коэффициентами, зависящими от одной переменной, к обыкновенным дифференциальным уравнениям. Поэтому настоящее исследование важно для уравнений в частных производных с неограниченными коэффициентами. В отличие от предыдущих работ, старший и промежуточный коэффициенты исследуемого уравнения могут быть сильно колеблющимися. При доказательстве основных теорем, авторы пользуются более ранним их результатом о корректной разрешимости указанного уравнения.

Ключевые слова: дифференциальное уравнение второго порядка, линейное дифференциальное уравнение, дифференциальное уравнение в неограниченной области, максимальная регулярность, колеблющиеся коэффициенты.

1 Introduction

In this paper, we consider the smoothness properties of the solution of a second-order singular differential equation

$$T_0 y = -\rho(x)\left(\rho(x)y'\right)' + r(x)y' + s(x)y = f(x), \tag{1}$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, ρ is a positive and twice continuously differentiable function, r is a continuously differentiable function, and s is a continuous function, $f \in L_2 = L_2(\mathbb{R})$.

By T_0 we denote the operator mapping from the set of twice continuously differentiable and finite functions $C_0^{(2)}(\mathbb{R})$ to L_2 by the following formula

$$T_0 y = -\rho(x)\left(\rho(x)y'\right)' + r(x)y' + s(x)y.$$

We denote by T the closure T_0 in L_2 space. The function $y \in D(T)$ such that $Ty = f$ is said to be solution of the equation (1).

The solution $y \in L_2$ of the equation (1) is said to be maximally regular if the following inequality holds

$$\|-\rho(\rho y')'\|_2 + \|ry'\|_2 + \|sy\|_2 \leqslant C\,\|f\|_2\,,$$

where $C > 0$ does not depend on y, $\|\cdot\|_2$ is a norm of L_2.

Some conditions for existence, uniqueness and maximal regularity of a solution of the equation (1) were obtained in our work [1]. There the relevance in theory and practical issues of studying this equation in the case when its coefficients can be unbounded functions were also covered, and the case of tending to zero function $\rho(x)$ in the leading term of the equation was also studied. Naturally, the correctness of the equation (1) assumes that there are some relations between its coefficients. The equation (1) is reduced to the well-known Sturm-Liouville equation if the intermediate coefficient $r(x)$ is absent or grows slowly, so that $r(x)\frac{dy}{dx}$ as operator is controlled by the sum of leading and free terms in the left part. When these conditions are not met, the equation (1) is investigated poorly.

The investigated equation (1) and its multidimensional generalizations arise in Brownian particle motion modeling, in biology and financial mathematics problems [2–6]. Their well-known representatives are the Ornstein-Ulenbeck and Fokker-Planck-Kolmogorov equations, which have been actively studied since the first half of the twentieth century.

In this paper, unlike [1] as well as [7], we will assume that the coefficients $\rho(x)$ and $r(x)$ do not follow the weak fluctuation conditions. Such conditions usually appear when evaluating the norm of the higher derivative of a solution to the second-order singular differential equation. In [8] there is an example of a Sturm-Liouville equation with an oscillating coefficient whose solution is not maximally regular.

The main result of the work is Theorem 2. We have proved the validity of the maximal regularity estimate of a solution of the equation (1) when the mentioned coefficients ρ and r can fluctuate rapidly.

2 Material and Methods

We rely on Lemma 1 obtained in [1], where the theorem of the existence and uniqueness of the solution of the equation (1) is proved and a uniform estimate for the norm of the solution and its first derivative was obtained.

An auxiliary binomial degenerate differential operator associated with the equation (1) was investigated. Applying the method of local estimates developed in the work of M. Otelbaev [9], we obtained a representation of the resolvent of its certain shift. Using this representation we have proved the separability of the above binomial differential operator. Then we applied the closed operator perturbation theorem in [10]. Here, the partition of the real axis chosen by us depends on the dominant intermediate coefficient, which allowed us to consider the case of strongly fluctuating coefficients.

3 Auxiliary statements

Consider the equation

$$l_0 y = -\rho\,(\rho y')' + r y' = F(x), \tag{2}$$

Let $D(l_0) = C_0^{(2)}(\mathbb{R})$, and l is a closure of the operator l_0 by the norm of L_2. A function $y \in D(l)$ such that $ly = f$ is said to be a solution of the equation (2). Let $u(x)$ and $v(x) \neq 0$ are some real continuous functions. We denote

$$\gamma_{u,v} = \max\left(\sup_{x>0}\left(\int_0^x u^2(t)dt\right)^{\frac{1}{2}}\left(\int_x^{+\infty} v^{-2}(t)dt\right)^{\frac{1}{2}}, \sup_{\tau<0}\left(\int_\tau^0 u^2(t)dt\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\tau} v^{-2}(t)dt\right)^{\frac{1}{2}}\right).$$

In [1, теорема 3.1] the following statement is proved.

Lemma 1 *Let $\rho(x) > 0$ is a twice continuously differentiable function, and $r(x) \geqslant 1$ is a continuously differentiable function. Let*

$$\frac{r}{\rho^2} \geqslant 1, \qquad \gamma_{1,\sqrt{r}} < +\infty, \tag{3}$$

and there also exists $a \in \mathbb{R}$ such that

$$\sup_{x<a}\left\{\rho(x)\exp\left(-\int_x^a \frac{r(t)}{\rho^2(t)}dt\right)\right\} < +\infty. \tag{4}$$

Then for any $F \in L_2$ the equation (2) has a unique solution y, and for y the following estimate holds

$$\left\|\sqrt{r}y'\right\|_2 + \|y\|_2 \leqslant C\,\|f\|_2\,.$$

When the condition (3) holds, the following inequality was also proved in [1]:

$$\left\|\sqrt{r}y'\right\|_2 \leqslant \left\|\frac{1}{\sqrt{r}}ly\right\|_2, \tag{5}$$

where $y \in D(l)$.

4 Main results

We use the following theorem in the proof of the main result which is Theorem 2. Meanwhile Theorem 1 is of independent interest.

Theorem 1 *Let $0 < \rho(x) < +\infty$ is a twice continuously differentiable function and $r(x) \geqslant 1$ is a continuously differentiable function for which the conditions (3) and (4) of Lemma 1 are satisfied. Suppose, moreover*

$$\sup_{|x-\eta|\leqslant\frac{k(\eta)}{r(\eta)}}\frac{\rho(x)}{\rho(\eta)} < +\infty, \qquad \sup_{|x-\eta|\leqslant\frac{k(\eta)}{r(\eta)}}\frac{r(x)}{r(\eta)} < +\infty, \tag{6}$$

where $k(\eta) \geqslant 4$ is continuous and $\lim\limits_{|\eta|\to+\infty} k(\eta) = +\infty$. Then the following estimate holds for the solution y of the equation (2):

$$\left\|-\rho\left(\rho y'\right)'\right\|_2 + \|ry'\|_2 + \|y\|_2 \leqslant C_1\,\|f\|_2\,. \tag{7}$$

Proof. By virtue of lemma 2.1 [9] and the condition (6) there is a cover of $\{\Delta_j\}_{j=1}^{+\infty}$ of the set $\mathbb{R}$ (i. e. $\bigcup\limits_{j=1}^{+\infty} \Delta_j = \mathbb{R}$) such that each interval $\Delta_j = (a_j, b_j)$, where

$$b_j - a_j \leqslant \frac{k\left(\frac{b_j-a_j}{2}\right)}{2r\left(\frac{b_j-a_j}{2}\right)},$$

can intersect with the others no more than ξ times. There exists also a set of functions $\{\varphi_j\}_{j=1}^{+\infty}$ such that

$$\sum_{j=1}^{+\infty} \varphi_j^2(x) = 1, \qquad \varphi_j \in C_0^\infty(\Delta_j).$$

Let $\rho_j(x)$, $r_j(x)$ and $F_j(x)$ $(j = 1, 2, \ldots)$ are restrictions on Δ_j of the functions $\rho(x)$, $r(x)$ and $F(x)$, respectively, and $\lambda \geqslant 0$. Consider the following problem

$$l_{0,j,\lambda} y = -\rho_j(x)\left(\rho_j(x) y'\right)' + [r_j(x) + \lambda] y' = F_j(x), \tag{8}$$

$$y(a_j) = y(b_j) = 0. \tag{9}$$

We define the solution to the problem (8), (9) as the function $y(x)$, for which there exists the sequence $\{y_k(x)\}_{k=1}^{+\infty}$ from the set $C_0^{(2)}(\Delta_j)$ of twice continuously differentiable and finite in Δ_j functions such that $\|y_k - y\|_{L_2(\Delta_j)} \to 0$ and $\|l_{0,j,\lambda} y_k - f_j\|_{L_2(\Delta_j)} \to 0$ as $k \to +\infty$. We denote by $l_{j,\lambda}$ $(j = 1, 2, \ldots)$ the closure of the operator $l_{0,j,\lambda}$ with $D(l_{0,j,\lambda}) = C_0^{(2)}(\Delta_j)$ in the space $L_2(\Delta_j)$. The function $y \in L_2(\Delta_j)$ is said to be the solution of the problem (8), (9) if $y \in D(l_{j,\lambda})$ and $l_{j,\lambda} y = F_j$. It follows from the general theory of differential equations that for any $F_j \in L_2(\Delta_j)$ the solution to the problem (8), (9) exists.

Let us introduce the following notation: $z = y'$ $(y \in D(l_{0,j,\lambda}))$, $L_{0,j,\lambda} z = -\rho_j(\rho_j z)' + (r_j + \lambda) z$, $\|\cdot\|_{2,\Delta_j} = \|\cdot\|_{L_2(\Delta_j)}$. Let $z \in D(L_{0,j,\lambda})$. Integrating by parts we obtain

$$\int\limits_{\Delta_j} z L_{0,j,\lambda} z dx = \int\limits_{\Delta_j} z(-\rho_j(\rho_j z)' + (r_j + \lambda) z) dx = \int\limits_{\Delta_j} (r_j + \lambda) z^2 dx = \left\|\sqrt{r_j + \lambda} z\right\|_{2,\Delta_j}^2. \tag{10}$$

On the other hand, according to Hölder's inequality

$$\int\limits_{\Delta_j} z L_{0,j,\lambda} z dx \leqslant \left(\int\limits_{\Delta_j} \left|(r_j + \lambda)^{-\frac{1}{2}} L_{0,j,\lambda} z\right|^2 dx\right)^{\frac{1}{2}} \left(\int\limits_{\Delta_j} \left|(r_j + \lambda)^{\frac{1}{2}} z\right|^2 dx\right)^{\frac{1}{2}} =$$

$$= \left\|\frac{1}{\sqrt{r_j + \lambda}} L_{0,j,\lambda} z\right\|_{2,\Delta_j} \left\|\sqrt{r_j + \lambda} z\right\|_{2,\Delta_j}. \tag{11}$$

From (10) and (11) it follows that

$$\left\|\sqrt{r_j+\lambda}z\right\|_{2,\Delta_j} \leqslant \left\|\frac{1}{\sqrt{r_j+\lambda}}L_{0,j,\lambda}z\right\|_{2,\Delta_j}, \qquad z \in D(L_{0,j,\lambda}). \tag{12}$$

Since $z = y'$, we have

$$\left\|\sqrt{r_j+\lambda}y'\right\|_{2,\Delta_j} \leqslant \left\|\frac{1}{\sqrt{r_j+\lambda}}l_{0,j,\lambda}y\right\|_{2,\Delta_j}, \qquad y \in C_0^{(2)}(\Delta_j). \tag{13}$$

Further, according to the well-known Friedrichs's inequality

$$\|y\|_{2,\Delta_j} \leqslant C\left\|\sqrt{r_j+\lambda}y'\right\|_{2,\Delta_j}, \qquad y \in C_0^{(2)}(\Delta_j). \tag{14}$$

By virtue of (13) and (14)

$$\|\sqrt{r_j+\lambda}y'\|_{2,\Delta_j} + \|y\|_{2,\Delta_j} \leqslant (C+1)\|F_j\|_{2,\Delta_j}, \qquad y \in C_0^{(2)}(\Delta_j).$$

According to (12)

$$\left\|\sqrt{r_j+\lambda}z\right\|_{2,\Delta_j} \leqslant \frac{1}{\inf\limits_{x\in\Delta_j}\sqrt{r_j+\lambda}}\left\|L_{0,j,\lambda}z\right\|_{2,\Delta_j}.$$

Hence, taking into account of the condition (6) and the choice of Δ_j, we have

$$\|(r_j+\lambda)z\|_{2,\Delta_j} \leqslant \sup_{x\in\Delta_j}\sqrt{r_j+\lambda}\left\|\sqrt{r_j+\lambda}z\right\|_{2,\Delta_j} \leqslant$$
$$\leqslant C\inf_{x\in\Delta_j}\sqrt{r_j+\lambda}\left\|\sqrt{r_j+\lambda}z\right\|_{2,\Delta_j} \leqslant C\left\|L_{0,j,\lambda}z\right\|_{2,\Delta_j}, \; z \in D(L_{0,j,\lambda}).$$

Then

$$\|-\rho_j(\rho_j z)'\|_{2,\Delta_j} + \|(r_j+\lambda)z\|_{2,\Delta_j} \leqslant C_1\left\|L_{0,j,\lambda}z\right\|_{2,\Delta_j}, \quad z \in D(L_{0,j,\lambda}). \tag{15}$$

Due to (8) и (14) we have

$$\|-\rho_j(\rho_j y')'\|_{2,\Delta_j} + \|(r_j+\lambda)y'\|_{2,\Delta_j} + \|y\|_{2,\Delta_j} \leqslant C_1\left\|l_{0,j,\lambda}y\right\|_{2,\Delta_j}, \quad y \in C_0^{(2)}(\Delta_j). \tag{16}$$

Since the $l_{j,\lambda}$ is closed, the inequality (16) holds for all $y \in D(l_{j,\lambda})$, in particular, for a solution of the problem (8), (9).

Let $L_{j,\lambda}$ is an operator from the set $D(L_{j,\lambda}) = \{z \in L_2(\Delta_j) : \exists y \in D(l_{j,\lambda}), z = y'\}$ in $L_2(\Delta_j)$ by the following formula

$$L_{j,\lambda}z = -\rho_j(x)(\rho_j(x)z)' + (r_j(x)+\lambda)z.$$

Since $R(L_{j,\lambda}) = R(l_{j,\lambda}) = L_2(\Delta_j)$, and for all $z \in D(L_{j,\lambda})$ the inequality (15) holds, the operator $L_{j,\lambda}$ is bounded invertible. We define the following operators for $f \in L_2$:

$$B_\lambda f := -\sum_{j=-\infty}^{+\infty}\rho^2(x)\varphi_j'(x)L_{j,\lambda}^{-1}\varphi_j f, \qquad M_\lambda f := \sum_{j=-\infty}^{+\infty}\varphi_j(x)L_{j,\lambda}^{-1}\varphi_j f.$$

For any point $x \in \mathbb{R}$ the sums in the right-hand sides are consisted of no more that $\xi + 1$ terms, thus B_λ and M_λ are well-defined.

Let $L_\lambda z = -\rho(x)(\rho(x)z)' + (r(x) + \lambda)z$ for any $z = y'$, where $y \in D(l)$. Consider the operator $L_\lambda M_\lambda$. The operators L_λ and $L_{j,\lambda}$ are same in the interval Δ_j, therefore taking into account the properties of φ_j $(j \in \mathbb{Z})$, we get

$$L_\lambda M_\lambda f = \sum_{j=-\infty}^{+\infty} L_{j,\lambda}(\varphi_j L_{j,\lambda}^{-1}\varphi_j f) = \sum_{j=-\infty}^{+\infty} \left(\varphi_j L_{j,\lambda} L_{j,\lambda}^{-1}\varphi_j f - \rho^2\varphi_j' L_{j,\lambda}^{-1}\varphi_j f\right) = (E + B_\lambda)f,$$

i. e.

$$L_\lambda M_\lambda = E + B_\lambda. \tag{17}$$

Let's estimate the norm of the operator B_λ.

$$\|B_\lambda f\|_2^2 = \int_{-\infty}^{+\infty} \left| \sum_{j=-\infty}^{+\infty} \rho^2\varphi_j' L_{j,\lambda}^{-1}\varphi_j f \right|^2 dx \leqslant$$

$$\leqslant \sum_{j=-\infty}^{+\infty} \int_{\Delta_j} \left|\rho^2\varphi_j' L_{j,\lambda}^{-1}\varphi_j f\right|^2 dx \leqslant \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left| \sum_{j=-\infty}^{+\infty} \rho^2\varphi_j' L_{j,\lambda}^{-1}\varphi_j f \right|^2 dx \leqslant$$

$$\leqslant C_3(\xi + 1) \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left|\rho^2(x)\varphi_k'(x) L_{j,\lambda}^{-1}\varphi_k(x) f(x)\right|^2 dx.$$

Due to (12)

$$\left\|\rho^2\varphi_k' L_\lambda^{-1}\varphi_k f\right\|_{2,\Delta_k} \leqslant \frac{C_4 \sup\limits_{x\in\Delta_k} \rho^2(x)}{\inf\limits_{x\in\Delta_k} (r_k(x) + \lambda)} \|\varphi_k f\|_{2,\Delta_k} \leqslant \frac{C_5}{1+\lambda}\|\varphi_k f\|_{2,\Delta_k},$$

therefore, using the properties of the functions $\varphi_k(x)$ $(k \in \mathbb{Z})$, we have

$$\sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left|\rho^2\varphi_k' L_{k,\lambda}^{-1}\varphi_k f\right|^2 dx \leqslant \frac{C_5^2}{(1+\lambda)^2} \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \varphi_k^2 |f|^2 dx =$$

$$= \frac{C_5^2}{(1+\lambda)^2} \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}} \varphi_k^2 |f|^2 dx = \frac{C_5^2}{(1+\lambda)^2} \int_{\mathbb{R}} \left(\sum_{k=-\infty}^{+\infty} \varphi_k^2 \right) |f|^2 dx = \left(\frac{C_5}{1+\lambda}\right)^2 \|f\|_2^2.$$

Thus,

$$\|B_\lambda f\|_2^2 \leqslant C_2(\xi + 1)\frac{C_5^2}{(1+\lambda)^2} \|f\|_2^2, \quad f \in L_2.$$

Hence $\|B_\lambda\| \to 0$ as $\lambda \to +\infty$, so there exists $\lambda_0 > 0$ such that $\|B_\lambda\| \leqslant \frac{1}{2}$ for all $\lambda \geqslant \lambda_0$. It follows from Lemma 1 that the operator L_λ^{-1}, the inverse of L_λ, exists and is bounded in L_2. From (17), by virtue of the well-known Banach theorem, it follows that

$$L_\lambda^{-1} = M_\lambda(E + B_\lambda)^{-1}, \ \left\|(E + B_\lambda)^{-1}\right\| \leqslant 2, \ \lambda \geqslant \lambda_0. \tag{18}$$

Let us now prove the estimate (7). According to (5) and (18), for $z \in D(L_\lambda)$, we have

$$\|(r+\lambda)z\|_2^2 = \left\|(r+\lambda)L_\lambda^{-1}f\right\|_2^2 \leqslant 2\,\|(r+\lambda)M_\lambda f\|_2^2 \leqslant$$
$$\leqslant C_7 \sum_{j=-\infty}^{+\infty} \left\|(r_j+\lambda)\varphi_j L_{j,\lambda}^{-1}\varphi_j f\right\|_{2,\Delta_j}^2 \leqslant C_7 \sum_{j=-\infty}^{+\infty} \sup_{x\in\Delta_j}(r_j+\lambda)\left\|L_{j,\lambda}^{-1}\varphi_j f\right\|_{2,\Delta_j}^2 \leqslant$$
$$\leqslant C_7 \sum_{j=-\infty}^{+\infty} \sup_{x\in\Delta_j}(r+\lambda)\frac{1}{\inf\limits_{x\in\Delta_j}(r+\lambda)}\|\varphi_j f\|_{2,\Delta_j}^2 \leqslant C_8 \sum_{j=-\infty}^{+\infty}\|\varphi_j f\|_{2,\Delta_j}^2 \leqslant C_9\|f\|_2^2.$$

So

$$\|(r+\lambda)z\|_2 \leqslant C_9\,\|f\|_2\,.$$

Hence, assuming $\lambda = 0$ and $z = y'$, we obtain an estimate (7). The theorem is proved.

