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Nonlinear differential equation with first order partial derivatives

Aldibekov T.M., Al-Farabi Kazakh National University,
Almaty, Kazakhstan, +77017477069, E-mail: tamash59@mail.ru
Aldazharova M.M., Scientific Research Institute of
the al-Farabi Kazakh National University,
Almaty, Kazakhstan, +77019870744, E-mail: a_maira77@mail.ru

The asymptotic behavior of solutions of a nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives is investigated. Each first-order partial differential equation under certain conditions has a fundamental system of integrals or an integral basis. We note that for a general linear partial differential equation of the first order there can be no nontrivial integral. For a linear homogeneous first-order partial differential equation, where the coefficients of the equation are given on an unbounded set and have continuous first-order partial derivatives, with the first coefficient equal to one, an integral basis exists. In this paper, a nonlinear partial differential equation of the first order, which is solved with respect to one of the derivatives, is estimated from two sides by first-order partial differential equations. Using differential inequalities it is proved that a nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives has a solution that tends to zero as one tends to infinity to one of the independent variables. At present, the theory of partial differential equations finds its application in various fields of natural science.
Key words:equation, first order partial derivatives.

Сызықты бірінші ретті дербес туындылы теңдеулер туралы

Алдибеков Т.М., Әл-Фараби атындағы қазақ ұлттық университеті,
Алматы қ., Қазақстан Республикасы, +77017477069, Электрондық пошта: tamash59@mail.ru
Алдажарова М.М., Әл-Фараби атындағы қазақ ұлттық университетінің Ғылыми зерттеу институты,
Алматы қ., Қазақстан Республикасы, +77019870744, Электрондық пошта: a_maira77@mail.ru

Туындылардың біреуіне байланысты шешілген бірінші ретті дербес туындылы сызықты емес дифференциалдық теңдеудің шешімдерінің асимптотикалық мінезі зерттеледі. Бірінші ретті дербес туындылы дифференциалдық теңдеудің әрқайсысының қандайда бір шарттарда фундаменталды интегрладар жүйесі немесе интегралдық базисі болады. Айта кететіні, жалпы бірінші ретті сызықты дербес туындылы дифференциалдық теңдеудің тривиалды емес интегралы болмауы да мүмкін. Бірінші ретті сызықты дербес туындылы дифференциалдық теңдеу үшін, оның коэффициенттері шенелмеген жиында беріліп, үзіліссіз бірінші ретті дербес туындылары болса және бірінші коэффициенті бірге тең болса, интегралды базис бар болады. Бұл жұмыста туындылардың біреуіне байланысты шешілген бірінші ретті дербес туындылы сызықты емес дифференциалдық теңдеу екі жағынан бірінші ретті дербес туындылы дифференциалдық теңдеулермен бағаланады. Дифференциалдық теңсіздіктерді пайдалана отырып, туындылардың біреуіне байланысты шешілген бірінші ретті дербес туындылы сызықты емес теңдеудің тәуелсіз айнымалыларнының біреуі плюс шексіздікке ұмтылған жағдайда нөлге ұмтылатын шешімі бар болатыны дәлелденген. Қазіргі таңда дербес туындылы дифференциалдық теңдеулер теориясы жаратылыс танудың түрлі салаларында өз қолданыстарын табуда.
Түйін сөздер: теңдеу, бірінші ретті дербес туындылар.

О нелинейном дифференциальном уравнении с частными производными первого порядка

Алдибеков Т.М., Казахский национальный университет имени аль-Фараби,
г. Алматы, Республика Казахстан, +77017477069, E-mail: tamash59@mail.ru
Алдажарова М.М., Научно-исследовательский институт
Казахского национального университета имени аль-Фараби,
г. Алматы, Республика Казахстан, +77019870744, E-mail: a_maira77@mail.ru

Рассматривается асимптотическое поведение решений нелинейного дифференциального уравнения, с частными производными первого порядка разрешенное относительно одной из производных. Каждое дифференциальное уравнение с частными производными первого порядка при некоторых условиях имеет фундаментальную систему интегралов или интегральный базис. Заметим, для общего линейного дифференциального уравнения с частными производными первого порядка может не существовать нетривиального интеграла. Для линейного однородного дифференциального уравнения с частными производными первого порядка, где коэффициенты уравнения заданы на неограниченном множестве и имеют непрерывные частные производные первого порядка, причем первый коэффициент равен единице, интегральный базис существует. В работе нелинейное дифференциальное уравнение с частными производными первого порядка, разрешенное относительно одной из производных, оцениваются с двух сторон дифференциальными уравнениями с частными производными первых порядков. Использованием дифференциальных неравенств доказано, что нелинейное дифференциальное уравнение, с частными производными первого порядка разрешенное относительно одной из производных имеет решение стремящейся к нулю при стремлении на плюс бесконечность одной из независимой переменной. В настоящее время теория дифференциальных уравнений с частными производными находит свое применение в различных областях естествознании.
Ключевые слова: уравнение, частные производные первого порядка.

1 Introduction

The Cauchy problem for a nonlinear partial differential equation of the first order solved with respect to one of the derivatives, as is well known, under certain conditions has a unique solution in a small neighborhood. The paper deals with a nonlinear partial differential equation of the first order solved with respect to one of the derivatives, and the solution of the Cauchy problem is assumed extending to the right to plus infinity. A nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives was estimated from two sides by partial differential equations of the first order, the behavior of the solutions of which are known. Using differential inequalities, the asymptotic behavior of the solution of a first-order partial differential equation solved with respect to one of the derivatives was studied and was proved that the nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives has a solution that tends to zero while one of the independent variables tends to plus infinity.

2 Literature review

The general theory is presented in the books [1-10]. The domain of existence of solutions was investigated by Kamke and data is contained in the reference books [11, 12]. The domain of existence of solutions was investigated in the works [13-15]. Non-analytic equations are considered in the papers[16-18]. In work of Kruzhkov generalized solutions was considered [19]. Kovalevskaya's theorem was published in [20-22]. An example of nonexistence of a solution constructed in [23-26]. Differential inequalities are considered by Nagumo [27-29].

3 Materials and research methods

Let us consider a nonlinear partial differential equation of the first order with n + 1 independent variables solved with respect to one of the derivatives

$$\frac{\partial u}{\partial t} + H\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right) = 0 \tag{1}$$

where
$H\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right) = \left[\frac{1}{2}\sum_{k=1}^{n} p_{1k}(t)y_k + f\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right)\right]\frac{\partial u}{\partial y_1} + \left[\sum_{k=1}^{n} p_{2k}(t)y_k\right]\frac{\partial u}{\partial y_2} + \dots + \left[\sum_{k=1}^{n} p_{nk}(t)y_k\right]\frac{\partial u}{\partial y_n}$

$$u(o, y_1, \dots, y_n) = \varphi(y_1, \dots, y_n). \tag{2}$$

We define the (t, y) set B as follows

$$B = \{(t, y) : 0 \le t < +\infty,\ c_k - L_k t \le y \le d_k + L_k t, k = 1,\ \dots, n\}$$

where $L_k > 0$, $c_k < 0 < d_k$. Function $H(u, t, y, q)$ is defined in $E \subseteq \mathbb{R}^{2+2n}$, whose projection onto the $(t, y)-$space contains B. $(\varphi(0), 0, 0, \varphi_y(0)) \in E$ and $\varphi_y \in C^2$. The problem (1), (2) for small $|t|$, $\|y\|$, has a unique solution $u(t, y)$ of class C^2.[12, p.173]. We take sufficiently small $|t_0|$, $\|y_0\|$, where $(t_0, y_0) \in B,\ t_0 > 0$ and we assume that the solution $u(t, y)$ satisfying the condition (2) defining on the point $(t_0, y_0) \in B$ and continuing on$t > t_0$. For definiteness, we denote this solution of equation (1) with $u(t, y; t_0, y_0)$.

Theorem 1. Suppose that the following conditions hold on the set $E \in \mathbb{R}^{2+2n}$ whose projection onto the $(x, y)-$space contains B:

A) $|H\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial \bar{y}_1}, \dots, \frac{\partial u}{\partial \bar{y}_n}\right) - H\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial \bar{\bar{y}}_1}, \dots, \frac{\partial u}{\partial \bar{\bar{y}}_n}\right)| \le \sum_{k=1}^{n} L_k |\frac{\partial u}{\partial \bar{y}} - \frac{\partial u}{\partial \bar{\bar{y}}}|$;

B) The inequality is fulfilled:

$$f\left(u, t, y_1, \dots, y_n, \frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_n}\right) < \frac{1}{2}\sum_{k=1}^{n} p_{1k}(t)y_k;$$

$p_{ik}(t) \in \mathbb{C}^2(I),\ i = 1, \dots, n,\ k = 1, \dots, n,\ I \equiv [0, +\infty)$ satisfy next conditions:

$a_1) p_{k-1,k-1}(t) - p_{kk}(t) \ge \alpha_1\psi(t),\ t \in I,\ k = 2, \dots, n.\ \alpha_1 > 0,$