Theorem 2 *Let the functions ρ and r satisfy the conditions of Theorem 1, and s is a continuous function such that $\gamma_{s,r} < +\infty$. Then for any $f \in L_2$ the equation (1) has a unique solution y, and for y the following estimate holds*

$$\left\|-\rho\,(\rho y')'\right\|_2 + \|ry'\|_2 + \|(1+|s|)y\|_2 \leqslant C\|f\|_2, \tag{19}$$

Proof. Let $x = at$ in (1), where $a > 0$. Let us introduce the following notations

$$\tilde{y}(t) = y(at), \quad \tilde{\rho}(t) = \rho(at), \quad \tilde{r}(t) = r(at), \quad \tilde{s}(t) = s(at), \quad \tilde{f}(t) = a^{-1}F(at).$$

Then the equation (1) is transformed to the following form

$$-\tilde{\rho}(\tilde{\rho}\tilde{y}')' + \tilde{r}\tilde{y} + a^{-1}\tilde{s}\tilde{y} = \tilde{f}. \tag{20}$$

Let us denote by l_a the closure in L_2 of the operator $-\tilde{\rho}(\tilde{\rho}\tilde{y}')' + \tilde{r}\tilde{y}$, defined in $C_0^{(2)}(\mathbb{R})$. The function $\tilde{y}(t) \in D(l_a)$ is said to be a solution of the equation (20) if it satisfies the equality $l_a\tilde{y} = \tilde{f}$. Clearly, if the function $y(x)$ is a solution to the equation (1), then $\tilde{y}(t)$ is a solution to the equation (20) and vice versa.

It is easy to show that $\tilde{\rho}$ and $\tilde{r}$ satisfy the conditions of Theorem 1, so

$$\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\|_2 + \|\tilde{r}\tilde{y}'\|_2 + \|\tilde{y}\|_2 \leqslant C_{l_a}\,\|l_a\tilde{y}\|_2\,, \qquad \forall\tilde{y} \in D(l_a).$$

According to lemma 2.1 [1] and the last inequality, we have

$$\left\|a^{-1}\tilde{s}\tilde{y}\right\|_2 \leqslant 2a^{-1}\gamma_{s,r}\,\|\tilde{r}\tilde{y}'\|_2 \leqslant 2a^{-1}\gamma_{s,r}C_{l_a}\,\|l_a\tilde{y}\|_2\,.$$

Let us choose $a = 4\gamma_{s,r}C_{l_a}$, then $\|a^{-1}\tilde{s}\tilde{y}\|_2 \leqslant \frac{1}{2}\,\|l_a\tilde{y}\|_2$. According to the theorem 1.16 in [10] (chapter 4) we get that the operator $l_a + a^{-1}\tilde{s}E$ (where E is an identity operator) is reversible and its range coincides with L_2. This means that the equation (20) is uniquely solvable for any $\tilde{f} \in L_2$, then the equation (1) also has a unique solution $y(x) = \tilde{y}(a^{-1}x)$ for any $f \in L_2$.

Applying Lemma 2.1 [1], we obtain the estimate

$$\|sy\|_2 \leqslant 2\gamma_{s,r}\,\|ry'\|_2\,,$$

from which, taking into account (7), it follows (19). The inequality (19) implies uniqueness of the solution of the equation (1). The theorem is proved.

Example 1 *Consider the following equation*

$$-\left(1+5\sin^2 e^x\right)\left[\left(1+5\sin^2 e^x\right)y'\right]'+(4+x^8+e^{2x}\sin^3 e^x)y'+(x^3+e^x\sin e^x)y=f(x). \quad (21)$$

It is easy to check that for $k(x)=4+x^2$ the coefficients of the equation (21) satisfy the conditions of Theorem 2, so for any $f\in L_2$ the equation (21) has a unique solution y and for y the following estimate holds

$$\left\|-\left(1+5\sin^2 e^x\right)\left[\left(1+5\sin^2 e^x\right)y'\right]'\right\|_2+\left\|(4+x^8+e^{2x}\sin^3 e^x)y'\right\|_2+$$
$$+\left\|(1+x^3+e^x\sin e^x)y\right\|_2\leqslant C\left\|f\right\|_2.$$

5 Conclusion

A singular second-order differential equation with an unbounded variable coefficient at the first derivative of the unknown function is investigated in the paper. Only positiveness is required from the leading coefficient, i.e. the equation can degenerate near of infinity. In addition, we have studied the case of rapidly fluctuating coefficients. We have obtained conditions for the summability with weight of a strong solution of the considered equation and its derivatives up to the second order. The obtained result theoretically extends the class of coercive solvable differential equations of the second order. They can find application in stochastic analysis, modeling problems in biology and financial mathematics.

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GREEN'S FUNCTIONS AND CORRECT RESTRICTIONS OF THE POLYHARMONIC OPERATOR

In this paper, for completeness of presentation, we give explicitly the Green's functions for the classical problems – Dirichlet, Neumann, and Robin for the Poisson equation in a multidimensional unit ball. There are various ways of constructing the Green's function of the Dirichlet problem for the Poisson equation. For many types of areas, it is built explicitly. Recently, there has been renewed interest in the explicit construction of Green's functions for classical problems. The Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed in an explicit form, and for the Neumann problem the construction of the Green's function remains an open problem. The paper gives a constructive way of constructing the Green's function of Dirichlet problems for a polyharmonic equation in a multidimensional ball. Finding general well-posed boundary value problems for differential equations is always an urgent problem. In this paper, we briefly outline the theory of restriction and extension of operators and describe well-posed boundary value problems for a polyharmonic operator.

Key words: Poisson equation, polyharmonic equations, Dirichlet problem, Neumann problem, Roben problem, correct restrictions of the operator.

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Полигармоникалық оператор үшін Грин функциялары және тиянақты тарылуы

Бұл жұмыста классикалық – Дирихле, Нейман және Робен есептерінің Грин функциялары айқын түрде көпөлшемді бірлік шарда Пуассон теңдеуі үшін көрсетілген. Пуассон теңдеуі үшін Дирихле есебінің Грин функциясын құрудың әртүрлі тәсілдері бар. Аудандардың көптеген түрлері үшін оның айқын түрі құрылған. Соңғы уақыттарда классикалық есептердің Грин функцияларын айқын түрде құруға деген қызығушылық қайта жандануда. Көпөлшемді шарда полигармоникалық теңдеу үшін Дирихле есебінің Грин функциясы айқын түрде құрылған, ал Нейман есебі үшін Грин функциясын құру ашық проблема болып қала беруде. Жұмыста көпөлшемді шарда полигармоникалық теңдеу үшін Дирихле есебінің Грин функциясын құрудың тиімді әдісі көрсетілген. Дифференциалдық теңдеулер үшін жалпы тиянақты шекаралық есептерді табу әрқашан өзекті мәселе болып табылады. Бұл жұмыста операторлардың тарылуы мен кеңеюі теориясы қысқаша сипатталған және полигармоникалық операторлар үшін тиянақты шеттік есептердің тарылуы сипатталған.

Түйін сөздер: Пуассон теңдеуі, Дирихле есебі, Нейман есебі, Робен есебі, оператордың тиянақты тарылуы

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Функций Грина и корректные сужения полигармонического оператора

В данной работе приведены в явном виде функций Грина классических задач – Дирихле, Неймана и Робена для уравнения Пуассона в многомерном единичном шаре. Существуют различные способы построения функции Грина задачи Дирихле для уравнения Пуассона. Для многих видов областей она построена в явном виде. В последнее время возобновился интерес к построению в явном виде функций Грина классических задач. Функция Грина задачи Дирихле для полигармонического уравнения в многомерном шаре построена в явном виде, а для задачи Неймана построение функции Грина остается открытой задачей. В работе дан конструктивный способ построения функции Грина задач Дирихле для полигармонического уравнения в многомерном шаре. Нахождение общих корректных краевых задач для дифференциальных уравнений всегда является актуальной задачей. В данной работе кратко изложена теория сужения и расширения операторов и описаны корректные краевые задачи для полигармонического оператора.

Ключевые слова: уравнение Пуассона, полигармонические уравнения, задача Дирихле, задача Неймана, задача Робена, корректные сужения оператора.

Introduction

The need to study boundary value problems for elliptic equations is dictated by numerous practical applications in the theoretical study of the processes of hydrodynamics, electrostatics, mechanics, thermal conductivity, elasticity theory, and quantum physics [1-4]. The distributions of the potential of the electrostatic field are described using the Poisson equation. When studying the vibrations of thin plates of small deflections, biharmonic equations arise.

This work is devoted to the construction of the Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball and to the description of well-posed boundary value problems for polyharmonic operators.

1 Materials and methods

The subject of this research is a constructive way of constructing the Green's function of boundary value problems for a polyharmonic equation in a ball of arbitrary dimension.

The research method is the representation of polyharmonic functions through the sum of harmonic functions with certain weights. When constructing explicitly the Green's function of the Dirichlet problem for a polyharmonic equation in a ball, the method of special expansion of the fundamental solutions of the polyharmonic equation and the method of reflection are essentially used. When describing new well-posed boundary value problems for an inhomogeneous polyharmonic equation in a ball, the method of restricting abstract operators was applied.

There are various ways to construct the Green Function of the Dirichlet problem for the Poisson equation. For many types of domains, it is constructed explicitly. And for the Neumann problem in multidimensional domains, the construction of the Green function is an open problem. For the ball, the Green function of the internal and external Neumann problem is constructed explicitly only for the two-dimensional and three-dimensional cases. In the general case, for a multidimensional ball, the explicit form of the Green function of the Neumann and Robin problems for the Poisson equation is constructed recently in [5,6].

2 Results and discussion

Note that recently there has been renewed interest in the explicit construction of Green's functions for classical problems. In [7-9], the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball is constructed explicitly. In [10], the Green harmonic functions of the Dirichlet, Neumann, and Robin problems are used to construct the Green functions of the biharmonic Dirichlet, Neumann, and Robin problems in a two-dimensional circle. Similar results in the class of inhomogeneous biharmonic and triharmonic functions in the sector were obtained in [11-13]. Note also that the construction of explicit Green functions of the Robin problem in a circle, when the parameter in the boundary condition is equal to one, is devoted to the work [14,15]. The results of these studies are based on the classical theory of integral representations for analytic, harmonic, and polyharmonic functions on the plane.

Finding general correct boundary value problems for differential equations is always an urgent problem. The abstract theory of operator contraction and expansion originates from the work of John von Neumann [16], in which a method for constructing self-adjoint extensions of a symmetric operator was described and a theory of extension of symmetric operators with finite defect indices was developed in detail. Many problems for partial differential equations lead to operators with infinite defect indices.

In [17,18] considered extensions of the minimal operator, rejecting its symmetry, and described the areas of definition of the extension that have certain solvability properties, here are investigated to general boundary value problems for general second-order elliptic differential equations. In [19] found a correct problem that is not contained among the problems described [18]. This type of problem for ordinary differential equations was studied in [20].

In the early 80s of the last century, M. Otelbaev and his students [21-23] constructed an abstract theory that allows us to describe all correct contractions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. This theory was extended to the case of Banach spaces [24].

In [25] certain estimates are obtained for the deviation upon domain perturbation of singular number of correct restrictions of elliptic differential operators.

Thus, this paper is devoted to the construction of the Green function of the classical Dirichlet, Neumann and Robin problems for the Poisson equation in a multidimensional ball, a constructive way to construct the Green function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball, and the description of correct boundary value problems for polyharmonic operators.

3 Green's function of the Dirichlet, Neumann, and Robin problem for the Poisson equation in a multidimensional unit ball

Let $\Omega \subseteq \mathbb{R}^n, n \geq 2$ be a bounded region with a smooth boundary $\partial\Omega$. Consider in domain Ω following the Dirichlet problem for the Poisson equation

$$-\Delta u(x) = f(x),\, x \in \Omega,\;\, u(x) = \varphi(x),\, x \in \partial\Omega. \tag{1}$$

The classical solution $u(x) \in C^2(\Omega) \cap C(\overline{\Omega})$ of the Dirichlet problem (1) exists, is unique, and is represented by the Green's function $G_D(x, y)$ in the following form [1]

$$u(x) = \int_\Omega G_D(x, y) f(y) dy - \int_{\partial\Omega} \frac{\partial G_D(x, y)}{\partial n_y} \varphi(y) dS_y, \tag{2}$$

where $\dfrac{\partial}{\partial n_y}$– the external normal of $\partial\Omega$, and is calculated by the formula

$$\frac{\partial}{\partial n_y} = \sum_{k=1}^{n} (n_y)_k \frac{\partial}{\partial x_k},\ n_y \equiv \overrightarrow{n}_y = \{(n_y)_1, (n_y)_2, ..., (n_y)_n\},\ |n_y| = 1.$$

The Green function of the Dirichlet problem (1) is defined as follows

$$-\Delta G_D(x, y) = \delta(x - y),\ x, y \in \Omega,$$

$$G_D(x, y) = 0,\ x \in \partial\Omega,\ x \in \Omega,$$

where $\delta(x - y)$ is the Dirac delta function.

In particular, when $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ is a unit ball, the Green function of the Dirichlet problem (1) can be constructed by the reflection method and has the form

$$G_D(x, y) = \frac{1}{\omega_n} \left[\varepsilon_n(x - y) - \varepsilon_n\Big(x|y| - \frac{y}{|y|}\Big) \right], \tag{3}$$

where $\omega_n = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$– the surface area of a unit ball, $\varepsilon_n(x - y)$ is the fundamental solution of the Laplace equation [2, 3]

$$\varepsilon_n(x - y) = \begin{cases} \ln \dfrac{1}{|x - y|}, & n = 2,\ |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \\ \dfrac{1}{n - 2} |x - y|^{2-n}, & n \geq 3,\ |x - y| = \sqrt{\sum\limits_{k=1}^{n} (x_k - y_k)^2}. \end{cases}$$

Along with the Dirichlet problem, the Neumann problem for the Poisson equation is a classical and well-studied one

$$-\Delta u(x) = f(x),\ x \in \Omega,\ \frac{\partial u(x)}{\partial n} = \psi(x),\ x \in \partial\Omega. \tag{4}$$

It is known that the solution of the Neumann problem (4) from class $C^2(\Omega) \cap C^1(\overline{\Omega})$ is not unique up to the constant term. For the existence of a solution to the problem, it is necessary and sufficient to fulfill the condition

$$\int_\Omega f(y) dy + \int_{\partial\Omega} \psi(y) dS_y = 0. \tag{5}$$

If a solution to problem (4) exists, then this solution can be represented in integral form using the Green function of the Neumann problem $G_N(x, y)$ according to the formula [1]

$$u(x) = \int_\Omega G_N(x, y) f(y) dy + \int_{\partial\Omega} G_N(x, y) \psi(y) dS_y + \text{const}. \tag{6}$$

The Green function of the Neumann problem (4) is understood as [1] a function that has the representation

$$G_N(x,y)=\frac{1}{\omega_n}\left[\varepsilon_n(x-y)+g(x,y)\right],$$

where $g(x,y)-$ the harmonic function in region Ω.

In this case, the boundary condition must be met

$$\frac{\partial G_N}{\partial n_y}(x,y)=-\frac{1}{\omega_n},\ y\in\partial\Omega. \tag{7}$$

If such a Green's function $G_N(x,y)$ exists, then it follows from (5) and (7) that function (6) satisfies all conditions of problem (4).

For a unit ball, the Green function of the Neumann problem is presented explicitly for cases $n=2$ and $n=3$

$$G_N(x,y)=\frac{1}{2\pi}\left[\ln\frac{1}{|x-y|}+\ln\frac{1}{\left|x|y|-\frac{y}{|y|}\right|}\right],\ n=2,$$

$$G_N(x,y)=\frac{1}{4\pi}\left[\frac{1}{|x-y|}+\frac{1}{\left|x|y|-\frac{y}{|y|}\right|}-\ln\left|1+(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|\right],\ n=3,$$

where $(x,y)=x_1y_1+...+x_ny_n-$ the scalar product in $\mathbb{R}^n$ of vectors x and y.

The Green function of the Neumann problem (4) has the following representation [5]

$$G_N(x,y)=\frac{1}{\omega_n}\left[\varepsilon_n(x-y)+\varepsilon_n\left(x|y|-\frac{y}{|y|}\right)+\widetilde{\varepsilon}(x,y)\right]+const,$$

where $\widetilde{\varepsilon}(x,y)$ expressed by the identity

$$\widetilde{\varepsilon}(x,y)=\int_0^1\left[(n-2)s|x|y|-\frac{y}{|y|}|-1\right]\frac{ds}{s}\equiv\int_0^1\left[s\left|x|y|-\frac{y}{|y|}\right|^{2-n}-1\right]\frac{ds}{s},n\geq 3,$$

and they are written through elementary functions

$$\widetilde{\varepsilon}(x,y)=\ln\frac{2}{\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|},\ n=3; \tag{i}$$

$$\widetilde{\varepsilon}(x,y)=\ln\frac{(x,y)}{\sqrt{|x|^2|y|^2-(x,y)^2}}\arctan\frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{1-(x,y)}-\ln\left|x|y|-\frac{y}{|y|}\right|,\ n=4; \tag{ii}$$

$$\widetilde{\varepsilon}(x,y)=\ln\frac{2}{\left|1-(x,y)+\left|x|y|-\frac{y}{|y|}\right|\right|}+\sum_{k=1}^{m-1}\frac{1}{(2k-1)}\left\{\left|x|y|-\frac{y}{|y|}\right|^{1-2k}-1\right\}+$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1}\frac{2^i(k+i-1)(2k-3)!!(x,y)|x|^{2i}|y|^{2i}}{(k-1)!(2k+2i-1)!!(|x|^2|y|^2-(x,y)^2)^{i+1}}\left[\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k-1}}+(x,y)\right],$$

$$n \geq 5,\ n = 2m+1,\ m \geq 2; \tag{iii}$$

$$\widetilde{\varepsilon}(x,y) = -\ln\left|x|y| - \frac{y}{|y|}\right| + \sum_{k=1}^{m-1}\frac{1}{2k}\left\{\left|x|y| - \frac{y}{|y|}\right|^{-2k} - 1\right\} +$$

$$(x,y)\arctan\frac{\sqrt{|x|^2|y|^2-(x,y)^2}}{1-(x,y)}\sum_{k=0}^{m-1}\frac{(2k-1)!!}{2^k k!}\frac{|x|^{2k}|y|^{2k}}{(|x|^2|y|^2-(x,y)^2)^{k-1/2}} +$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{m-k-1}\frac{(2k+2i-1)!!(k+1)!\,(x,y)|x|^{2i}|y|^{2i}}{2^{i+1}(2k-1)!!(k+i)!(|x|^2|y|^2-(x,y)^2)^{i+1}}\left[\frac{|x|^2|y|^2-(x,y)}{\left|x|y|-\frac{y}{|y|}\right|^{2k}} - (x,y)\right],$$

$$n \geq 6,\ n = 2m+2,\ m \geq 2. \tag{iv}$$

Along with the Dirichlet and Neumann problems, the Robin problem (the third boundary value problem) for the Poisson equation is a classical and well-studied one

$$-\Delta u(x) = f(x),\ x \in \Omega,\ \frac{\partial u(x)}{\partial n} + au(x) = \psi(x),\ x \in \partial\Omega. \tag{8}$$

The solution of the problem Robin (8) from class $C^2(\Omega)\cap C^1(\overline{\Omega})$ is represented as follows

$$u(x) = \int_\Omega G_a(x,y)f(y)dy - \int_{\partial\Omega}\frac{\partial G_a(x,y)}{\partial n_y}\varphi(y)dS_y. \tag{9}$$

The Green function of the Robin problem (8) has the form [6]

a) if $a > 0$, then

$$G_a(x,y) = G_D(x,y) + \frac{1}{2\pi}\int_0^1 s^{a-1}P(r\rho s,\gamma)ds =$$

$$\varepsilon(x-y) - \varepsilon\Big(x|y| - \frac{y}{|y|}\Big) + \frac{n-2-2a}{\omega_n}\int_0^1 s^{a-1}\varepsilon\Big(sx|y| - \frac{y}{|y|}\Big)ds,$$

where $\gamma = \theta - \varphi$ и $P(r\rho s,\gamma) = \dfrac{1-t^2}{1-2tcos\gamma+t^2}-$ the Poisson kernel;

b) if $a < 0$ and $a-$ non-integer, then

$$G_a(x,y) = G_D(x,y) + \frac{1}{2\pi}\left[\frac{2}{a} + 2\sum_{k=1}^{m}\frac{1}{k+a}(r\rho)^s\cos k\gamma +\right.$$

$$\left.\int_0^1 s^{a-1}\Big(P(r\rho s,\gamma) + 1 - 2\sum_{k=0}^{m}(r\rho s)^k cosk\gamma\Big)ds\right],$$

where $m = -[a]+1$.