$\psi(t) \in \mathbb{C}(I),\ \psi(t) > 0,\ \int\limits_{t_0}^{+\infty} \psi(s)ds = +\infty;$

$b_1) \lim\limits_{t\to+\infty} \frac{|p_{ik}(t)|}{\psi(t)} = 0,\ i \ne k,\ i = 1, 2, \dots, n,\ k = 1, 2, \dots, n;$

$c_1) \lim\limits_{t\to+\infty} \frac{1}{\nu(t)}\int\limits_{t_0}^{t}(-p_{kk}(s))ds = \beta_k,\ k = 1, 2, \dots, n.$ Where $\nu = \int\limits_{t_0}^{t} \psi(s)ds \uparrow +\infty$ and the inequality performs: $\beta_1 < 0$;

M) next inequalities are true:

$p_{ik}(t) \ge b_{ik}(t)$, $b_{ik}(t) \in \mathbb{C}^2(I)$, $i = 1, \dots, n$, $k = 1, \dots, n$, where $b_{ik}(t)\, i = 1, \dots, n$, $k =$

$1, \ldots, n$ satisfy next conditions:
$a_2) b_{k-1,k-1}(t) - b_{kk}(t) \geq \alpha_2 \psi(t),\ t \in I,\ k = 2, \ldots, n.\ \alpha_2 > 0,$
$b_2) \lim\limits_{t\to+\infty} \frac{|b_{ik}(t)|}{\psi(t)} = 0,\ i \neq k,\ i = 1, 2, \ldots, n,\ k = 1, 2, \ldots, n;$
$c_2) \lim\limits_{t\to+\infty} \frac{1}{\nu(t)} \int\limits_{t_0}^{t} b_{kk}(s)ds = \mu_1,\ k = 1, 2, \ldots, n.$ Where $\mu_1 < 0$.
Inequality is true:

$$f(u, t, y, q) \geq h(u, t, y)$$

Where $h \in \mathbb{C}^2(D)$, $D \in E$, $t \in [t_0, +\infty)$, $|h(\theta, t, y)| \leq \delta(t)\|y\|$ and

$$\delta(t) \in \mathbb{C}(I),\ \lim_{t\to+\infty} \frac{\delta(t)}{\psi(t)} = 0;$$

Then for the solution $u(t, y; t_0, y_0)$ of equation (1) there exists a limit

$$\lim_{t\to+\infty} u(t, y; t_0, y_0) = 0$$

Proof. On the set $E \in \mathbb{R}^{2+2n}$ whose projection onto $(x, y)-$ space contains B, we consider the equation

$$\frac{\partial\vartheta}{\partial t} - f_1\left(\vartheta, t, y, \frac{\partial\vartheta}{\partial y}\right) = 0 \tag{3}$$

where $f_1\left(\vartheta, y, t, \frac{\partial\vartheta}{\partial y}\right) = \left(\sum\limits_{k=1}^{n} p_{1k}(t)y_k\right) \frac{\partial\vartheta}{\partial y_1} + \left(\sum\limits_{k=1}^{n} p_{2k}(t)y_k\right) \frac{\partial\vartheta}{\partial y_2} + \ldots + \left(\sum\limits_{k=1}^{n} p_{nk}(t)y_k\right) \frac{\partial\vartheta}{\partial y_n}$.
For equation (3), the characteristic system of differential equations has the form

$$\frac{dy_1}{dt} = -\sum_{k=1}^{n} p_{1k}(t)y_k,\ \frac{dy_i}{dt} = -\sum_{k=1}^{n} p_{ik}(t)y_k\ i = 2, \ldots, n. \tag{4}$$

The characteristic system (4) is considered for initial values

$$y_k|_{t=t_0} = \bar{y}_k^0,\ k = 1, \ldots, n.$$

The solution of the characteristic system (4) exists

$$y_k = \varphi_k(t, t_0, \bar{y}_1^0, \ldots, \bar{y}_n^0),\ k = 1, \ldots, n \tag{5}$$

for arbitrary initial values $\bar{y}_k^0, (k = 1, \ldots, n)$.
Let $(t_0, \bar{y}_k^0) \in B$. (5) are solvable with respect to $\bar{y}_1^0, \ldots, \bar{y}_n^0$ and holds

$$\bar{y}_k^0 = \varphi_k(t_0, t, y_1, \ldots, y_n),\ k = 1, \ldots, n \tag{6}$$

(6) forms an integral basis of equation (3). By B), inequality

$$\vartheta_t > H(u, t, y, \vartheta_y)$$

Indeed
$\vartheta_t - H(u, t, y, \vartheta_y) = \left(\sum\limits_{k=1}^{n} p_{1k}(t)y_k\right) \frac{\partial\vartheta}{\partial y_1} + \left(\sum\limits_{k=1}^{n} p_{2k}(t)y_k\right) \frac{\partial\vartheta}{\partial y_2} + \ldots + \left(\sum\limits_{k=1}^{n} p_{nk}(t)y_k\right) \frac{\partial\vartheta}{\partial y_n} -$