4 Green's function of the Dirichlet problem for a polyharmonic equation in a multidimensional ball

Let m be a natural number and in the $n-$ dimensional ball $\Omega = \{x \in \mathbb{R}^n : |x| < r \}$ consider Dirichlet problem for a polyharmonic equation

$$\Delta^m u(x) = f(x), \quad x \in \Omega, \tag{10}$$

$$\frac{\partial^j u(x)}{\partial n_x^j} = \varphi_j(x), \;\; 0 \le j \le m-1, \;\; x \in \partial\Omega. \tag{11}$$

The classical solution $u(x) \in C^{2m}(\Omega) \cap C^{m-1}(\overline{\Omega})$ to the Dirichlet problem (10), (11) exists, is unique, and it is represented by the Green's function $G_{2m,n}(x,y)$ in the following form [3]

$$u(x) = \int_\Omega G_{2m,n}(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}(x,y) \cdot \Delta_y^{m-1-j} \varphi(y) - \right.$$

$$\left. - \Delta_y^j G_{2m,n}(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} \varphi(y) \right] dS_y, \tag{12}$$

where $\dfrac{\partial}{\partial n_y}-$ external normal $\partial\Omega$.

The Green function of the Dirichlet problem (10), (11) is defined as follows

$$\Delta^m G_{2m,n}(x,y) = \delta(x-y), \;\; x, y \in \Omega, \tag{13}$$

$$\frac{\partial^j G_{2m,n}(x,y)}{\partial n_x^j} = 0, \; x \in \partial\Omega, \; y \in \Omega, \; 0 \le j \le m-1, \tag{14}$$

where $\delta(x-y)-$ the Dirac delta function.

Theorem 1 *[7–9] a) In the case of odd n, as well as for even n, if $2m < n$ the Green's function of the Dirichlet problem (13), (14) can be represented in the form*

$$G_{2m,n}(x,y) = \varepsilon_{2m,n}(x,y) - g^0_{2m,n}(x,y) - \sum_{k=1}^{m-1} g^k_{2m,n}(x,y), \tag{15}$$

where

$$\varepsilon_{2m,n}(x,y) = d_{2m,n} |x-y|^{2m-n},$$

$$g^0_{2m,n}(x,y) = d_{2m,n} \left[\left| \frac{y}{r} \right| \cdot \left| x - \frac{y}{|y|^2} r^2 \right| \right]^{2m-n},$$

$$g^k_{2m,n}(x,y) = d_{2m,n}(2m-n)...(2(m-k+1)-n) \cdot \left[\left| \frac{y}{r} \right| \cdot \left| x - \frac{y}{|y|^2} r^2 \right| \right]^{2m-n-2k} \cdot$$

$$\cdot \left(1 - \left| \frac{y}{r} \right|^2 \right)^k \left(1 - \left| \frac{x}{r} \right|^2 \right)^k \left(\frac{r^2}{-2} \right)^k \frac{1}{k!}, \quad k = 1, 2, ..., m-1,$$

$$d_{2m,n} = \frac{1}{(m-1)!\,(2m-n)(2(m-1)-n)...(4-n)(2-n)} \cdot \frac{\Gamma(n/2)}{2^m \pi^{n/2}},$$

$\Gamma(\cdot)-$ *gamma function;*

b) In the case of even n and $2m \geq n$, the Green's function of the Dirichlet problem (13), (14) can be represented in the form (15), where

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n}\ln|x-y|,$$

$$g^0_{2m,n}(x,y) = d_{2m,n}\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}\,r^2\right|\right]^{2m-n} \ln\left[\left|\frac{y}{r}\right|\left|x - \frac{y}{|y|^2}r^2\right|\right],$$

$$g^k_{2m,n}(x,y) = d_{2m,n}\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}r^2\right|\right]^{2m-n-2k} \left(1 - \left|\frac{y}{r}\right|^2\right)^k \left(1 - \left|\frac{x}{r}\right|^2\right)^k r^{2k}\cdot$$

$$\left[\frac{(-2)^k}{k!}(2m-n)(2(m-1)-n)...(2(m-k+1)-n)\ln\left[\left|\frac{y}{r}\right|\left|x - \frac{y}{|y|^2}r^2\right|\right] -\right.$$

$$\left.\frac{2^{2k}}{2}\left(\frac{1}{k} + \sum_{i=1}^{k-1}\frac{1}{i}\frac{(-1)^{k-i}}{(k-i)!}\frac{(2m-n)}{2}...\frac{(2m-n-2(k-i)+2)}{2}\right)\right],\ \ k = 1, 2, ..., m-1,$$

$$d_{2m,n} = \frac{(-1)^{n/2-1}}{\Gamma(m)\Gamma(m-n/2+1)\cdot 2^{2m-1}\pi^{n/2}}.$$

Lemma 1 *a) It is known [3] that in the case of odd n and even n, when $2m \leq n$, the function*

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n},$$

and in the case of even n, when $2m \geq n$, the function

$$\varepsilon_{2m,n}(x,y) = d_{2m,n}|x-y|^{2m-n}\ln|x-y|$$

is a fundamental solution to equation (10);

b) for all $0 \leq k \leq m-1$ functions

$$g^k_{2m,n}(x,y) = d_k\left[\left|\frac{y}{r}\right| \cdot \left|x - \frac{y}{|y|^2}r^2\right|\right]^{2m-n-2k} \cdot \left(1 - \left|\frac{y}{r}\right|^2\right)^k \left(1 - \left|\frac{x}{r}\right|^2\right)^k r^{2k},$$

where

$$d_k = \frac{1}{(-2)^k k!}\,d_{2m,n}(2m-n)(2(m-1)-n)...(2(m-k+1)-n)$$

are solutions to the homogeneous polyharmonic equation

$$\Delta^m g^k_{2m,n}(x,y) = 0,\ \ x, y \in \Omega. \tag{16}$$

Proof 1 *Indeed, the function $g^k_{2m,n}(x,y)$ can be represented in the form $g^k_{2m,n}(x,y) = g(x,y)f_{2k}(|x|,|y|)$, where $f_{2k}(|x|,|y|)-$ polynomial of degree $2k$ in $|x|$ for fixed $|y|$, and $g(x,y)$ satisfy the equation $\Delta^{m-k}g(x,y)=0$.*

By Almanzi's theorem [3], the function $g(x,y)$ can be represented as

$$g(x,y) = \sum_{j=0}^{m-k-1} |x|^{2j}\Psi_j(x,y),$$

where $\Psi_j(x,y)-$ harmonic functions, i.e. $\Delta_x\Psi_j(x,y) = 0$. Then the function $g^k_{2m,n}(x,y)$ satisfies the representation

$$g^k_{2m,n}(x,y) = \sum_{j=0}^{m-k-1} |x|^{2j}\Psi_j(x,y)f_{2k}(|x|,|y|) = \sum_{j=0}^{m-k-1} |x|^{2j}\widetilde{\Psi}_j(x,y),$$

where $\widetilde{\Psi}_j(x,y)-$ some harmonic functions.

Therefore, according to Almanzi's theorem, the function $g^k_{2m,n}(x,y)$ for all $0\le k\le m-1$ satisfies homogeneous polyharmonic equation (16).

It is easy to show that in the following notation

$$|x-y|^2 = X^2(x,y) = X^2,\ \left|\frac{y}{r}\right|^2\left|x-\frac{y}{|y|^2}r^2\right|^2 = Y^2(x,y) = Y^2,$$

$$\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)r^2 = Z^2(x,y) = Z^2, \tag{17}$$

we have the identity

$$X^2 - Y^2 = -Z^2,\ \forall x,y\in\Omega. \tag{18}$$

Proof 2 *a) Using equality (18) and the expansion of functions $f(x)=(1-x)^\alpha$, $0<x\le 1$ [4], we represent the fundamental solution of equation (10) as a series*

$$\varepsilon_{2m,n}(x,y) = X^{2m-n} = Y^{2m-n}\Big(1-\frac{Z^2}{Y^2}\Big)^{\frac{2m-n}{2}} =$$

$$Y^{2m-n} + \sum_{k=1}^{m-1}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}+$$

$$\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}.$$

Moving the m terms to the left, we get the required Green's function in the following form:

$$G_{2m,n}(x,y) = \mathfrak{G}^m_{2m,n}(x,y) = \mathfrak{G}^\infty_{2m,n}(x,y),$$

where

$$\mathfrak{G}^m_{2m,n}(x,y) = d_{2m,n}\Big[X^{2m-n} - Y^{2m-n} - \sum_{k=1}^{m-1}\frac{(-1)^k}{k!}(m-\frac{n}{2})...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}\Big],$$

$$\mathfrak{G}^{\infty}_{2m,n}(x,y)=d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}.$$

Because

$$\Big(X^2-Y^2\Big)\Big|_{x\in\partial\Omega,y\in\Omega}=-Z^2\Big|_{x\in\partial\Omega,y\in\Omega}=-r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)\Big|_{x\in\partial\Omega,y\in\Omega}=0,$$

then using the equalities

$$\frac{\partial^j}{\partial n_x^j}Z^{2m}\Big|_{x\in\partial\Omega,y\in\Omega}=0,\ j=\overline{0,m-1},$$

it is easy to show that the function

$$\mathfrak{G}^{\infty}_{2m,n}(x,y)=$$

$$Z^{2m}\Big[d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+m+1\Big)Y^{2m-n-2k}Z^{2k-2m}\Big]=$$

$$\left(r^2\left(1-\left|\frac{y}{r}\right|^2\right)\left(1-\left|\frac{x}{r}\right|^2\right)\right)^m\times$$

$$\Big[d_{2m,n}\sum_{k=m}^{\infty}\frac{(-1)^k}{k!}(m-\frac{n}{2})(m-\frac{n}{2}-1)...\Big(m-\frac{n}{2}-k+m+1\Big)Y^{2m-n-2k}Z^{2k-2m}\Big]$$

satisfies the boundary condition (14).

According to Lemma 1 and the last equality, we have

$$(-\Delta_x)^mG_{2m,n}(x,y)=(-\Delta_x)^m\mathfrak{G}^m_{2m,n}(x,y)=\delta(x-y),\ x,y\in\Omega,$$

$$\frac{\partial^j}{\partial n_x^j}G_{2m,n}(x,y)\Big|_{x\in\partial\Omega}=\frac{\partial^j}{\partial n_x^j}\mathfrak{G}^{\infty}_{2m,n}(x,y)\Big|_{x\in\partial\Omega}=0,\ j=\overline{0,m-1}.$$

By virtue of the uniqueness of the solution to the Dirichlet problem for the polyharmonic equation, the Green's function of problem (13), (14) is

$$G_{2m,n}(x,y)=d_{2m,n}\Big[X^{2m-n}-Y^{2m-n}-\sum_{k=1}^{m-1}\frac{(-1)^k}{k!}(m-\frac{n}{2})...\Big(m-\frac{n}{2}-k+1\Big)Y^{2m-n-2k}Z^{2k}\Big].$$

b) Using Lemma 1 and the expansion of functions $f(x)=\ln(1-x)$, $0<x\le 1$ [4], we represent the fundamental solution of equation (10) as a series

$$\varepsilon_{2m,n}(x,y)=|x-y|^{2m-n}\ln|x-y|=X^{2m-n}\ln X=$$

$$\Big[Y^{2m-n}\ln Y+\sum_{\mu=1}^{m-\frac{n}{2}}\frac{1}{2\mu}\Big]\cdot\Big(1-\frac{Z^2}{Y^2}\Big)^{m-\frac{n}{2}}+\frac{1}{2}\Big[\ln\Big(1-\frac{Z^2}{Y^2}\Big)-\sum_{\mu=1}^{m-\frac{n}{2}}\frac{1}{2\mu}\Big]\cdot\Big(1-\frac{Z^2}{Y^2}\Big)^{m-\frac{n}{2}}=$$

$$Y^{2m-n}\ln Y+\sum_{\nu=1}^{m-\frac{n}{2}}C_\nu^{m-\frac{n}{2}}Z^{2\nu}Y^{2(m-\nu)-n}\ln Y+\sum_{\nu=1}^{m-\frac{n}{2}}(-1)^\nu C_\nu^{m-\frac{n}{2}}\sum_{\mu=m-\nu+1-\frac{n}{2}}^{m-\frac{n}{2}}\frac{1}{2\mu}Z^{2\nu}Y^{2(m-\nu)-n}+$$

$$(-1)^{m+1-\frac{n}{2}}\sum_{\nu=1}^{\infty}\frac{1}{2\nu C_\nu^{\nu+m-\frac{n}{2}}}Z^{2(m+\nu)-n}Y^{-2\nu}.$$

Moving the $m-1$ terms to the left, we get the equality

$$G_{2m,n}(x,y)=\mathfrak{F}^m_{2m,n}(x,y)=\mathfrak{F}^\infty_{2m,n}(x,y),$$

where

$$\mathfrak{F}^m_{2m,n}(x,y)=d_{2m,n}\Big[X^{2m-n}\ln X-Y^{2m-n}\ln Y-$$

$$\sum_{\nu=1}^{m-n/2}(-1)^\nu C_\nu^{m-n/2}\Big[\ln Y+\widetilde{C}\Big]Z^{2\nu}Y^{2m-2\nu-n}+(-1)^{m-n/2}\sum_{\nu=1}^{n/2-1}\frac{2^{2m+2\nu-n}}{2\nu C_{\nu+n/2}^{m+\nu}}Z^{2(m+\nu)}Y^{-2\nu-n}\Big],$$

$$\mathfrak{F}^\infty_{2m,n}(x,y)=-d_{2m,n}\sum_{\nu=0}^{\infty}\frac{2^{2(m+\nu)}}{(2\nu+n)C_{\nu+n/2}^{m+\nu}}Y^{-n-2\nu}Z^{2(m+\nu)},\ \ \widetilde{C}=\sum_{\mu=m-n/2+1-\nu}^{m-n/2}\frac{1}{2\mu}.$$

Using this representation, just as in the proof of assertion a), we make sure that $G_{2m,n}(x,y)$ is the required Green's function for even n for $2m\geq n$. The theorem is proved.

5 Correct constrictions and extensions of differential operators

In the early 80s of the last century, M.O. Otelbaev and his students [21-23] constructed an abstract theory that allows us to describe all correct constrictions of a certain maximum operator and separately - all correct extensions of a certain minimum operator, independently of each other, in terms of the inverse operator. Moreover, this theory was extended to the case of Banach spaces and it was possible to partially abandon the linearity of operators. We give a brief summary of this theory in the case of Hilbert spaces.

Let the Hilbert space H be a linear operator L with a domain of definition $D(L)$ and a domain of value $R(L)$. The kernel of operator L is the set

$$KerL=\{f\in D(L):Lf=0\}.$$

Definition 1 *A linear closed operator $\widehat{L}$ in a Hilbert space H is called* maximal, *if $R(\widehat{L})=H$ and $Ker\widehat{L}\neq\{0\}$.*

Definition 2 *A linear closed operator L_0 in a Hilbert space H is called* is called, *if $\overline{R(L_0)}\neq H$ and there is a bounded inverse operator L_0^{-1} by $R(L_0)$.*

Definition 3 *A linear closed operator L in a Hilbert space H is called* correct, *if there is a bounded inverse operator L^{-1} defined on all H.*

Definition 4 *Operator L is called* a contraction *of operator L_1, and operator L_1 is called* an extension *of operator L, and briefly write $L\subset L_1$, if*

1) $D(L)\subset D(L_1)$,

2) $Lf=L_1f,\ \forall f\in D(L)$.

Definition 5 *The correct operator L in the Hilbert space H is called* the correct contraction of the maximum operator $\widehat{L}$ *(*the correct extension of the minimum operator L_0*), if $L \subset \widehat{L}$ $(L_0 \subset L)$.*

Definition 6 *A correct operator L in a Hilbert space H is called* a boundary-correct extension*, if L is both a correct contraction of the maximum operator $\widehat{L}$ and a correct extension of the minimum operator L_0, i.e. $L_0 \subset L \subset \widehat{L}$.*

Theorem 2 *[21, 22] Let $\widehat{L}$ be a maximal linear operator in a Hilbert space H, $L-$ a known correct narrowing of operator $\widehat{L}$ and $K-$ an arbitrary linear operator bounded in H that satisfies the following condition*

$$R(K) \subset Ker\widehat{L}. \tag{19}$$

Then the operator L_K^{-1}, defined by the formula

$$L_K^{-1} f = L^{-1} f + K f, \ \forall f \in H, \tag{20}$$

is the inverse of some correct narrowing of L_K of the maximal operator $\widehat{L}$, i.e. $L_K \subset \widehat{L}$. Conversely, if L_1 is some correct narrowing of the maximal operator $\widehat{L}$, then there exists a linear operator K_1 bounded in H that satisfies condition (19), such that the equality holds

$$L_1^{-1} f = L^{-1} f + K_1 f, \ \forall f \in H.$$

As a rule, it is difficult to describe the kernel of the maximal operator. Therefore, often the following Theorem 3 is more effective than Theorem 2.

Theorem 3 *[23] Let $\widehat{L}$ be the maximal operator, L_ϕ be the known correct constriction of $\widehat{L}$, and K be the continuous operator acting from H to $D(\widehat{L})$ be the domain of the definition of operator $\widehat{L}$. Then operator L_K^{-1}, defined by the formula*

$$L_K^{-1} f = L_\phi^{-1} f + (E - L_\phi^{-1} \widehat{L}) K f \tag{21}$$

is the inverse of some correct narrowing $\widehat{L}$, i.e. $L_K \subset \widehat{L}$.
Conversely, any correct narrowing of operator $\widehat{L}$ is represented as (21) with some operator K.