$\left(\frac{1}{2}\sum\limits_{k=1}^{n}p_{1k}(t)y_k+f(\nu,t,y,\vartheta_y)\right)\frac{\partial\vartheta}{\partial y_1} - \left(\sum\limits_{k=1}^{n}p_{2k}(t)y_k\right)\frac{\partial\vartheta}{\partial y_2} - \dots - \left(\sum\limits_{k=1}^{n}p_{nk}(t)y_k\right)\frac{\partial\vartheta}{\partial y_n} = \left(\frac{1}{2}\sum\limits_{k=1}^{n}p_{1k}(t)y_k-f(\nu,t,y,\vartheta_y)\right)\frac{\partial\vartheta}{\partial y_1} > 0$

Let $\vartheta(t_0,t,y)=\varphi_k(t_0,t,y_1,...,y_n), k\in 1,...,n$ be a solution of the equation $(\vartheta,t,y,\vartheta_y)\in E$ satisfying the condition $\vartheta(t_0,t_0,y)=\omega_1(y)$, $\omega_1(y)\in C^2$ and such that $\omega_1(y)>\varphi(y)$. Then everywhere on B next inequality is true:

$$\vartheta(t_0,t,y)>u(t,y,t_0,y_0),\ t\in I. \tag{7}$$

In fact, if inequality (7) is not true, then there is a point $(t_1,y_1)\in B$, where $t_1>t_0$ is such that inequality (7) is true (t_0,t_1), and at (t_1,y_1) will have equality, i.e. $\vartheta(t_0,t_1,y_1)=u(t_1,y_1;t_0,y_0)$. Integrating equations (1) and (3), we obtain

$$u(t_1,y_1;t_0,y_0)-u(t_0,y_0;t_0,y_0)+\int\limits_{t_0}^{t_1}Hds=0$$

and

$$\vartheta(t_0,t_1,y_1)-\vartheta(t_0,t_0,y_0)-\int\limits_{t_0}^{t_1}f_1ds=0$$

This implies

$$u(t_0,y_0;t_0,y_0)+\int\limits_{t_0}^{t_1}(-H-f_1)ds-\vartheta(t_0,t_0,y_0)=0$$

The difference $\vartheta-u$ is positive for $t_0<t<t_1$ and is zero for $t=t_1$. Hence, the derivative of the difference $\vartheta-u$ at the point $t=t_1$ is nonpositive, i.e. $((\vartheta-u)_t)_{t=t_1}=0$, then this implies the inequality

$$u(t_0,y_0;t_0,y_0)-\vartheta(t_0,t_0,y_0)\geq 0.$$

This contradicts the inequality $\omega_1(y)>f(y)$. Consequently, we have (7). Consider the equation

$$\frac{\partial\theta}{\partial t}-f_2\left(\theta,t,y,\frac{\partial\theta}{\partial y}\right)=0 \tag{8}$$

where
$f_2\left(\theta,t,y,\frac{\partial\theta}{\partial y}\right)=\left(\sum\limits_{k=1}^{n}b_{1k}(t)y_k+h(\theta,t,y)\right)\frac{\partial\theta}{\partial y_1}+\left(\sum\limits_{k=1}^{n}b_{2k}(t)y_k\right)\frac{\partial\theta}{\partial y_2}+\dots+\left(\sum\limits_{k=1}^{n}b_{nk}(t)y_k\right)\frac{\partial\theta}{\partial y_n}.$
For the equation (8), the characteristic system has the form

$$\frac{dy_1}{dt}=-\left(\sum_{k=1}^{n}b_{1k}(t)y_k+h(\theta,t,y)\right),\ \frac{dy_i}{dt}=-\sum_{k=1}^{n}b_{ik}(t)y_k\ i=2,\dots,n.$$