In what follows, this theorem will be applied to the polyharmonic operator.

6 Correct boundary value problems for a polyharmonic operator in a multidimensional ball

In this section $\Omega = \{x \in \mathbb{R}^n : |x| < r\}$. On $D(\widehat{L}) = W_2^{2m}(\Omega)$ we define the maximal operator $\widehat{L}$ by the formula

$$\widehat{L} u = \Delta^m u, \ \forall u \in D(\widehat{L}).$$

By definition $R(\widehat{L}) = L_2(\Omega)$, and $Ker\widehat{L}$ is not trivial.

In the previous section, it was proved that the Dirichlet boundary value problem for the polyharmonic equation

$$L_\phi u := \begin{cases} \Delta_x^m u(x) = f(x), \quad x \in \Omega = \{x : |x| < r\ \}, \\ \dfrac{\partial^j u(x)}{\partial n_x^j} = 0, \ \ 0 \leq j \leq m-1, \ \ x \in \partial\Omega \end{cases}$$

has a unique solution $u(x)$ for any $f \in L_2(\Omega)$, which has an integral representation

$$L_\phi^{-1} f = u(x) = \int_\Omega G_{2m,n}^D(x,y) f(y) dy, \tag{22}$$

where $G_{2m,n}^D(x,y) \equiv G_{2m,n}(x,y)-$ Green's function of the Dirichlet problem from (15).

Note that the zero Dirichlet boundary conditions for a polyharmonic equation are equivalent to the following boundary conditions for the same equation.

Theorem 4 *a) For any $f \in L_2(\Omega)$, the function $u(x)$ given by formula (22) with $m = 2p$ is a solution to the boundary value problem*

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega, \tag{23}$$

$$u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} u(x) \bigg|_{\partial\Omega} = 0, \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} \Delta_x u(x) \bigg|_{\partial\Omega} = 0,$$

$$\ldots\ldots\ldots\ \Delta_x^{p-1} u(x)\,\big|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} \Delta_x^{p-1} u(x) \bigg|_{\partial\Omega} = 0. \tag{24}$$

b) For any $f \in L_2(\Omega)$, the function $u(x)$ given by formula (22) with $m = 2p+1$ is a solution to the boundary value problem

$$\Delta_x^m u(x) = f(x), \quad x \in \Omega,$$

$$u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} u(x) \bigg|_{\partial\Omega} = 0, \ \Delta_x u(x)\,|_{\partial\Omega} = 0, \ \frac{\partial}{\partial n_x} \Delta_x u(x) \bigg|_{\partial\Omega} = 0,$$

$$\ldots\ldots\ldots\ \frac{\partial}{\partial n_x} \Delta_x^{p-1} u(x) \bigg|_{\partial\Omega} = 0, \ \Delta_x^p u(x)\,|_{\partial\Omega} = 0. \tag{25}$$

Proof 3 *Let us show that* $\Delta u|_{\partial\Omega} = \dfrac{\partial^2}{\partial n^2} u|_{\partial\Omega} = 0,$ *if* $u|_{\partial\Omega} = 0$ *u* $\dfrac{\partial}{\partial n} u|_{\partial\Omega} = 0.$
This fact follows from the following identity

$$\Delta u = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} u + \frac{1}{r^2} \Delta_\theta u, \ x = r \cdot \theta \in \mathbb{R}^n.$$

In the case of the ball Ω, the direction of the outward normal to the boundary $\partial\Omega$ coincides with the direction of the radius of the vector $\overrightarrow{r}$, therefore the derivative with respect to the outward normal at the boundary $\partial\Omega$ coincides with the derivative in the direction of the radius. From here we get

$$\Delta u|_{\partial\Omega} = \frac{\partial^2}{\partial r^2} u|_{\partial\Omega} + \frac{n-1}{r} \frac{\partial}{\partial r} u|_{\partial\Omega} + \frac{1}{r^2} \Delta_\theta u|_{\partial\Omega} = 0,$$

because $\Delta u|_{\partial\Omega} = 0, \ \dfrac{\partial}{\partial n} u|_{\partial\Omega} = 0, \ u|_{\partial\Omega} = 0.$

In this section, based on the representation of the solution (12) of the Dirichlet problem, we present other well-posed boundary value problems for an inhomogeneous polyharmonic equation. For this we apply Theorem 3 to describe correct restrictions of the maximal operator $\widehat{L}$.

Lemma 2 *For any $h \in W_2^{2m}(\Omega)$ fair representation*

$$(E - L_\phi^{-1}\widehat{L})h(x) =$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

Proof 4 *For this purpose, we introduce into consideration the integral*

$$I(x) = L_\phi^{-1}\widehat{L}h = \int_\Omega G_{2m,n}^D(x,y)\Delta_y^m h(y)dy, \tag{26}$$

where $h(y)$ is sufficiently smooth, for example, from the class $W_2^{2m}(\Omega)$, and the rest is an arbitrary function.

Taking into account the second Green's formula for the polyharmonic equation, the integral (26) can be written in the form

$$I(x) = \int_\Omega h(y)\Delta_y^m G_{2m,n}^D(x,y)dy -$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y =$$

$$h(x) - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

From here, on the one hand, we get

$$h - L_\phi^{-1}\widehat{L}h =$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y) \cdot \Delta_y^{m-1-j} h(y) - \Delta_y^j G_{2m,n}^D(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y.$$

Lemma 2 is proved.

Lemma 3 *The Green function of the Dirichlet problem $G_{2m,n}(x,y)$ on the boundary $\partial\Omega$ has the following properties*

$$\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,m-1,\ \forall y \in \partial\Omega, \tag{27.1}$$

$$\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,m-2,\ \forall y \in \partial\Omega, \tag{27.2}$$

$$\frac{\partial}{\partial n_y} \Delta_y^{m-1} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y \in \partial\Omega \tag{27.3}$$

and $(28.i) - (30.i)$ with $m = 2p$; $(28.i) - (31.i)$ with $m = 2p+1$, $i = 1,2,3$.

Proof 5 *It follows from representation (26) that $I(x)$ satisfies the Dirichlet boundary conditions. Therefore, for $x \in \partial\Omega$, taking into account (24) or (25), we obtain the relation*

$$0 \equiv I(x)|_{x\in\partial\Omega} = h(x)|_{x\in\partial\Omega} - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y,$$

i.e. $I(x) \equiv 0$, $\forall x \in \partial\Omega$, which is valid for all sufficiently smooth $h(x)$. Since the function $h(y)$ is arbitrary, we conclude from this that the relations $(27.i)$, $i = 1, 2, 3$.

Using the second boundary condition of Dirichlet, arguing similarly, we obtain the relation

$$0 \equiv \frac{\partial}{\partial n_x} I(x)|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x} h(x)|_{x\in\partial\Omega} - \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \frac{\partial}{\partial n_x} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$. Since the values $\{\Delta_y^{m-1-j} h(y), \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y), j = 0, 1, ..., m-1\}$ are linearly independent from each other, therefore

$$\frac{\partial}{\partial n_x} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-2,\ \forall y \in \partial\Omega, \tag{28.1}$$

$$\frac{\partial}{\partial n_x} \frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-1,\ \forall y \in \partial\Omega, \tag{28.2}$$

$$\frac{\partial}{\partial n_x} \Delta_y^{m-1} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \forall y \in \partial\Omega. \tag{28.3}$$

Taking into account the above statement, the third Dirichlet boundary condition allows us to write out the relation

$$0 \equiv \frac{\partial^2}{\partial n_x^2} I(x)\Big|_{x\in\partial\Omega} = \Delta_x I(x)\Big|_{x\in\partial\Omega} = \Delta_x h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\Delta_x \frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. \Delta_x \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$. Therefore

$$\Delta_x \frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0, 1, ..., m-1,\ j \neq m-2,\ \forall y \in \partial\Omega, \tag{29.1}$$

$$\Delta_x \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,m-1,\ \forall y \in \partial\Omega, \tag{29.2}$$

$$\Delta_x \frac{\partial}{\partial n_y} \Delta_y^{m-2} G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \ \forall y \in \partial\Omega. \tag{29.3}$$

Similarly, we write out other conditions that the Green's function $G_{2m,n}^D(x,y)$ on the boundary $\partial\Omega$ satisfies

for $m = 2p$

$$0 \equiv \frac{\partial^{2p-1}}{\partial n_x^{2p-1}} I(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1} I(x)\Big|_{x\in\partial\Omega} = \frac{\partial}{\partial n_x}\Delta_x^{p-1} h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{2p-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_x}\Delta_x^{p-1} \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{2p-1-j} h(y) - \right.$$

$$\left. \frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y}\Delta_y^{2p-1-j} h(y) \right] dS_y,$$

for $m = 2p+1$

$$0 \equiv \frac{\partial^{2p}}{\partial n_x^{2p}} I(x)\Big|_{x\in\partial\Omega} = \Delta_x^p I(x)\Big|_{x\in\partial\Omega} = \Delta_x^p h(x)\Big|_{x\in\partial\Omega} -$$

$$\sum_{j=0}^{2p} \int_{\partial\Omega} \left[\Delta_x^p \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \Delta_y^{2p-j} h(y) - \right.$$

$$\left. \Delta_x^p \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} \cdot \frac{\partial}{\partial n_y}\Delta_y^{2p-j} h(y) \right] dS_y,$$

for an arbitrary sufficiently smooth function $h(y)$. Therefore, for $m = 2p$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1} \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,2p-1,\ \ \forall y \in \partial\Omega, \tag{30.1}$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,2p-1,\ j \neq p, \forall y \in \partial\Omega, \tag{30.2}$$

$$\frac{\partial}{\partial n_x}\Delta_x^{p-1}\Delta_y^p G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = -\delta(x-y)|_{x\in\partial\Omega},\ \ \forall y \in \partial\Omega; \tag{30.3}$$

for $m = 2p+1$

$$\Delta_x^p \frac{\partial}{\partial n_y}\Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,2p,\ j \neq p\, \forall y \in \partial\Omega, \tag{31.1}$$

$$\Delta_x^p \Delta_y^j G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = 0,\ j = 0,1,...,2p,\ \forall y \in \partial\Omega, \tag{31.2}$$

$$\Delta_x^p \Delta_y^p G_{2m,n}^D(x,y)|_{x\in\partial\Omega} = \delta(x-y)|_{x\in\partial\Omega},\ \ \forall y \in \partial\Omega. \tag{31.3}$$

Lemma 3 is proved.

Now we can describe the domain of the maximum operator $\widehat{L}$ in terms of the Green's function $G_{2m,n}$.

Lemma 4 *The domain of the maximum operator $\widehat{L}$ has the representation*

$$D(\widehat{L}) = \{w : \, w(x) = \int_{\Omega} G_{2m,n}(x,y) f(y) dy + \sum_{j=0}^{m-1} \int_{\partial\Omega} \left[\frac{\partial}{\partial n_y} \Delta_y^j G_{2m,n}(x,y) \cdot \Delta_y^{m-1-j} h(y) - \right.$$

$$\left. - \Delta_y^j G_{2m,n}(x,y) \cdot \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y) \right] dS_y, \, \forall f \in L_2(\Omega), \, \forall h \in W_2^{2m}(\Omega)\}. \tag{32}$$

In particular, if

$$\Delta_y^{m-1-j} h(y)|_{y\in\partial\Omega} = 0, \, \frac{\partial}{\partial n_y} \Delta_y^{m-1-j} h(y)|_{y\in\partial\Omega} = 0, \, j = 0, ..., m-1,$$

then we get $D(L_\phi)$ domain of the operator L_ϕ.

Now the question arises: how to describe the domains of definition of other possible correct restrictions of the maximal operator $\widehat{L}$?

Let K be an operator that puts each function $f(x) \in L_2(\Omega)$ in there is a unique function $h(x) \in W_2^{2m}(\Omega)$, such that $\|Kf\|_{L_2(\Omega)} \leq C\|f\|_{L_2(\Omega)}$. Using the chosen operator K, construct the set

$$D_K = \{ \, w(x) \in D(\widehat{L}) : \, h = Kf \}.$$

On the set D_K we define the operator

$$\widehat{L}\Big|_{D_K} = L_K.$$

It follows from Theorem 3 that L_K is a correct restriction of the maximal operator $\widehat{L}$. In conclusion, we give another description of the operator L_K in terms of boundary conditions.

Theorem 5 *Let K be an arbitrary continuous operator acting from $L_2(\Omega)$ to $D(\widehat{L})$. Then the inhomogeneous operator equation $L_K w = f$ is equivalent to the following boundary value problem*

a) for $m = 2p$

$$\Delta_x^m w(x) = f(x), \quad x \in \Omega,$$

$$w\,|_{\partial\Omega} = K(\Delta_x^m w\,)|_{\partial\Omega}, \, \frac{\partial}{\partial n_x} w\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x} K(\Delta_x^m w\,)\Big|_{\partial\Omega}, \ldots,$$

$$\Delta_x^{p-1} w\,|_{\partial\Omega} = \Delta_x^{p-1} K(\Delta_x^m w\,)\big|_{\partial\Omega}, \, \frac{\partial}{\partial n_x} \Delta_x^{p-1} w\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x} \Delta_x^{p-1} K(\Delta_x^m w\,)\Big|_{\partial\Omega};$$

b) for $m = 2p+1$

$$w\,|_{\partial\Omega} = K(\Delta_x^m w\,)|_{\partial\Omega}, \, \frac{\partial}{\partial n_x} w\,\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x} K(\Delta_x^m w\,)\Big|_{\partial\Omega}, \ldots,$$

$$\frac{\partial}{\partial n_x} \Delta_x^{p-1} w\,\Big|_{\partial\Omega} = \frac{\partial}{\partial n_x} \Delta_x^{p-1} (K\Delta_x^m w\,)\Big|_{\partial\Omega}, \, \Delta_x^p w\,|_{\partial\Omega} = \Delta_x^p (K\Delta_x^m w\,)|_{\partial\Omega}.$$

Other applications of M.Otelbaev's results in various branches of the theory of differential equations can be found in the works [26–29].

7 Conclusion

The studies carried out in this article are of significant importance in the theory of boundary value problems of linear and nonlinear partial differential equations, spectral theory, and the theory of numerical methods for approximate solutions of certain classes of boundary value problems for differential equations.

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3-бөлім	**Раздел 3**	**Section 3**
Информатика	**Информатика**	**Computer Science**

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DETERMINATION OF THE SLIDING STATE OF THE SOILS OF THE SHYMBULAK SLOPE WITH THE DEVELOPMENT OF THE CRITERION OF FAILURE

On the basis of the known criterion for the destruction of Coulomb Mor applied to soils of isotropic structure, developed the general criteria for the destruction of stratified, anisotropic, inclined layer of soils. New expressions were obtained to determine the parameters of the mechanics of disruption allowing not only the direction of disruption, but also the possible distribution of dislocations along the layers of soil and perpendicular. Applying the proposed criterion, the problem of determining the stressful condition of the soils of the dangerous rock slope of the slope, and on the joints of soils of different geological structure, such as eluvium and delta. The results of the analysis are presented in the form of tables, diagrams and graphs. Deductions are made on the reliability of the proposed approach to the solution of the problem on the determination of the precondition of the soil of mountain slopes. The first part of the work describes the different categories of communication conditions. The second section gives various examples of the construction of individual elastic curved joints and their connections, supplemented by certain connection conditions of the limit problem. The result is a numerical calculation of the natural frequency of free oscillations of the joints of elastic bent joints.
Key words: slope, soil, layers, stress, anisotropy, landslide.

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ШЫМБҰЛАҚ БӨКТЕРІ ТОПЫРАҚТАРЫНЫҢ КӨШКІН ЖАҒДАЙЫНДА БҰЗЫЛУ КРИТЕРИЙІН ӘЗІРЛЕУ АРҚЫЛЫ АНЫҚТАУ

Изотропты топырақтарда қолданылатын белгілі Кулон Мордың бұзылу критерийі негізінде құрылымы қатпарлы, анизотропты, көлбеу қабатты топырақтар үшін жалпыланған бұзылу критериі жасалды. Бұзылу механикасының параметрлерін анықтайтын жаңа өрнектер алынды, бұл тек бұузылудың таралу бағытын ғана емес, сонымен қатар бұзылу сызығының топырақ қабаттары бойымен және оған перпендикуляр бойынша таралуын да анықтауға мүмкіндік береді.ұсынылған критерийді қолдана отырып,көлбеу төсеніштердің көшкінге бейімді тау беткейіндегі және элювий мен делювий сияқты әртүрлі геологиялық құрылымдардың топырақтарының түйіскен жерлеріндегі топырақтардың стресс жағдайын анықтау мәселесі шешілді. Талдау нәтижелері кестелер, сызбалар және графиктер түрінде өсынылған. Тау беткейлері топырақтарының көшкін алдындағы жағдайын анықтау мәселесін шешуге ұсынылған тәсілдің сенімділігі туралы қорытынды жасалады.

Жұмыстың бірінші бөлімі байланыс шарттарының әртүрлі категорияларын сипаттайды. Екінші бөлімде шектік есептің белгілі бір байланыс шарттарымен толықтырылғандағы жұқа серпімді иілген өзектер мен олардың байланыстарының контрукцияларының әртүрлі мысалдары келтірілген. Қорытындыда серпімді жұқа иілген өзектердің түйіспелерінің бос тербелістерінің табиғи жиілігін сандық есептеуі келтірілген.
Түйін сөздер: көлбеу, топырақ, қабаттар, стресс, анизотропия, көшкін.

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ОПРЕДЕЛЕНИЕ ОПОЛЗНЕВОГО СОСТОЯНИЯ ГРУНТОВ СКЛОНА ШЫМБУЛАК С РАЗРАБОТКОЙ КРИТЕРИЯ РАЗРУШЕНИЯ

На базе известного критерия разрушения Кулона Мора применяемого для грунтов изотропного строения, разработан обобщенный критерий разрушения для грунтов слоистого, анизотропного, наклонно слоистого строения. Получены новые выражения для определения параметров механики разрушения позволяющих определить не только направление распространения разрушений, но и возможного распространения линии разрушения вдоль слоев грунта и перпендикулярно к нему. Применением предложенного критерия решена задача об определений напряженного состояния грунтов оползнеопасного горного склона наклонной слоистости, и на стыках грунтов различного геологического строения, типа элювия и делювия. Приводятся результаты анализа в виде таблиц, эпюр и графика. Делаются выводы о надежности предложенного подхода решения задачи по определению предоползневого состояния грунтов горных склонов. В первой части работы описаны разные категории условий сообщения. Во втором разделе приведены различные примеры построения отдельных упругих криволинейных соединений и их соединений, дополненные некоторыми условиями соединения предельной задачи. Результатом является численный расчет собственной частоты свободных колебаний соединений упругих гнутых соединений.

Key words: склон, грунт, слои, напряжение, анизотропия, оползень.