Let

$$y_i=\theta_i(t,t_0,\bar{\bar{y}}_1^0,\dots,\bar{\bar{y}}_n^0),\ i=1,\dots,n;\ t\in(t_0,+\infty)$$

be a solution of the characteristic system, where $(t_0, \bar{\bar{y}}_0) \in B$. This system of solutions is solvable with respect to $\bar{\bar{y}}_1^0, \ldots, \bar{\bar{y}}_n^0$, therefore

$$\theta_i(t_0, t, y) = \theta_i(t_0, t, y_1, ..., y_n), k \in 1, ..., n$$

forms an integral basis of equation (8), for which in the whole domain the functional determinant

$$\frac{\partial(\theta_1, \ldots, \theta_n)}{\partial(y_1, \ldots, y_n)} > 0$$

By condition M), inequality

$$\theta_t \le H(\theta, t, y, \theta_y).$$

Indeed,

$$\theta_t - H(\theta, t, y, \theta_y) = \left(\sum_{k=1}^{n} b_{1k}(t)y_k + h(\theta, t, y)\right)\frac{\partial\theta}{\partial y_1} + \left(\sum_{k=1}^{n} b_{2k}(t)y_k\right)\frac{\partial\theta}{\partial y_2} + \ldots + \left(\sum_{k=1}^{n} b_{nk}(t)y_k\right)\frac{\partial\theta}{\partial y_n} - \left(\sum_{k=1}^{n} p_{1k}(t)y_k + f(\theta, t, \theta, \theta_y)\right)\frac{\partial\theta}{\partial y_1} - \left(\sum_{k=1}^{n} p_{2k}(t)y_k\right)\frac{\partial\theta}{\partial y_2} - \ldots - \left(\sum_{k=1}^{n} p_{nk}(t)y_k\right)\frac{\partial\theta}{\partial y_n} = \left(\sum_{k=1}^{n}(b_{1k}(t) - p_{1k}(t))y_k + h(\theta, t, y) - f(\theta, t, \theta, \theta_y) +\right)\frac{\partial\theta}{\partial y_1} + \left(\sum_{k=1}^{n}(b_{2k}(t) - p_{2k}(t))y_k\right)\frac{\partial\theta}{\partial y_2} + \ldots + \left(\sum_{k=1}^{n}(b_{nk}(t) - p_{nk}(t))y_k\right)\frac{\partial\theta}{\partial y_n} \le 0.$$

Let $\theta(t_0, t, y) = \theta_i(t_0, t, y), i \in 1, \ldots, n$ be a solution of the equation (8) $(\theta, t, y, \theta_y) \in E$ satisfying the condition $\theta(t_0, t_0, y) = \omega_2(y)$, where $\omega_2(y) \in \mathbb{C}^2$ and such that $\omega_2(y) < f(y)$. We have the inequality

$$u(t, y, t_0, y_0) > \theta(t_0, t, y) \; t \in I. \tag{9}$$

Indeed, if inequality (9) is not true, then there is a point $(t_1, y_1) \in B$, where $t_1 > t_0$ is such that inequality (9) is true in the interval (t_0, t_1), and at (t_1, y_1) have the equality $u(t_1, y_1; t_0, y_0) = \theta(t_0, t_1, y)$. Integrating equations (1) and (8), we obtain

$$u(t_1, y_1; t_0, y_0) - u(t_0, y_0; t_0, y_0) + \int_{t_0}^{t_1} H ds = 0$$

and

$$\theta(t_0, t_1, y_1) - \theta(t_0, t_0, y_0) - \int_{t_0}^{t_1} f_2 ds = 0$$

This implies

$$-u(t_0, y_0; t_0, y_0) + \int_{t_0}^{t_1} (H + f_2) ds + \theta(t_0, t_0, y) = 0$$

The difference $u - \theta$ is positive for $t_0 < t < t_1$ and is zero for $t = t_1$. Therefore, the derivative of the difference $u - \theta$ at the point $t = t_1$ is nonpositive, i.e. we have the inequality

$((u-\theta)_t)_{t=t_1} \le 0$. From this and the inequality $\theta_t = H(\theta, t, y, \theta_y)$ it follows that equality $((u-\theta)_t)_{t=t_1} = 0$. Then equality

$$-u(t_0, y_0; t_0, y_0) + \theta(t_0, t_0, y)ds = 0$$

This contradicts the inequality $\omega_2(y) < f(y)$. Therefore, (9) holds. By assumption, conditions A) and the integrals $\vartheta(t_0, t, y), \theta(t_0, t, y)$ of equations (3), (8) with initial values $\vartheta(t_0, t_0, y) = \omega_1(y)$, $\theta(t_0, t_0, y) = \omega_2(y)$ belong to B classC^1 and satisfy the following conditions:
1)$(\vartheta, t, y, \vartheta_y) \in E, (\theta, t, y, \theta_y) \in E$;
2)$\vartheta_t > H(u, t, y, \vartheta_y)$, $\theta_t = H(\theta, t, y, \theta_y) in E$;
3)$\omega_1(y) > f(y) > \omega_2(y)$ Everywhere on B, inequality

$$\vartheta(t_0, t, y) > u(t, y; t_0, y_0) > \theta(t_0, t, y),\ t \in I \tag{10}$$

By condition a_1, b_1, c_1 of B), the characteristic system (4) has a generalized upper central exponent equal to $\beta_1 < 0$. Therefore system (4) is asymptotically stable in the sense of Lyapunov as $t \to +\infty$. From which it follows that

$$\lim_{t_0 \to +\infty} \vartheta(t_0, t, y) = 0,\ \ t > t_0 \tag{11}$$

A linear homogeneous system of differential equations

$$\frac{dy_i}{dt} = -\sum_{k=1}^{n} b_{ik}(t) y_k, i = 1, \dots, n.$$

due to the condition a_2, b_2, c_2 of M) has a generalized upper central exponent equal to $\mu_1 < 0$. Therefore, the system is asymptotically stable in the sense of Lyapunov on $t \to +\infty$. Moreover, system (8)

$$\frac{dy_1}{dt} = -\left(\sum_{k=1}^{n} b_{1k}(t) y_k + h(\theta, t, y)\right),\ \frac{dy_i}{dt} = -\sum_{k=1}^{n} b_{ik}(t) y_k,\ i = 2, \dots, n.$$

by the condition on $h(\theta, t, y)$ has an asymptotically stable zero solution. Hence we will have

$$\lim_{t_0 \to +\infty} \theta(t_0, t, y) = 0,\ \ t > t_0 \tag{12}$$

Consequently, it follows from (10), (11), (12) that the solution $u(t, y; t_0, y_0)$ of equation (1) has a limit

$$\lim_{t \to +\infty} u(t, y; t_0, y_0) = 0.$$

The theorem is proved.

4 Results and discussion

The paper deals with a nonlinear differential equation with partial derivatives of the first order, solved with respect to one of the derivatives and the asymptotic behavior of the solution. Using differential inequalities it is proved that a nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives has a solution tending to zero as one of the independent variables tending to infinity.

5 Conclusion

A condition for a nonlinear differential equation with first-order partial derivatives solved with respect to one of the derivatives was found, for which the equation has a solution that tends to zero as one of the independent variables tends to infinity.

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Necessary and Sufficient Conditions for Oscillations of Functional Differential Equations

Moremedi[1] G.M., Stavroulakis[1,2] I.P., Zhunussova[3] Zh.Kh.
[1]Department of Mathematical Sciences, University of South Africa,
0003 South Africa, E-mail: moremgm@unisa.ac.za
[2]Department.of Mathematics, University of Ioannina,
451 10 Ioannina, Greece, E-mail: ipstav@uoi.gr
[3]Faculty of Mechanics and Mathematics,
Al-Farabi Kazakh National University,
Almaty, Kazakshtan, E-mail: Zhunussova777@gmail.com

In this survey, necessary and sufficient conditions for the oscillation of solutions of retarded, advanced and neutral differential equations of first and higher order with one or several constant coefficients and constant arguments, in terms of the characteristic equation, are presented. Explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions are also presented in the case of one argument. In the case of nth order equations necessary and sufficient conditions for the oscillation of all solutions are presented when n is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions are presented when n is even. In this case explicit sufficient conditions for the oscillation of all solutions are presented when n is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when n is even. In the case of several arguments explicit but sufficient conditions only are given and the results are also extended to equations with several variable coefficients.
Key words: Oscillation; Delay, Necessary and sufficient conditions, Characteristic equation, Difference Equations.

Необходимые и достаточные условия для колебательных функциональных дифференциальных уравнений

Моремеди[1] Г.М., Ставроулакис[1,2] И.П., Жунусова[3] Ж.Х.
[1]Департамент математики, Университет Южной Африки,
0003 Южная Африка, E-mail: moremgm@unisa.ac.za
[2]Департамент математики, Университет Иоаннина,
451 10 Иоаннина, Греция, E-mail: ipstav@uoi.gr
[3]Механико-математический факультет, КазНУ имени аль-Фараби,
Алматы, Казахстан, E-mail: zhunussova777@gmail.com