1 Introduction

Due to the lack of a reliable criterion for the destruction of soils on slopes of complex structure, it is still not possible to predict the landslide hazard of mountain slopes, at the immediate foot of which there are buildings, houses, objects of the national economy and densely populated areas. It is assumed that soil failure occurs when the value of stress concentration near the heterogeneity of the soil structure reaches the maximum possible breaking stress. Despite the large number of works on fracture mechanics, research in this area cannot be considered complete, all the more complete, especially in the field of fracture mechanics on mountain slopes, under the foundations of civil and engineering structures for various purposes. The rapid development of computer technologies and methods of computer-mathematical modeling makes it possible to solve specific practical problems with a high degree of reliability in the application of the results. Among these methods of mathematical modeling, the most widespread is the finite element method (FEM). Instantaneous, unexpected landslide masses on the slopes represent a great danger and cause significant human and material damage. To apply the results of the development of a new criterion, one of the specific mountain slopes of the Northern Tien Shan, the Shym Bulak mountain slope, is considered. Landslides of various types have often occurred here over the past decades. These include landslides, avalanches, embankments, and mixed-type landslides. This slope is located on a mountain gorge on a mountain road between the Medeu dam and the Shym Bulak sports complex. The landslides listed above occurred here in 2009, 2011, and in 2015, which are shown in figures1. The types

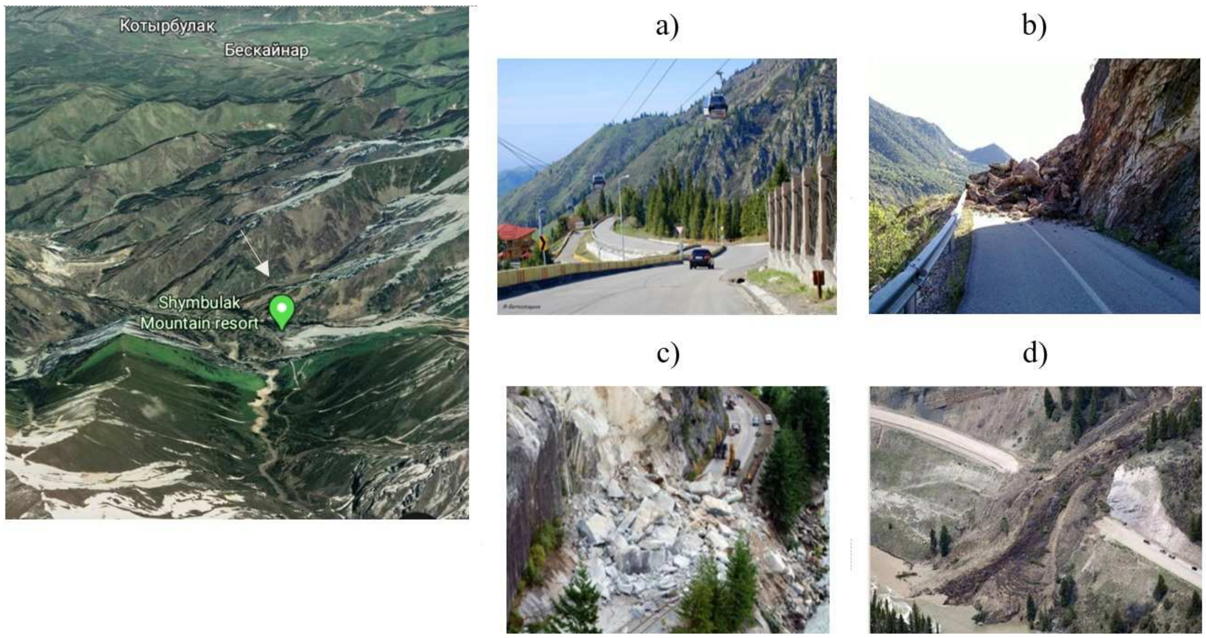

Figure 1: View from space of the Shym Bulak mountain gorges together with the Kishi Almaty River and the road from the Medeu dam to the sports complex

and characters of landslides are as follows: a -2009, species before the landslide; b - and c - landslides and collapses of boulder soils in 2011; d - landslide deluvial soil masses 2015.

2 Materials and Methods

After finding the FEM values of the stress components in the elements σ_x, σ_z and τ_{xz}, the values of the principal stresses and the directions of the main areas $\ alpha$ are calculated using the following well-known expressions of the theory of elasticity [6].

$$\sigma_{max} = \frac{\sigma_{\text{ya}} + \sigma_{\text{ch}}}{2} + \frac{1}{2}\sqrt{(\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2},$$

$$\sigma_{min} = \frac{\sigma_z + \sigma_x}{2} - \frac{1}{2}\sqrt{(\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2}, \tag{1}$$

$$\tan 2\alpha = \frac{2\tau_{zx}}{\sigma_z - \sigma_x}, \tau_{max} = \frac{\sigma_{max} + \sigma_{min}}{2}.$$

Values of normal stress components $\sigma_{n,\overline{\varphi}}, \sigma_{t,\overline{\ varphi}}, \tau_{nt,\overline{\varphi}}$ across and along layers of isotropy planes of transtropic array $\overline{\varphi}$, calculated using the following relations, after applying transformation formulas /7/

$$\sigma_{n,\overline{\varphi}} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\overline{\varphi} + \tau_{xy}\sin 2\overline{\varphi}, \tag{2}$$

$$\sigma_{t,\overline{\varphi}} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\overline{\varphi} - \tau_{xy}\sin 2\overline{\varphi}, \tag{3}$$

$$\tau_{nt,\overline{\varphi}} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\overline{\varphi} + \tau_{xy}\cos 2\overline{\varphi} \tag{4}$$

The values of the normal stress components σ_n, σ_t, τ_{nt} at the main sites are similarly calculated for the angle $alpha$ using the following formulas

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha,$$

$$\sigma_t = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha - \tau_{xy}\sin 2\alpha,$$

$$\tau_{nt} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha + \tau_{xy}\cos 2\alpha.$$

For any angle $0 \le \overline{\alpha}\, le 90^0$, which defines an arbitrary direction measured from the isotropy plane $\overline{\varphi}$ to the perpendicular to them, for breaking shear stresses, we determine using the Casagrande Carrillo condition in the form [6].

$$\tau_{\overline{\alpha}} = \tau_{max,\perp} + (\tau_{max,\coprod} - \tau_{max,\perp})cos^2\overline{\alpha} \tag{5}$$

or

$$\tau_{\overline{\alpha}} = \tau_{max,90^0} + (\tau_{max,0} - \tau_{max,90^0})cos^2\overline{\alpha} \tag{6}$$

where $\tau_{max,\coprod}$, $tau_{max,\perp}$ are the experimentally determined critical values of layered rocks for the cases $overline\alpha = 0$ and $overline\alpha = 90^0$. Comparing the calculated values of the maximum tangential stresses τ_{max} according to (1) with their critical values according to (4), we determine the zones of destruction of soils and rocks.

If $\tau_{max} < \tau_{\overline{\alpha}}$, then the state of the array is stable.

If $\tau_{max} > \tau_{\overline{\alpha}}$, then the array is destroyed in the direction of the $\overline{\alpha}$ corner.

3 Simulation of the Shymbulak slope using FEM

For the transition to mathematical modeling of the Shym Bulak mountain slope, the following calculation scheme was developed with a West - East view shown in Figures 2 below. Here, the red dotted line shows the slope surface before the landslide, the blue dotted line after the landslide, that is, the current state of the slope. Pebbles and other lines show the movements of the landslide mass and their accumulation on the road. Figure 3 shows the design scheme of the slope consisting of the joints of eluvium, diluvium and proluvium. It also shows the

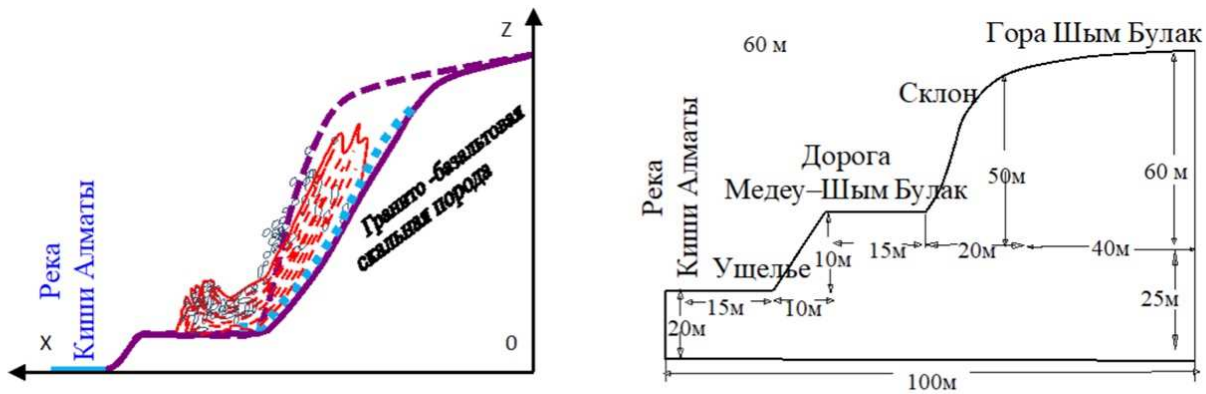

Figure 2: Cross-sectional diagram of the "Shym Bulak"landslide, which consists of granite-basalt rock, soil deposits on a steep slope, a road and the Kishi Almaty river

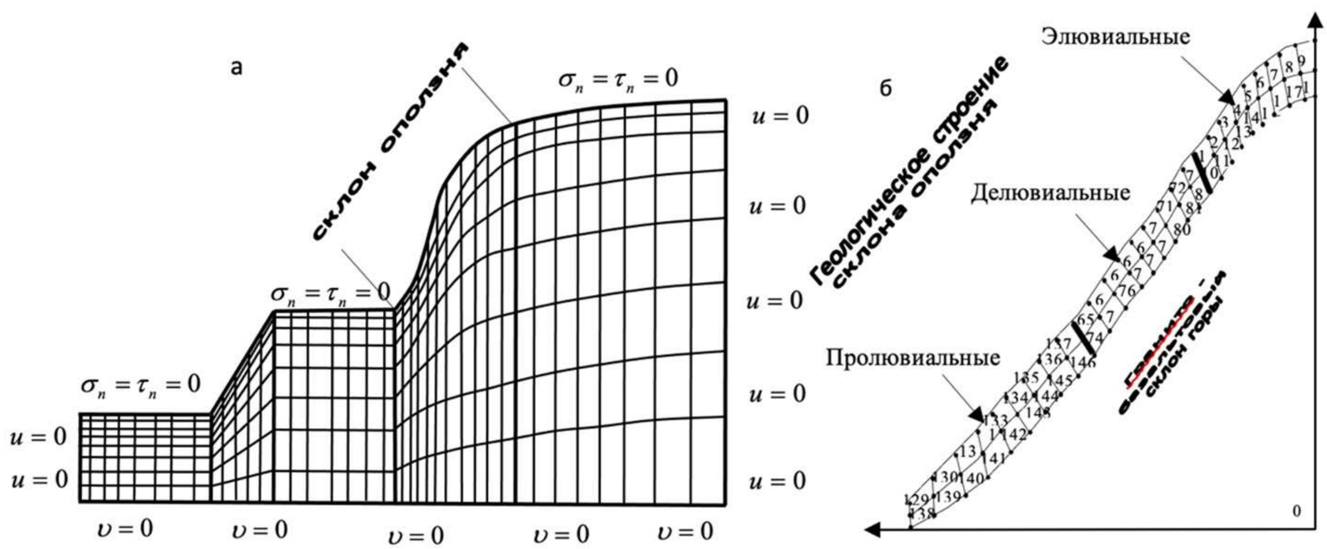

Figure 3: Design scheme of the "Shym Bulak"landslide, prepared for finite element modeling

geometrical dimensions prepared for mathematical modeling. It covers all real natural objects with exact dimensions that are located near landslides and avalanches, shown in Figures 1.

Boundary conditions of the problem. To solve this problem, mixed boundary conditions in stresses and displacements are set. On the two lateral boundaries there are no horizontal displacement components ($U = 0$), based on the computational domain, there are no vertical displacement components ($V = 0$), everywhere at the upper boundaries, there are no normal stress components on the free day surface ($\alpha_n = \tan_n = 0$).

The area shown in Figure 3 is divided into 1260 isoparametric quadrangular elements with a total number of subdivisions of 1334. Slope soils consist of varieties of loam obtained with mixtures of eluvial, deluvial deposits. Therefore, it has an anisotropic structure. The strength properties of the soils of the slope are given in Table 1. Calculations for determining the stress-strain state of soils on the ShymBulak slope were carried out by FEM algorithms and work programs [6] - [9].

Soils	Young's Modules, MPa		Poisson's Ratios		Shift modules, MPa	Volume weight	Forces adhesion, MPa		Angles internal friction, degree	
	E_1	E_2	ν_1	ν_2	G_1	γ	C_1	C_2	φ_1	φ_2
Loam	30.0	15.0	0.36	0.24	7.60	2.00	0.03	0.06	19	23
Loam	12.0	8.0	0.39	0.35	3.40	0.94	0.010	0.014	20	24

4 Results and Discussion

Results of calculated stress values σ_x, σ_z τ_{xz} and σ_{max}, σ_{min}, τ_{max} [6] for different zones and slope layers are shown in Table 2.

The northern slope of the Shym Bulak landslide												
zone	layer	elem №	Stress components in elements, MPa			zone	layer	elem №	Components stresses at the main sites, MPa			Main sites, degree
			σ_x	σ_z	τ_{xz}				σ_{max}	σ_{min}	τ_{max}	α
№ I	№ в 1	8 9	-1.18	-1.05	0.45 -0.42	№ I	№ в 1	8 9	-0.62	-1.53	0.59	31 -69
	2	16 17 18	-1.67	-2.23	-0.44		2	16 18	-1.47	-2.16	0.85	46 -82
II	1	69 73	0.95	-0.49	-0.90	II	1	69 73	1.57			-51
	2	77 81 82	0.41	-1.24	-1.08		2	82		-0.55	1.01	-70
III	1	131 132	-2.57	-2.49	2.29	III	1	131 132 133	2.59	-3.86 -3.48	3.18	52 48 44
	2	139 140	-2.0	-2.61	1.82		2	139 140		-3.48	2.65	56 43

5 Conclusion

By generalizing the critical criteria of fracture mechanics known for an isotropic medium, analytical expressions are obtained that make it possible to determine the type of fracture and opening of cracks propagating along and across the layers of the isotropic plane in a transtropic massif. Thus, the proposed model, research methodology, calculations performed, results obtained for one of the real slopes of the Northern Tien Shan show its reliability.

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PARALLEL CUDA IMPLEMENTATION OF THE ALGORITHM FOR SOLVING THE NAVIER-STOKES EQUATIONS USING THE FICTITIOUS DOMAIN METHOD

An important direction of development of numerical modeling methods is the study of approximate methods for solving problems of mathematical physics in complex multidimensional domains. To solve many applied problems in irregular domains, the fictitious domain method is widely used, the idea of which is to solve the problem not in the original, but in a simpler domain. This approach allows to create application software packages for numerical modeling of processes in arbitrary computational domains. In this paper, we develop a computational method for solving the Navier-Stokes equations in the Boussinesq approximation in two-connected domains by the fictitious domain method with continuation by lower coefficients. The problem formulation in the current function, velocity vortex variables is considered. A computational algorithm for solving the auxiliary problem of the fictitious domain method based on the finite difference method is developed. A parallel implementation of the algorithm using the CUDA parallel computation architecture is developed, which was tested on various configurations of the computational mesh. The results of computational experiments for the problem under consideration are presented.
Key words: Navier-Stokes equations, stream function, velocity vortex, fictitious domain method, boundary conditions, CUDA, parallel algorithm, high performance computing.

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Жалған облыстар әдісін пайдаланып Навье-Стокс теңдеулерін шешу алгоритмінің параллельді CUDA жүзеге асырылуы

Сандық модельдеу әдістерін дамытудың маңызды бағыты - математикалық физика есептерін күрделі көп өлшемді облыстарда шешудің жуық әдістерін зерттеу болып табылады. Көптеген қолданбалы есептерді күрделі облыстарда шешу үшін жалған облыстар әдісі кеңінен қолданылады. Оның идеясы есепті бастапқы облыста емес, қарапайым облыста шешуге негізделген. Бұл тәсілдеме еркін есептеу облыстарында үрдістерді сандық модельдеуге арналған қолданбалы бағдарламалар пакеттерін жасауға мүмкіндік береді. Бұл жұмыста кіші коэффициенттері бойынша жалғастырылған жалған облыстар әдісімен екі байланысты облыста Буссинеск жақындатуындағы Навье-Стокс теңдеулерін шешудің есептеу әдісі жасалды. "Ток функциясы, құйын жылдамдығы" айнымалыларындағы есептің қойылымы қарастырылады. Жалған облыстар әдісінің көмекші есебін шешудің ақырлы айырымдық әдісіне негізделген есептеу алгоритмі жасалды. CUDA параллельді есептеу архитектурасын қолдана отырып, параллельдік алгоритм жүзеге асырылды, ол есептеу торының әртүрлі конфигурацияларында сыналды. Қарастырылып отырған есеп үшін есептеу эксперименттерінің нәтижелері келтірілді.

Түйін сөздер: Навье-Стокс теңдеулері, ток функциясы, жылдамдық құйыны, жалған облыстар әдісі, шекаралық шарттар, CUDA, параллель алгоритм, жоғары өнімді есептеулер.

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Параллельная CUDA-реализация алгоритма решения уравнений Навье-Стокса с использованием метода фиктивных областей

Важным направлением развития методов численного моделирования являются исследования приближенных методов решения задач математической физики в сложных многомерных областях. Для решения многих прикладных задач в нерегулярных областях широко применяется метод фиктивных областей, идея которого заключается в решении задачи не в исходной, а в более простой области. Данный подход позволяет создавать пакеты прикладных программ для численного моделирования процессов в произвольных расчетных областях. В настоящей работе разработан вычислительный метод решения уравнений Навье-Стокса в приближении Буссинеска в двухсвязных областях методом фиктивных областей с продолжением по младшим коэффициентам. Рассматривается постановка задачи в переменных "функция тока, вихрь скорости". Разработан вычислительный алгоритм решения вспомогательной задачи метода фиктивных областей на основе конечно-разностного метода. Осуществлена параллельная реализация алгоритма с использованием архитектуры параллельных вычислений CUDA, которая была протестирована на различных конфигурациях вычислительной сетки. Приведены результаты вычислительных экспериментов для рассматриваемой задачи.

Ключевые слова: уравнения Навье-Стокса, функция тока, вихрь скорости, метод фиктивных областей, граничные условия, CUDA, параллельный алгоритм, высокопроизводительные вычисления.

1 Introduction

The growth of computer technology productivity and the development of parallel computing contributed to the solution of important practical problems of the industry. One example of such problems is the assessment of efficiency and forecasting of oil field development indicators. Due to the complexity of the mathematical models describing these processes, calculations for a single field can last from several hours to several days. Therefore, the issue of developing effective parallel algorithms that can significantly speed up calculations becomes relevant.

Along with the classical model of fluid flow in porous media based on Darcy's law, a number of other models are widely used in the study of fluid flows in oil reservoirs, such as the models of N. E. Zhukovsky [1], Forchheimer [2], Navier-Stokes [1, 2]. The use of these models is associated with a violation of the Darcy law under certain conditions, the need for a detailed study of flow processes near wells [3], etc.

The aim of this paper is to construct an algorithm for the numerical implementation of the model of an incompressible fluid motion using the CUDA parallel computing software and hardware architecture. The initial boundary value problem for the Navier-Stokes equations in the current function, velocity vortex variables in a two-dimensional two-connected domain is considered. To solve this problem, we consider an approximate method based on the fictitious domain method with continuation by lower coefficients. The discretization of the obtained equations is carried out by the finite difference method, but the obtained results will be used in parallel implementation of finite element methods in subsequent works. In conclusion, the results of computational experiments for various mesh configurations and acceleration analysis of the calculations are presented.