В этой статье представлены необходимые и достаточные условия для колебаний всех решений запаздывающих, продвинутых и нейтральных дифференциальных уравнений первого и высшего порядка с одним или несколькими постоянными коэффициентами и постоянными аргументами в терминах характеристического уравнения. Явные (только по постоянному коэффициенту и постоянному аргументу) необходимые и достаточные условия также представлены в случае одного аргумента. В случае уравнения n-го порядка необходимые и достаточные условия для колебаний всех решений представлены когда n является нечетным, а необходимые и достаточные условия для колебаний всех граничных решений представлены когда n является четным. В этом случае явные достаточные условия для колебаний всех решений представлены когда n является нечетным, а явные достаточные условия для колебаний всех граничных решений для уравнений с запаздыванием и всех неграничных решений для продвинутых уравнений представлены когда n является нечетным. В случае нескольких аргументы явные, но достаточные условия даются, и результаты также распространяется на уравнения с несколькими переменными коэффициентами.
Ключевые слова: Колебание; запаздывание, необходимые и достаточные условия, характеристическое уравнение, разностные уравнения.

1 Introduction

Consider the first-order linear functional differential equations with several deviating arguments of the retarded

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0 \tag{1}$$

and the advanced type

$$x'(t) - \sum_{i=1}^{n} p_i x(t + \tau_i) = 0. \tag*{(1)$'$}$$

In the special case that $n = 1$ the above equations reduce to the following retarded

$$x'(t) + px(t - \tau) = 0 \tag{2}$$

and advanced differential equation

$$x'(t) - px(t + \tau) = 0. \tag*{(2)$'$}$$

Equations of higher order and equations with variable coefficients are also studied. Several sufficient and necessary and sufficient conditions under which all solutions oscillate are presented.

As it is customary, a solution is said to be oscillatory if it has arbitrarily large zeros. Otherwise it is called non-oscillatory and in this case it is eventually positive or eventually negative. Solutions are assumed to be defined for all $t \geq 0$.

The oscillation theory of differential equations was initiated by Sturm [26] in 1836. Since then many papers have been published on the subject. See, for example, the references [1-26] and the papers cited therein. For the general theory of delay equations the reader is referred to the monographs [8,9,6,4].

2 Literature review

The oscillation theory of Ordinary Differential Equations (ODEs) was originated by Sturm in 1836. Since then hundreds of papers have been published studying the oscillation theory of ODEs.

The oscillation theory of Delay Differential Equations (DDEs) was mainly developed after the 2nd world war. It was during the war that the admirals and officers in Navy (Fleet) observed that the ships were vibrating and asked the engineers and the scientists to solve the problem. Investigating the problem of vibrations (oscillations) the scientists found out that the equation which was to be taken into consideration was not an ODE (a usual equation without delays) but it was a differential equation with delays.

In the decade of 1970 a great number of papers were written extending known results from ODEs to DDEs. Of particular importance, however, has been the study of oscillations which are caused by the delay and which do not appear in the corresponding ODE. In recent years there has been a great deal of interest in the study of oscillatory behavior of the solutions to DDEs and also the discrete analogue Delay Difference Equations (DΔEs). See, for example, [1-26] and the references cited therein.

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_{0,}\infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is non-decreasing, $\tau(t) < t$ for $t \geq t_0$, and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations. See, for example, [4-6, 8-12, 14-17, 19, 21, 22, 24] and the references cited therein.

By a solution of Eq. (1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that (1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory*.

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1$'$}$$

where $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a sequence of integers such that $\tau(n) < n - 1$ for $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$, has also attracted growing attention in the recent few years. The reader is referred to [1-3, 7, 13, 18, 20, 23, 25] and the references cited therein.

By a solution of Eq. $(1)'$ we mean a sequence $x(n)$ which satisfies $(1)'$ for $n \geq 0$. A solution $x(n)$ of $(1)'$ is said to be *oscillatory* if the terms of the solution are not eventually positive or eventually negative. Otherwise the solution is called *nonoscillatory*.

3 Materials and methods

3.1 Necessary and sufficient conditions

In this section we present necessary and sufficient conditions under which all solutions of the equations under consideration oscillate.

3.1.1 First-order Equations

Consider the first-order linear retarded differential equation (1) with constant coefficients. In the following theorem a necessary and sufficient condition for the oscillation of all solutions of (1) in terms of the characteristic equation associated with (1) is given.

Theorem 1. ([15]) Consider the equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0 \tag{1}$$

where the coefficients p_i are real numbers and the delays τ_i are non-negative real numbers. Then all solutions of (1) oscillate if and only if its characteristic equation

$$\lambda + \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0 \tag{3}$$

has no real roots.

In the special case of Eq. (2) and (2)′, we have the following theorem.

Theorem 2. ([17,8])Consider the equation with one constant coefficient and one constant delay

$$x'(t) + px(t - \tau) = 0, \tag{2}$$

where p, τ are real numbers. Then all solutions of (2) oscillate if and only if its characteristic equation

$$\lambda + pe^{-\lambda \tau} = 0 \tag{4}$$

has no real roots.