2 Literature review

Let us first give a literature review of recent works devoted to solving problems of fluid motion in complex domains by the fictitious domain method. This method is applied for solving a wide class of problems in computational fluid dynamics, including the problem of flow around obstacles of a viscous incompressible fluid with boundary slip condition using the Navier law [4], the two-phase Stokes problem with surface tension forces [5], the problem of the non-Newtonian incompressible fluid motion [6], the problem of flow with arbitrary particle density [7], the problem of modeling the interaction of movable or deformable structure with an internal or surrounding fluid flow [8, 9], simulations of superquadric particles in fluid flows [10]. In [11], the fictitious domain method with H^1-penalty for the Stokes problem with the Dirichlet boundary condition is studied. [12] is devoted to the application of the fictitious domain method in the numerical simulation of a pulse oscillation converter.

The papers [13, 14] are devoted to the study of the fictitious domain method for problems with discontinuous coefficients. In [15,16], an elliptic equation with strongly varying coefficients is considered. The interest in the study of such equations is caused by the fact that equations of this type are obtained using the fictitious domain method. A special method for the numerical solution of an elliptic equation with strongly varying coefficients is proposed. A theorem is proved for estimating the convergence rate of the developed iterative process. A computational algorithm is developed and numerical calculations are performed to illustrate the effectiveness of the proposed method. In [17], modifications of well-known iterative methods for solving grid problems are constructed that arise is the fictitious domain method. The possibilities of the fictitious domain method are illustrated by examples of solving the problems of ideal and viscous incompressible fluid, fluid flow in porous media under a hydraulic structure.

Parallel implementations of the fictitious domain method are also known. For example, in [18], a parallel fictitious domain method is constructed for the three-dimensional Helmholtz equation, in [19] - for modeling particle-loaded flows and turbulent flow in a channel, in [20] - for biomechanics problems.

Currently, high-performance computing is widely used in the field of scientific research. Computer technologies and fluid dynamics models are developing every day, which allow to evaluate and analyze various technological processes. In this regard, the efficiency of solving scientific problems increases. Supercomputing technologies are widely used in many industries. Calculations that are performed on graphics devices significantly speed up the calculation of these "large" problems due to their unique architecture [21, 22].

Many papers have been devoted to the study of the applicability of the CUDA parallel computing architecture to various applied problems which allows increasing computing performance through the use of graphics processors. Its numerous applications to computational fluid dynamics problems are known, including the problems of the oil industry [21, 23], the problems of the motion of a viscous incompressible fluid [24], the problems of underground hydrogen storage [25], and others [22, 26].

3 Material and methods

3.1 The formulation of the problem

To model convective flows, we consider the Navier-Stokes equations in the Boussinesq approximation [27] in the two-dimensional domain $\overline{D} = D \cup \partial D$:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{1}{\mathrm{Re}}\Delta u, \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = \frac{1}{\mathrm{Re}}\Delta v - \mathrm{Gr}\theta, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{RePr}}\Delta \theta, \quad (x,y) \in D, \ \ t \in (0,T] \tag{4}$$

with the following initial and boundary conditions:

$$u = u_0\left(x,y\right), \quad v = v_0\left(x,y\right), \quad \theta = \theta_0\left(x,y\right), \quad (x,y) \in \overline{D}, \ \ t = 0, \tag{5}$$

$$u = a_x\left(x,y,t\right), \quad v = a_y\left(x,y,t\right), \quad \theta = \xi\left(x,y,t\right), \quad (x,y) \in \partial D, \ \ t = [0,T], \tag{6}$$

where u, v are components of the velocity, p is the pressure, θ is the temperature, Re is the Reynolds number, Gr is the Grashof number, Pr is the Prandtl number, $\partial D = \gamma_1 \cup \gamma_2$ is the boundary of the domain $\overline{D}$.

We introduce the current function ψ and the velocity vortex ω, which are related with the velocity components u, v by the following relations:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{7}$$

The problem (1)-(6) in the variables ψ, ω is written as follows [12]:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{\mathrm{Re}}\Delta \omega + \mathrm{Gr}\frac{\partial \theta}{\partial x}, \tag{8}$$

$$\Delta \psi = \omega, \tag{9}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{\mathrm{RePr}}\Delta \theta, \quad (x,y) \in D, \ \ t \in (0,T], \tag{10}$$

$$\omega = \alpha\left(x,y\right), \quad \theta = \varphi\left(x,y\right), \quad (x,y) \in \overline{D}, \quad t = 0, \tag{11}$$

$$\psi = \xi_1\left(x,y,t\right), \quad \frac{\partial \psi}{\partial \vec{n}} = \eta_1\left(x,y,t\right), \quad (x,y) \in \gamma_1, \quad t \in (0,T], \tag{12}$$

$$\psi = \xi_2\left(x,y,t\right) + \lambda\left(t\right), \quad \frac{\partial \psi}{\partial \vec{n}} = \eta_2\left(x,y,t\right), \quad (x,y) \in \gamma_2, \quad t \in (0,T], \tag{13}$$

$$\theta = \beta_l\left(x,y,t\right), \quad (x,y) \in \gamma_1, \quad l = 1,2, \quad t \in (0,T], \tag{14}$$

α, φ, ξ_i, η_i, β_i, $i = 1, 2$ are given functions.

Introduce a uniform mesh in $\overline{D}$:

$$D_h = \{(x_i, y_j)\,,\ \ x_i = (i-1)\,h_1,\ \ y_j = (j-1)\,h_2,\ \ i = 1, ..., n_1,$$

$$\left. j = 1, ..., n_2,\ \ h_1 = \frac{l_1}{n_1 - 1},\ \ h_2 = \frac{l_2}{n_2 - 1} \right\}.$$

Assume that the inner subdomain D_0 is a rectangle:

$$D_{0h} = \{(x, y)\,,\ x_{k1} \le x \le x_{k2},\ \ y_{m1} \le y \le y_{m2}\}\,.$$

Consider the domain D_1 covering the domain D_0, that is $D_0 \subset D_1$. $D_1 = \{(x, y)\,,\ x_{k3} \le x \le x_{k4},\ \ y_{m3} \le y \le y_{m4}\}$.

Consider the fictitious domain method for solving the problem (8)-(14):

$$\frac{\partial \omega^\varepsilon}{\partial t} + \frac{\partial \psi^\varepsilon}{\partial y}\frac{\partial \omega^\varepsilon}{\partial x} - \frac{\partial \psi^\varepsilon}{\partial x}\frac{\partial \omega^\varepsilon}{\partial y} = \frac{1}{\mathrm{Re}}\Delta\omega^\varepsilon + \mathrm{Gr}\frac{\partial \theta^\varepsilon}{\partial x} - \operatorname{div}\left(k\,(x, y)\,\nabla\psi\right), \tag{15}$$

$$\Delta\psi^\varepsilon = \omega^\varepsilon, \tag{16}$$

$$\frac{\partial \theta^\varepsilon}{\partial t} + \frac{\partial \psi^\varepsilon}{\partial y}\frac{\partial \theta^\varepsilon}{\partial x} - \frac{\partial \psi^\varepsilon}{\partial x}\frac{\partial \theta^\varepsilon}{\partial y} = \frac{1}{\mathrm{RePr}}\Delta\theta^\varepsilon, \tag{17}$$

$$\psi^\varepsilon\Big|_{\gamma_1} = 0, \quad \frac{\partial \psi^\varepsilon}{\partial \vec{n}}\Big|_{\gamma_1} = 0, \tag{18}$$

$$\theta^\varepsilon = \beta_l\,(x, y, t)\,, \quad (x, y) \in \gamma_l,\ \ l = 1, 2,\ \ t \in (0, T]\,,$$

where

$$k\,(x, y) = \begin{cases} 1, & (x, y) \in D_0, \\ 0, & (x, y) \in \overline{D} \backslash D_0. \end{cases}$$

For the numerical solution of the obtained problem (15)-(18), the following explicit scheme and the iterative method of successive over-relaxation are constructed. For simplicity, we exclude the superscript ε. Replace the differential problem with its difference analog of the following form:

$$\frac{\omega_{ij}^{n+1} - \omega_{i,j}^n}{\tau} + \Lambda_{1,h}\omega_{i,j}^n + \Lambda_{2,h}\omega_{i,j}^n = \frac{1}{\mathrm{Re}}\Lambda_{11,h}\omega_{i,j}^n + \frac{1}{\mathrm{Re}}\Lambda_{22,h}\omega_{i,j}^n + \mathrm{Gr}\,\Phi_h\theta_{i,j}^n - \Lambda_{12,h}\psi_{i,j}^n, \tag{19}$$

$$\Lambda_{11,h}\psi_{i,j}^{n+1} + \Lambda_{22,h}\psi_{i,j}^{n+1} = \omega_{i,j}^{n+1}, \tag{20}$$

$$u_{i,j}^{n+1} = \frac{\psi_{i,j+1/2}^{n+1} - \psi_{i,j-1/2}^{n+1}}{h_2}, \tag{21}$$

$$v_{i,j}^{n+1} = -\frac{\psi_{i+1/2,j}^{n+1} - \psi_{i-1/2,j}^{n+1}}{h_1}, \tag{22}$$

$$\frac{\theta_{i,j}^{n+1} - \theta_{i,j}^n}{\tau} + \Lambda_{1,h}\theta_{i,j}^n + \Lambda_{2,h}\theta_{i,j}^n = \frac{1}{\mathrm{Re\,Pr}}\Lambda_{11,h}\theta_{i,j}^n + \frac{1}{\mathrm{Re\,Pr}}\Lambda_{22,h}\theta_{i,j}^n. \tag{23}$$

The difference analogs of the corresponding differential operators are as follows:

$$\Lambda_{1,h}\omega_{i,j}^n = \frac{1}{2}\left[\left(u_{i+1/2,j}^n - \left|u_{i+1/2,j}^n\right|\right)\frac{\omega_{i+1,j}^n - \omega_{i,j}^n}{h_1} + \left(u_{i-1/2,j}^n + \left|u_{i-1/2,j}^n\right|\right)\frac{\omega_{i,j}^n - \omega_{i-1,j}^n}{h_1}\right],$$

$$\Lambda_{2,h}\omega_{i,j}^n = \frac{1}{2}\left[\left(v_{i,j+1/2}^n - \left|v_{i,j+1/2}^n\right|\right)\frac{\omega_{i,j+1}^n - \omega_{i,j}^n}{h_2} + \left(v_{i,j-1/2}^n + \left|v_{i,j-1/2}^n\right|\right)\frac{\omega_{i,j}^n - \omega_{i,j-1}^n}{h_2}\right],$$

$$\Lambda_{11,h}\omega_{i,j}^n = \frac{\omega_{i+1,j}^n - 2\omega_{i,j}^n + \omega_{i-1,j}^n}{h_1^2},$$

$$\Lambda_{22,h}\omega_{i,j}^n = \frac{\omega_{i,j+1}^n - 2\omega_{i,j}^n + \omega_{i,j-1}^n}{h_2^2},$$

$$\Lambda_{12,h}\psi_{i,j}^n = \left[k_{i+1/2,j}\frac{\psi_{i+1,j}^n - \psi_{i,j}^n}{h_1^2} - k_{i-1/2,j}\frac{\psi_{i,j}^n - \psi_{i-1,j}^n}{h_1^2}\right] +$$
$$+\left[k_{i,j+1/2}\frac{\psi_{i,j+1}^n - \psi_{i,j}^n}{h_1^2} - k_{i,j-1/2}\frac{\psi_{i,j}^n - \psi_{i,j-1}^n}{h_1^2}\right],$$

$$\Phi_h\theta_{i,j}^n = \frac{\theta_{i+1,j}^n - \theta_{i-1,j}^n}{2h_1}.$$

The numerical implementation algorithm is performed as follows: first, the values of $\omega_{i,j}^{n+1}$ are calculated using the formula (19); then the values of $\psi_{i,j}^{n+1}$ are found by (20). The obtained values of $\psi_{i,j}^{n+1}$ are used to determine the values of $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$ using the formulas (21), (22); after that, using the new values of $u_{i,j}^{n+1}$ and $v_{i,j}^{n+1}$, the values of $\theta_{i,j}^{n+1}$ are calculated using (23). The iterative process is continued until the following condition is met:

$$\max_{\substack{1\le i\le n_1\\ 1\le j\le n_2}} \left|\omega_{i,j}^{n+1} - \omega_{i,j}^n\right| < \varepsilon.$$

The algorithm described above is implemented using the CUDA parallel computing architecture. The grids are divided into blocks, and each block copies the data to the shared memory, after which each node of the individual block performs the calculation and saves the calculated data to the global memory. In each subdomain, it is required to use data from the neighboring subdomain, i.e. it is necessary to copy the boundary data from the global memory, therefore, the size of each subdomain will be increased. The first stage (19), the recalculation stage (21), (22) and the temperature calculation stage (23) are parallelizable, since an explicit scheme is used to implement these stages. The second stage (20) is calculated in global memory, since the neighboring values of the same iterative process are needed to determine the current function.

4 Results and discussion

The method given above is used to numerically solve the test problem (1)-(6) for the Navier-Stokes equations describing the motion of a viscous incompressible fluid in the current function, velocity vortex variables in the Boussinesq approximation.

Below, temperature distributions and current functions are presented as numerical results. The results are obtained for different cavity sizes, temperature conditions at the boundary, and values that determine the flow of dimensionless parameters, the Grashof Gr and Prandtl Pr numbers.

Figures 1-4 show the results of solving the problem by the method of fictitious domain with continuation by the lowest coefficients.

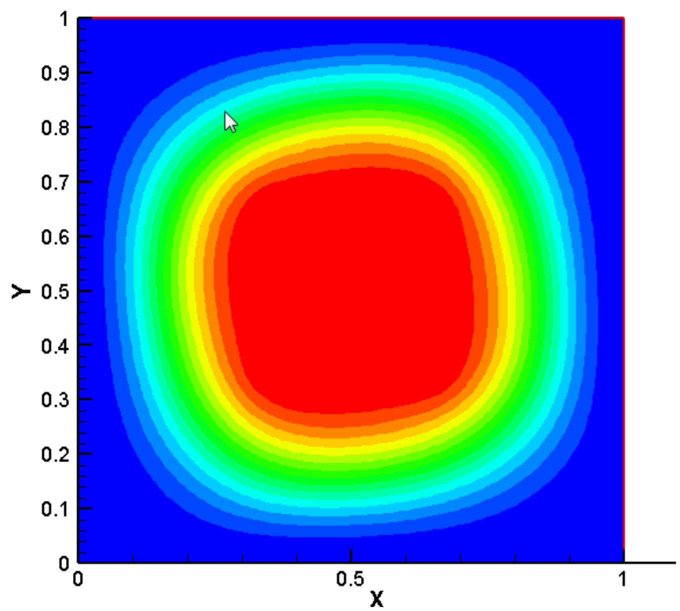

Figure 1: Isolines of the current function. The cavity size is 1.0×1.0; $\theta = 0.5$, $\Pr = 5.39$, $\mathrm{Gr} = 100$ on internal borders

The parallel algorithm of this problem was implemented using the CUDA architecture. When implementing the parallel algorithm on CUDA, two optimization methods were used:

1. Computational data was copied to the internal subdomains, then it was copied from global memory to shared memory. At the end of the optimization process, the boundary data is copied from the global memory. In such cases, the size of the subdomain remains unchanged [27].

2. In our case, it is impossible to avoid re-copying data at the border from the global memory. In these cases, columns and rows are copied at the boundaries of the subdomain. Therefore, the two-dimensional decomposition must be changed to a one-dimensional one. As a result, we do not make repeated copies.

A uniform grid with dimensions of 128x128, 256x256, 512x512, 1024x1024 and 2048x2048 is used in the calculations. All data was represented as single-precision real numbers. The computational experiment was conducted on a personal computer with an Intel Core i7-3770 3.40 GHz quad-core processor and an Nvidia GeForce GTX 550 Ti graphics card. The test result is shown in Figure 5. During the calculation, the following optimal block size was chosen: 16x16 (the number of threads in one block is 256). Figure 6 shows the performance gain compared to the sequential algorithm, depending on the mesh size.

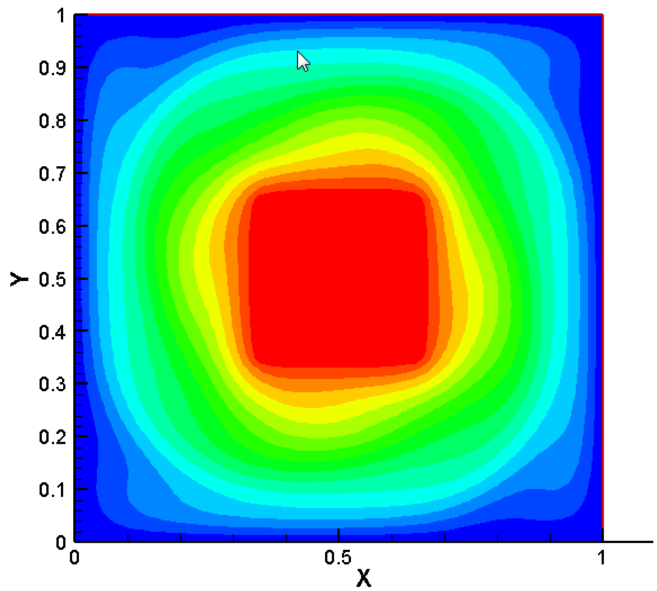

Figure 2: Isotherms. The cavity size is 1.0×1.0; $\theta = 1$, Pr $= 5.39$, Gr $= 100$ on internal borders

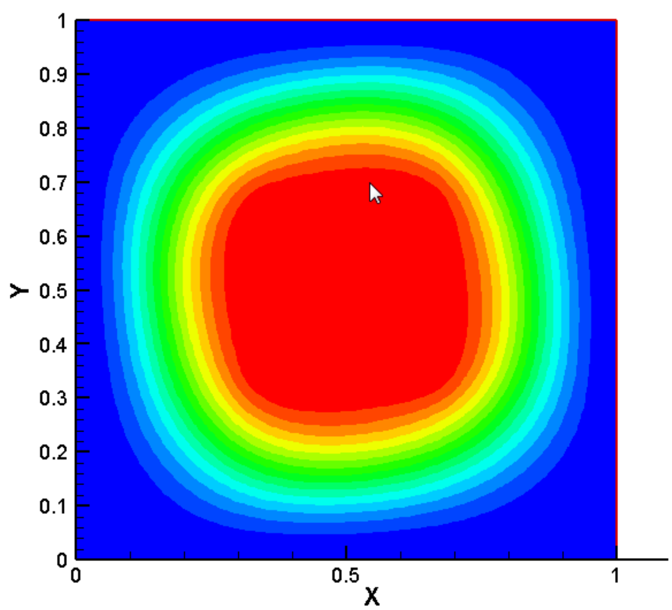

Figure 3: Isolines of the current function. The cavity size is 1.0×1.0; $\theta = 0.5$, Pr $= 5.39$, Gr $= 100$ on internal borders

5 Conclusion

Thus, the paper deals with the numerical implementation of the Navier-Stokes equations in a two-dimensional two-connected domain using the CUDA parallel computing architecture.

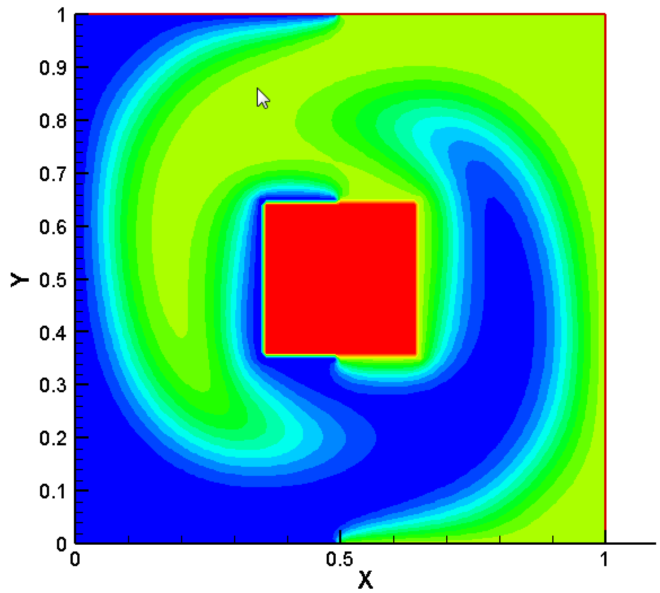

Figure 4: Isotherms. The cavity size is 1.0×1.0; the temperature on the right part of the border is $\theta = 0.5$, the temperature on the left part of the border is $\theta = -0.5$, $\mathrm{Pr} = 5.39$, $\mathrm{Gr} = 100$

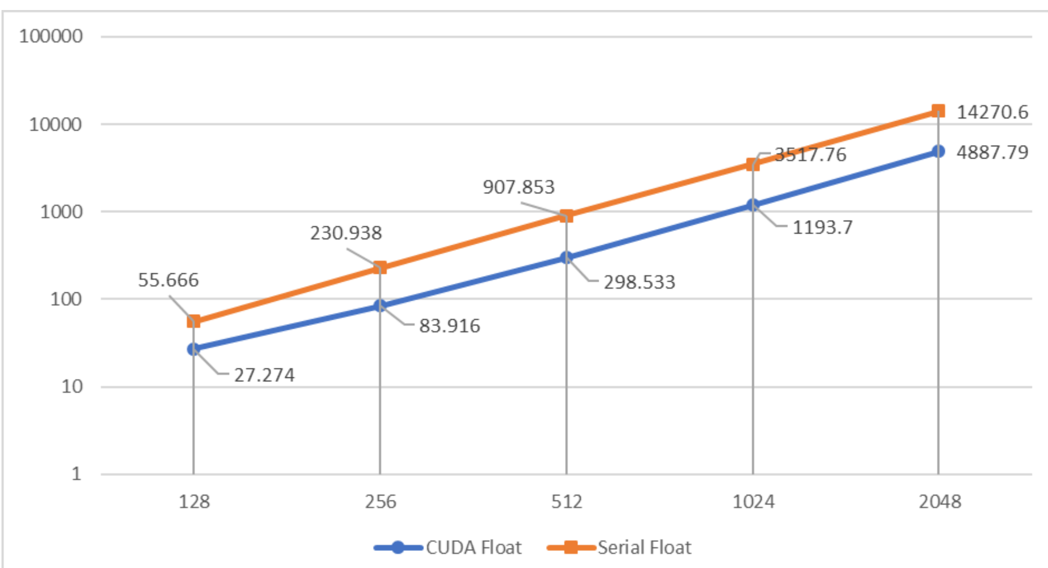

Figure 5: Execution time of a parallel algorithm on different computational meshes

The results of computational experiments show that the use of parallel algorithms using CUDA for this kind of tasks gives a good acceleration.

Further research will focus on the creation of parallel algorithms and acceleration of calculations related to the solution of nonlinear problems of multiphase fluid flows in porous

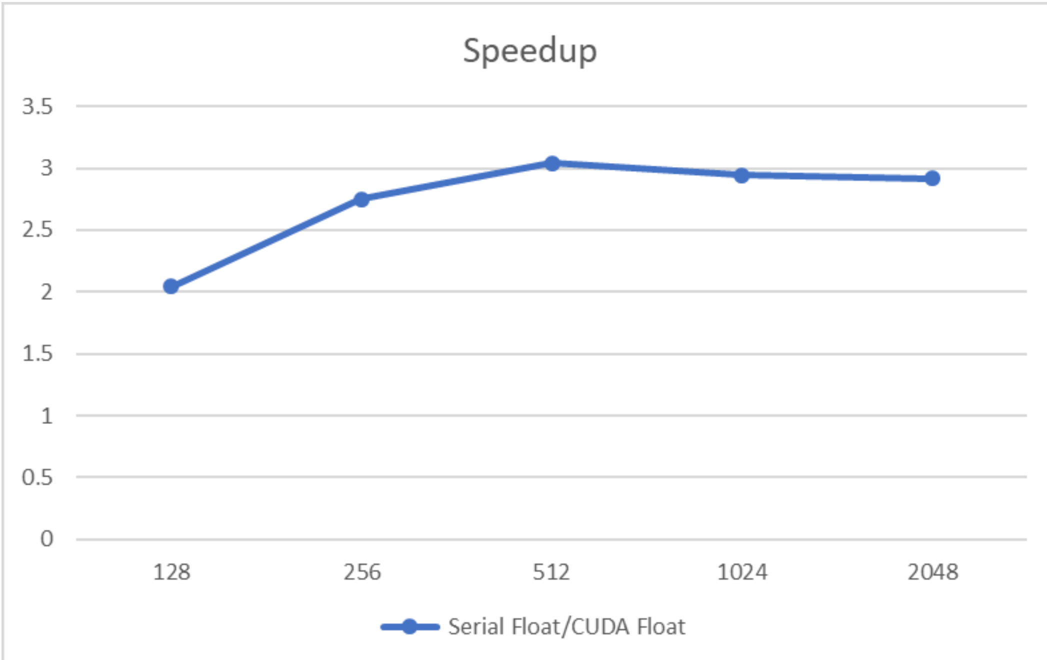

Figure 6: Parallel algorithm acceleration on CUDA

media considered in [28] by finite element methods using the CUDA software and hardware architecture of parallel computing.

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4-бөлім | Раздел 4 | Section 4

Қолданбалы математика | Прикладная математика | Applied Mathematics

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NUMERICAL SIMULATION OF CONTAMINANTS TRANSPORT IN HUMAN SETTLEMENTS TAKING INTO ACCOUNT CHEMICAL REACTIONS

In this paper, we performed a numerical simulation of the spread of pollutants due to a chemical reaction near the roadway inside an urban street canyon. In the course of our study, we studied the dispersion properties of the gas when it collides with the idealized buildings that make up the urban canyon. In conclusion, a qualitative assessment was given that characterizes the nature of the distribution of the concentration of the pollutant and the process of the appearance of zones with increased turbulence, in which vortices are formed that interfere with the ventilation properties of the horizontal flow, which significantly affects the health and life of people. NO and NO_2 released into the canyon area were chosen as the considered reactive substances, and ozone O_2, which is present in the moving air stream, was chosen as the third reactive substance. The results obtained can be used in the future for use by transport designers and road engineers, whose goal is to reduce the concentration of nitrogen oxides near the pedestrian zone of the city. All the results obtained were first tested on test problems, the results of which are in excellent agreement with the numerical and experimental values of other authors.
Key words: urban street canyon, turbulent model, Navier-Stokes equations, LES model, chemical reaction, air pollution, concentration, polluting emission, ozone.

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Химиялық реакцияларды ескере отырып, елді мекендерде ластаушы заттардың тасымалдануын сандық модельдеу

Бұл жұмыста қалалық көше каньонының ішіндегі жолдың жанында химиялық реакция нәтижесінде ластаушы заттардың таралуын сандық модельдеу жүргізілді. Біздің зерттеу барысында қалалық каньонды құрайтын идеализацияланған ғимараттармен соқтығысқан кезде газдың дисперсиялық қасиеттері зерттелді. Осы жұмысты қорытындылай келе, ластаушы концентрацияның таралу қозғалысы мен бағытын сипаттайтын нәтижелері берілді және көлденең ағынның желдету қасиеттеріне кедергі келтіретін құйындар пайда болатын турбуленттілігі жоғары аймақтардың пайда болу процесі талданды және де бұл адамдардың денсаулығымен өміріне айтарлықтай әсер ететінін дәлелдеді. Қарастырылған химиялық белсенді заттар ретінде каньон аймағына шығарылатын NO және NO_2 таңдалды, және де үшінші реактивті зат ретінде қозғалатын ауа ағынының құрамында болатын озон O_2 таңдалынып алынды. Қала жаяу жүргіншілер аймағына жақын таралатын улы газдардың, яғни азоток-сидінің концентрациясын төмендету мақсаты болып табылатын көлік жобалаушыларымен жол инженерлерінің пайдалануы үшін алынған нәтижелерді пайдалануға болады. Алынған барлық нәтижелер алдымен тест тапсырмаларында сыналып зерттелді және де олардың нәтижелері басқа да ғалымдардың жұмыстарымен салыстырылып, олардың сандық және эксперименттік нәтижелері жақсы сәйкестік көрсетті.
Түйін сөздер: қалалық көше шатқалы, турбулентті модель, Навье-Стокс теңдеулері, LES моделі, химиялық реакция, ауаның ластануы.

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Численное моделирование переноса загрязняющих веществ в населенных пунктах с учетом химических реакций

В настоящей работе было произведено численное моделирование распространения загрязняющих веществ вследствие протекания химической реакции вблизи проезжей части внутри городского уличного каньона. В ходе нашего проведенного исследования были изучены дисперсионные свойства газа при столкновении с идеализированными зданиями, составляющими городской каньон. В заключении была дана качественная оценка, характеризующая характер распространение концентрации загрязнителя и проанализирован процесс появления зон с повышенной турбулентностью, в которых образовываются вихри, препятствующие вентиляционным свойствам горизонтального потока, что в значительной степени сказывается на здоровье и жизнедеятельности людей. В качестве рассматриваемых химически активных веществ были выбраны NO и NO_2, выбрасываемые в область каньона, третьим реакционным веществом был выбран озон O_2, присутствующий в составе движущегося потока воздуха. Полученные результаты могут быть использованы в дальнейшем для использования транспортными проектировщиками и дорожными инженерами, целью которых является снижение концентрации оксидов азота вблизи пешеходной зоны города. Все полученные результаты были сначала апробированы на тестовых задач, результаты которых отлично согласуются с численными и экспериментальными значениями других авторов.
Ключевые слова: городской уличный каньон, турбулентная модель, уравнения Навье-Стокса, модель LES, химическая реакция, загрязнение воздуха.

1 Introduction

Every year, with the growth of technological progress and widespread urbanization of cities, we can observe a colossal increase in vehicles and the growth of industrial zones, which in turn led to the maximization of daytime traffic and a commensurate increase in the concentration of exhaust gases in the pedestrian zone [1], [2]. Transport is a source of toxic gases such as sulfur dioxide SO_2, nitrogen N_2, nitrogen oxides NO_x, carbon monoxide CO, carbon dioxide CO_2, aldehydes, heavy metal compounds, benzene C_6H_6, carcinogenic benzopyrene C_2OH_12, as well as particulate matter and soot hazardous to health. The height of the building in the street canyon is much greater than the width of the road, which creates a harmful environment in the space without air circulation due to weak gusts of wind. The high concentration of pollutants from exhaust gases in the air pool is harmful to human health, especially asthma, chronic diseases of the digestive, cardiac, nervous and respiratory systems [3]. Tominaga Y., Stathopoulos T. [4], [5] conducted a number of studies to study the mechanism of the formation of pollutants in the pedestrian zone. In these studies, wind tunnel experiments were performed and numerical simulations were performed using computational fluid dynamics (CFD).

One of the most ambitious problems in the field of chemical reactions is the interaction of ozone with NOx molecules: as you know, the air contains a fraction of ozone, which periodically interacts with nitrogen mono- and dioxides, entering into an exchange reaction. The resulting substances have a detrimental effect on the health of all living organisms. This problem has been studied by Carpenter, L.J., Clemitshaw, K.C. [6], in addition, recommendations and analysis of the quality of the air in the areas adjacent to the canyon were given. Also in Baker, J., et al. [7] took into account the photochemical properties of the ongoing reactions. In studies Zhong J. et al. [8] the numerical results that were solved using the LES model coincide with the experimental values. Also, with the LES model, good results were seen for the emerging product with a compound of ozone O3 and nitrogen oxide NO. This problem was studied by Kim M. et al. [9]. The paper [10] examines the sensitivity of O3 to NOx and VOC emissions. This study is an attempt to analyze the spread of dozens of reactive pollutants in and over a street canyon using a CFD model.

In this article, numerical results were presented, in which the course of a chemical reaction was considered, a comparative analysis of the results obtained with experimental data was made, after which a qualitative conclusion was made about the consistency of the obtained data with empirical values. In the task, a mixture of ozone with carbon monoxide was investigated, which, as a result of chemical reactions, decomposes into carbon dioxide and oxygen. All numerical calculations were carried out in an idealized street canyon, which consists of two buildings, in addition, a pollutant emerges in the middle of the street along the entire length of the roadway.

2 Formulation of the problem

Denev J, A., Frohlich J., Bockhorn H. [11] carried out a direct numerical simulation (DNS) of a transverse flow with a jet emerging from a circular cylindrical tube; as a result of collision of flows, a chemical reaction occurs in the forward direction. This simulation was calculated with a small value of the Reynolds number, equal to $Re = 275$. For the study, a three-dimensional model of a microreactor was built in the center, consisting of two transverse flows. All dimensions are reduced to dimensionless, with the value of the pipe diameter, where $D = 8mm$ (D is the pipe diameter). The vertical axis is located at a $3D$ distance from the horizontal axis. Height and width of horizontal pipe $13,5D$, length $20D$, pipe height $2D$. The tube configuration is shown in detail in figure(Рис.1).

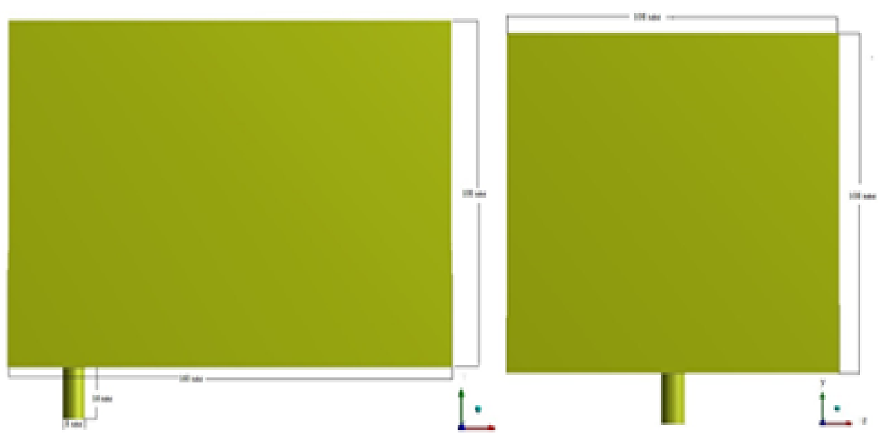

Figure 1: Parameter of the calculated area

An unstructured mesh was created for the simulation. In the geometry, a computational subdomain was created, to which the mesh was refined, the location of the subdomain was chosen near the cylindrical pipe. The total number of elements is 1104850, the number of nodes is 192787. Figure (Рис.2) show the groups of the studied territories.

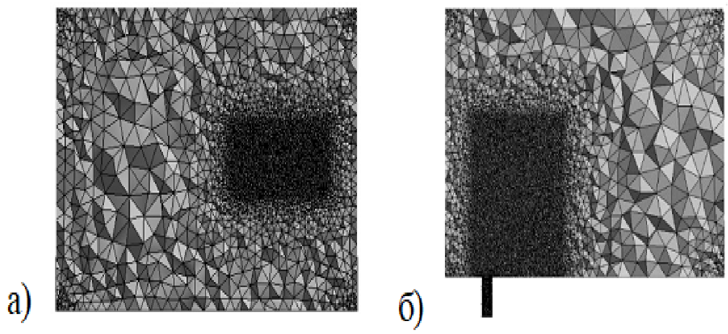

Figure 2: Through the calculated area in the plane (a) XOZ (b) XOY

Damkehler's number $Da = 1.0$ is chosen so that the simulation estimate is available. As the calculated indicators in the calculation, it was assumed that the kinematic viscosity of air $\nu = 1.40610^{-5}$ m^2/s, density $\rho = 1.225$ kg/m^3, velocity is $u = 0.48$ m/s. All values are obtained empirically and are completely physical [12].

3 Materials and Methods

Determination of the reactive flow velocity with the main velocity is denoted as $R = U_{b,jet}/U = 2.42$. The volumetric flow rate is $U_{b,jet} = 1.16$ m/s. The flow velocity from the cylindrical pipe is characterized by the following velocity profile:

$$w\left(r\right)/U_{b,jet} = 2\left[1 - \left(r/\left(D/2\right)^2\right)\right], \tag{1}$$

here, r - is the radial coordinate, w - is the vertical velocity component. The velocity profile at the flow boundary can be described as follows:

$$u\left(d_n\right) = 1.0 - \exp(-3.0 d_n), \tag{2}$$

here, d_n - the closest distance to the channel wall, that is [11]:

$$d_n = min(x, L_x - x). \tag{3}$$

All limited conditions The reference area is shown in more detail in the pictures(Рис.3).

In this work, the numerical results were compared with the experimental and numerical values of Denev J, A. [11]. Figure (Рис.4) shows the average velocity of the horizontal velocity component.

Figure (Рис.5) shown is the average speed v of the speed component. The figure (Рис.5) compares the profiles of the middle conceptions of the B version, with a freely selectable

To solve the problem inside the urban canyon, the study by Baker, J. [7] was used. It was accepted that $H/W = 1$. When creating a computational model, a three-dimensional

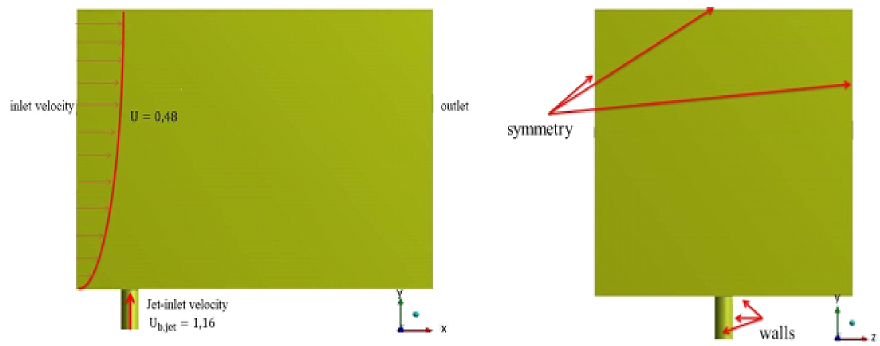

Figure 3: Basic conditions for the vertical view of the reactor in the OXY plane; Basic conditions of the vertical view of the reactor in the plane of OYZ

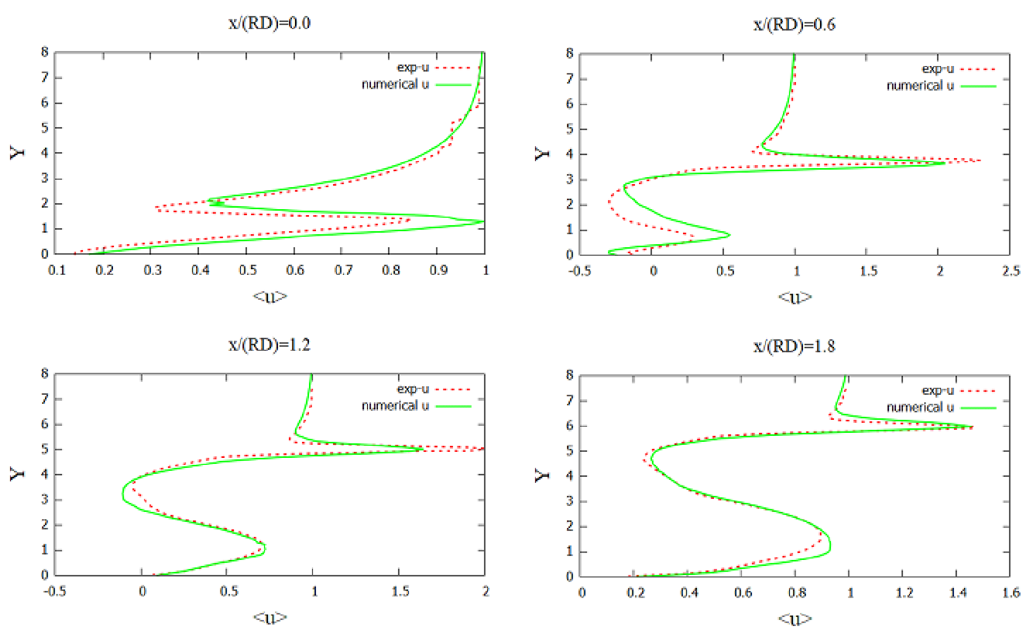

Figure 4: Comparison of the profiles of the average velocity u at the indicated points

computational domain was adopted. Figure (Рис.7) 8 illustrates the configuration of the computational domain. The height of the buildings is the same and equal to the value of $H = 18$, the width of the canyon is equal to $W = 18$. In the center of the pedestrian part, there is a source of pollutant, 0.3 m by 0.3 m in size. Gas CO and NO, come out of the source, which enters into a chemical reaction with ozone O_3, moving in a cross-flow.

$$O_3 + NO = NO_2 + O_2 \tag{4}$$

$$O_3 + CO = CO_2 + O_2 \tag{5}$$

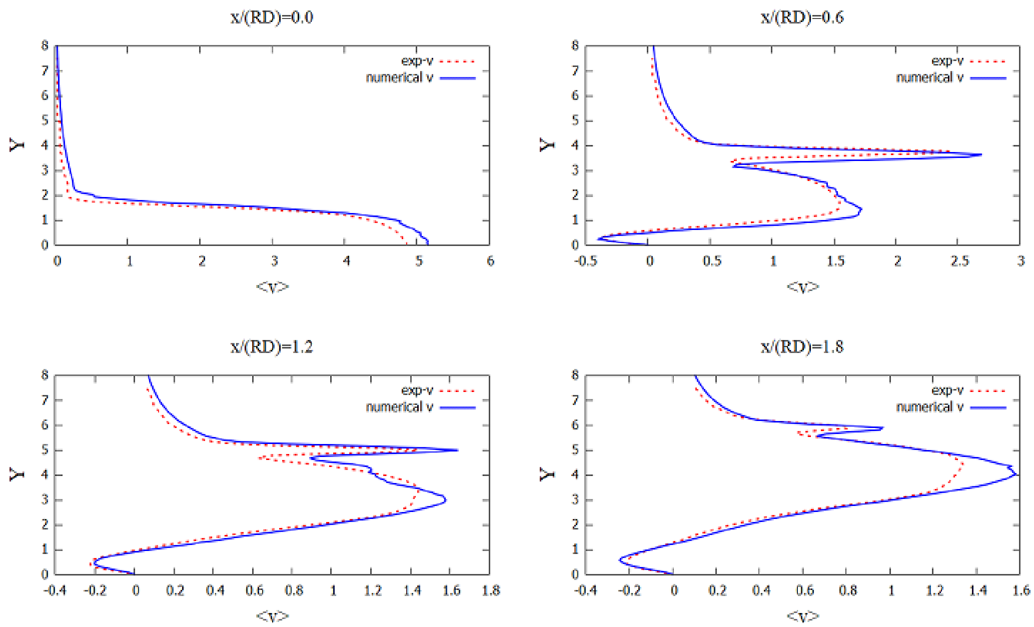

Figure 5: Comparison of the profiles of the middle v speed components at the specified points

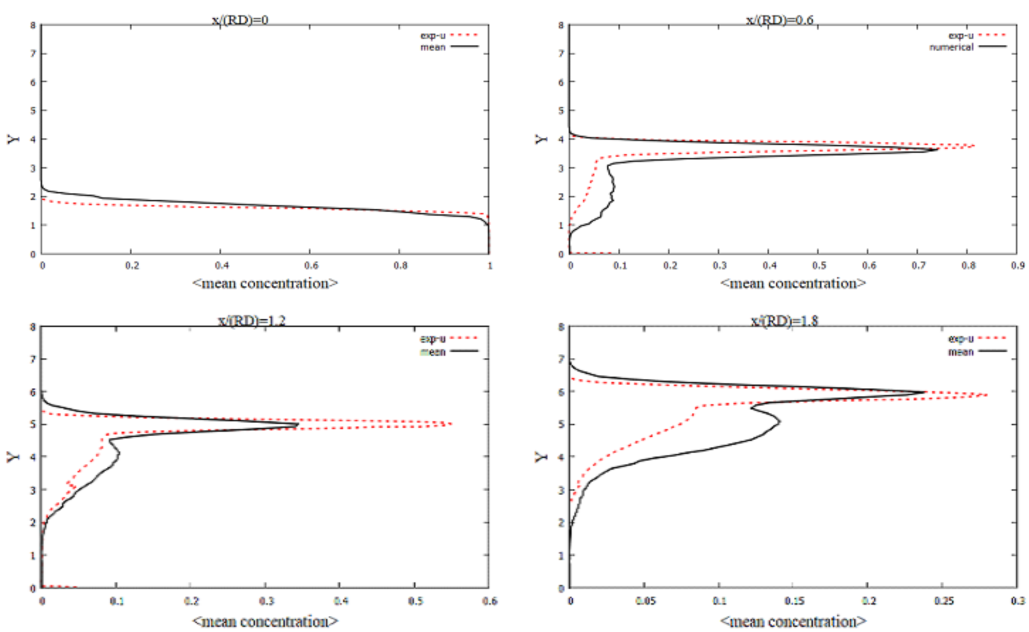

Figure 6: Comparison of the profiles of the mid-range values ??of the time B

Unlike the initial gases, the reaction products have a detrimental effect on the health of living organisms, as a result of which the substitution reaction is the object of close study.

Figure (Рис.8) shows an image of the computational domain. Thickening was carried out along the street of the canyon at a height of 30 above the source of pollution. The total

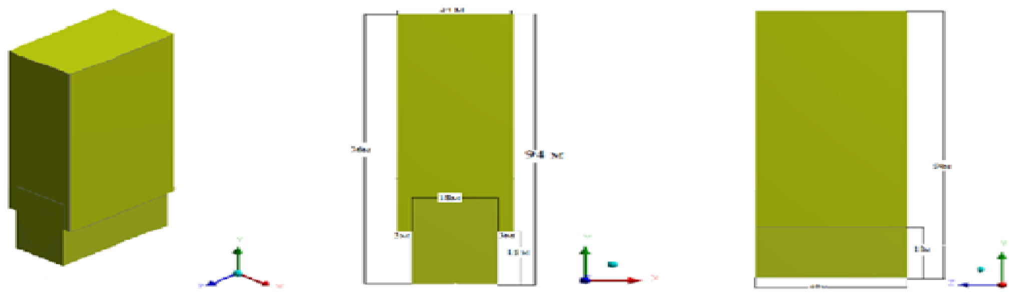

Figure 7: 3D model of the investigated term; Parameter of the calculated range from the XOY and YOZ planes

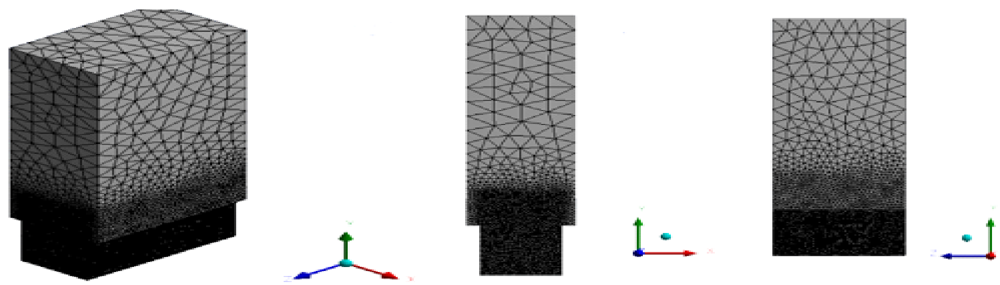

Figure 8: Powerful area

number of elements and nodes is 1526580 and 264265 respectively.

4 Numerical results

The study was carried out using the ANSYS Fluent. The time step size is 1. The total computation time was one hour, and the result was visualized. The LES model was used in conjunction with calculations of the chemical reaction caused by pollutants within the canyon. The numerical calculation was carried out using the SIMPLE algorithm (partially closed method for pressure-dependent equations).

The chemical reaction was solved using the Smagorinsky model. Smagorinsky's constant is equal to $C_s = 0.1$. According to reaction (4) figure (Рис.9) shows the mass fraction of mean concentrations in the windward and lateral relief of NO, NO_2, O_3.

The mass fraction profile of the average concentration of CO, CO_2 and O_3 -coating at a height of 0,3 m from the pollution source is shown in figure (Рис.11). According to reaction (5) figure (Рис.11) shows the mass fraction of the average concentrations of CO, CO_2 and O_3 in the windward and lateral relief. In figures (Рис.10) and (Рис.12) the detection of pollutant concentrations was visualized using ANSYS Fluent Volume Rendering.

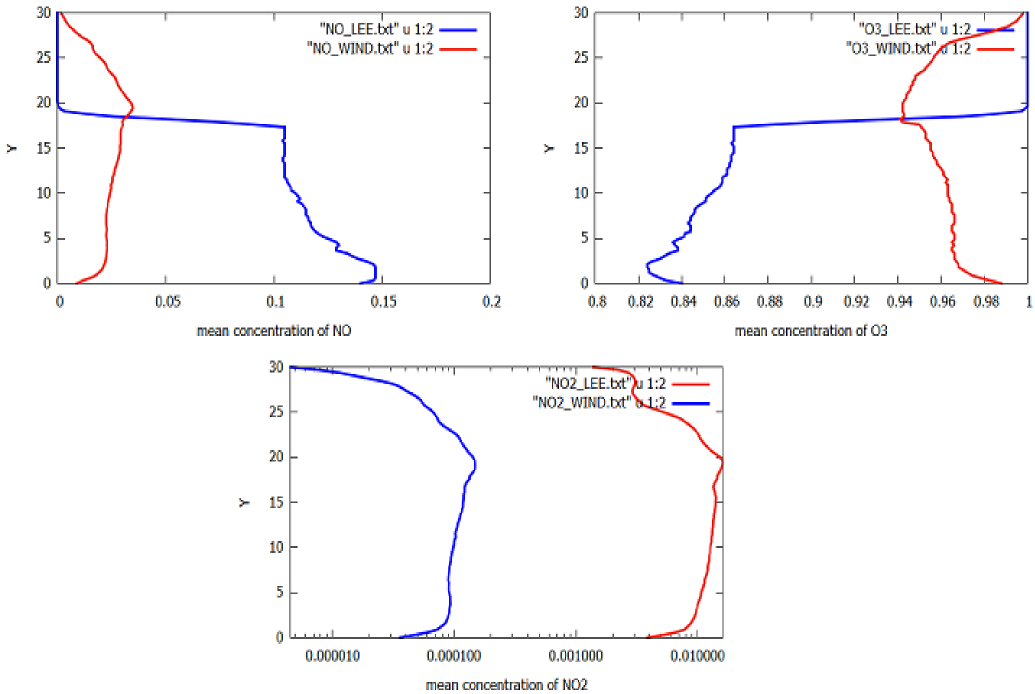

Figure 9: Equivalence of the mass content of the average concentrations of the nitric oxide, the ozone and the dioxide of the nitrogen oxide in the current $z/H = 0$

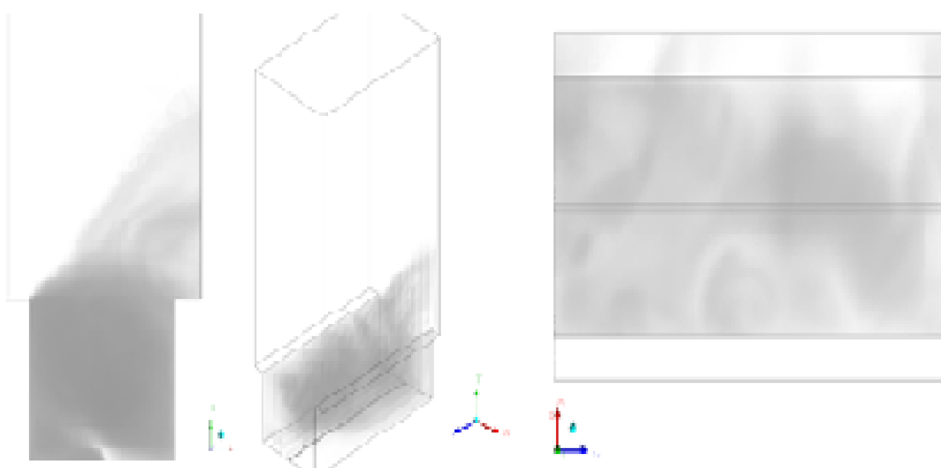

Figure 10: Distribution of concentration NO_2

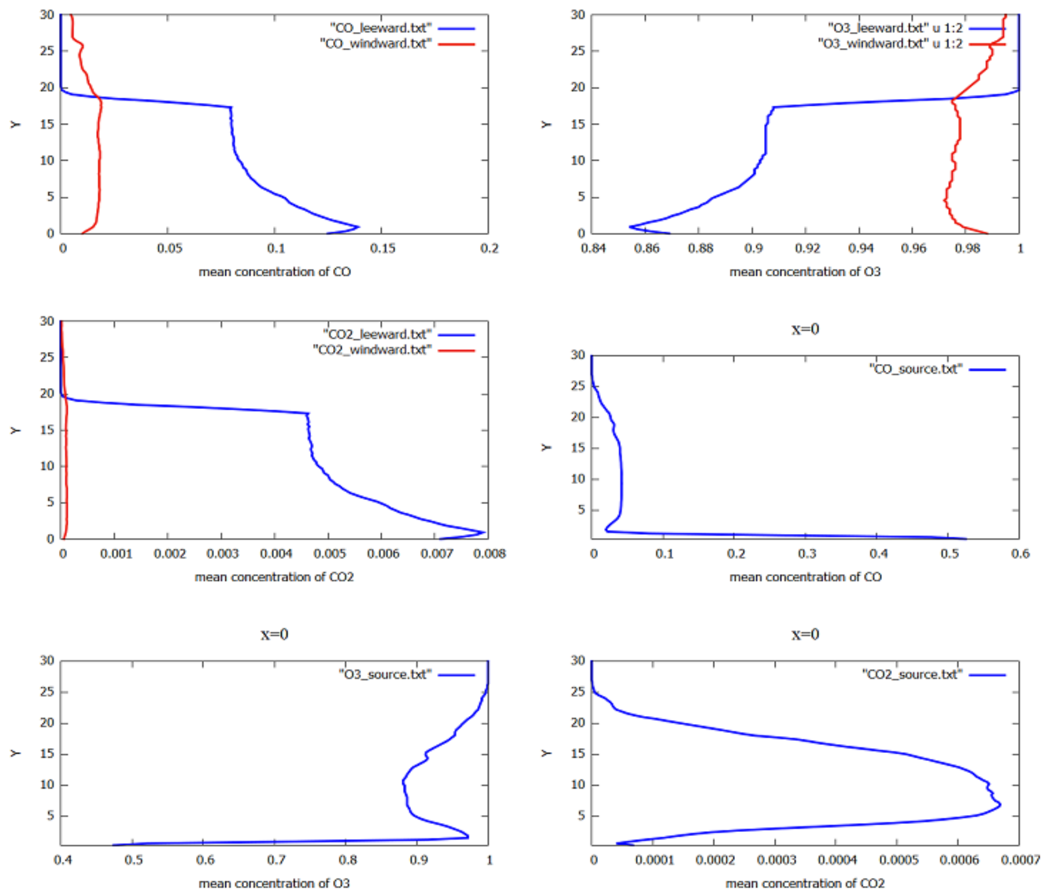

Figure 11: Comparison of the concentration of the upwind and leeward headlands at the point $z/H = 0$ and the concentration of the pollutant , $_3$, $_2$ at the point $z/H = 0$

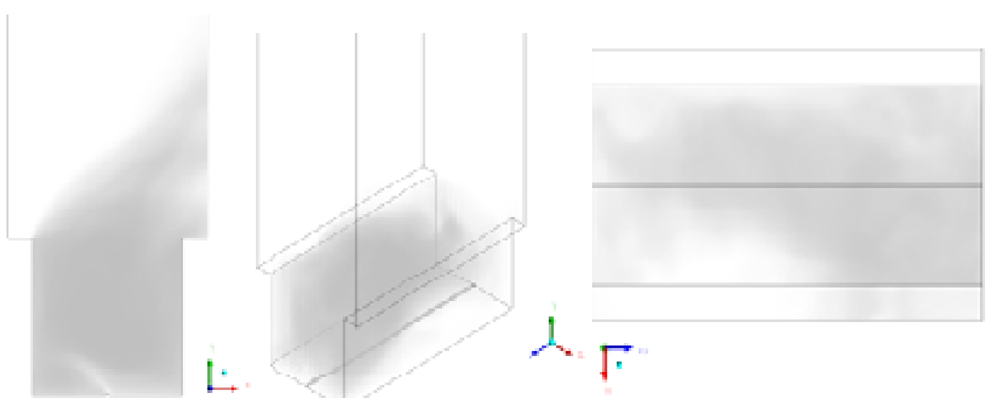

Figure 12: Distribution of concentration CO_2

5 Conclusion

At the beginning of the work, a literature review was carried out on modeling the chemical reaction that occurs when pollutants from vehicles are detected in the city street gorge in order to identify the main problems in this industry. A mathematical model was developed to describe the flow. To ensure the correctness of the mathematical model and numerical algorithm, the test problem was solved using the ANSYS Fluent software package. The results of the study show that to check the growth rate and determine the mass of the formed chemical products of a chemical reaction, a test task was performed, and then checked and compared with the results of experimental and numerical studies by well-known authors; the search for the most efficient turbulence model was carried out; the geometry of the real dimensions of the city street canyon was created, divided into groups and the computational domain was condensed; reactive substances NO, CO, interacting with ozone from the source of pollution and emitting toxic gases such as NO_2, CO_2, harmful to human health, were quantified and visualized.

In street canyons, depending on the length of buildings and the configuration of the street, gases such as NO_2, CO_2, are dispersed, which leads to disruption of the normal circulation of the wind flow. Even in the gorge, this leads to an increase in temperature and an accumulation of concentrations of life-threatening gases. The direction of the wind can change and intensify, causing a hurricane. In addition, the slope of the gorge is 4 times more polluted with ultrafine particles than the wind slope. This means that the most polluted air is breathed in the pedestrian zone. The research results can be applied to numerical modeling in the future to solve new questions and problems in the studied area of knowledge.

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