Consider now the first-order neutral differential equation

$$\frac{d}{dt}[x(t) + px(t - \tau)] + qx(t - \sigma) = 0. \tag{5}$$

The following theorem holds.

Theorem 3. ([24,8]**)** Consider Eq.(5), where p, q, τ and σ are real numbers. Then all solutions of Eq. (5) oscillate if and only if its characteristic equation

$$\lambda + p\lambda e^{-\lambda \tau} + qe^{-\lambda \sigma} = 0 \tag{6}$$

has no real roots.

In the general case of the first-order neutral differential equation with several coefficients we have the following.

Theorem 4. ([7]**)** Consider the neutral differential equation

$$\frac{d}{dt}[x(t) + \sum_{i=1}^{n} p_i x(t - \tau_i)] + \sum_{i=1}^{n} q_i x(t - \sigma_i) = 0. \tag{7}$$

where p_i, q_i, τ_i and σ_i are real numbers. Then all solutions of Eq.(7) oscillate if and only if the characteristic equation associated with (7)

$$\lambda + \lambda \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} + \sum_{i=1}^{n} q_i e^{-\lambda \sigma_i} = 0 \tag{8}$$

has no real roots.

3.1.2 Higher-order Equations

Consider now the nth-order delay equation

$$x^{(n)}(t) + (-1)^{n+1} p x(t-\tau) = 0, \quad p, \tau > 0; \; n \geq 1. \tag{9}$$

In this case the characteristic equation of Eq. (9) is

$$\lambda^n + (-1)^{n+1} p e^{-\lambda \tau} = 0. \tag{10}$$

We have the following theorem.

Theorem 5. **([16])** For n odd [n even] the following statements are equivalent.
(a) All solutions of Eq.(9) oscillate [All bounded solutions of Eq.(9) oscillate].
(b) The characteristic equation (10) has no real roots [Eq.(10) has no real roots in $(-\infty, 0]$].

In the general case of the nth-order differential equation with several coefficients of the form

$$x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{n} p_i x(t-\tau_i) = 0, \; p_i, \tau_i > 0 \text{ and } n \geq 1 \tag{11}$$

the characteristic equation of Eq.(11) is

$$\lambda^n + (-1)^{n+1} \sum_{i=1}^{n} p_i e^{-\lambda \tau_i} = 0. \tag{12}$$

and we have the following.

Theorem 6. **([16])** For n odd [n even] the following statements are equivalent.
(a) All solutions of Eq.(11) oscillate [All bounded solutions of Eq.(11) oscillate].
(b) The characteristic equation (12) has no real roots [Eq.(12) has no real roots in $(-\infty, 0]$].

Consider now the general case of the nth-order neutral differential equation with several coefficients

$$\frac{d^n}{dt^n}[x(t) + \sum_{\mathcal{J}} p_i x(t-\tau_i)] + \sum_{\mathcal{K}} q_k x(t-\sigma_k) = 0, \quad n \geq 1 \tag{13}$$

where $\mathcal{J}, \mathcal{K}$ are initial segments of natural numbers and $p_i, \tau_i, q_k, \sigma_k \in R$ for $i \in \mathcal{J}$ and $k \in \mathcal{K}$.

Theorem 7. **([2])**A necessary and sufficient condition for the oscillation of all solutions of Eq.(13) is that the characteristic equation associated with (13)

$$\lambda^n + \lambda^n \sum_{\mathcal{J}} p_i e^{-\lambda \tau_i} + \sum_{\mathcal{K}} q_i e^{-\lambda \sigma_k} = 0 \tag{14}$$

has no real roots.

3.2 Explicit Oscillation Conditions

In this section we present explicit (in terms of the coefficients and the arguments only) oscillation conditions. In the case of equations with one delay an explicit necessary and sufficient condition is also derived.

3.2.1 First-order Equations

Theorem 8. ([19,1,10]) Consider the differential equation with several constant retarded arguments

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0 \tag{1}$$

and the differential equation with several constant advanced arguments

$$x'(t) - \sum_{i=1}^{n} p_i x(t + \tau_i) = 0 \tag{1$'$}$$

where p_i and τ_i, $i = 1, 2, ..., n$ are positive constants. Then each one of the following conditions

(i) $p_i\ \tau_i > \frac{1}{e}$ for some i, $i = 1, 2, ..., n$,

(ii) $\left(\sum_{i=1}^{n} p_i\right) \tau > \frac{1}{e}$, where $\tau = \min\{\tau_1, \tau_2, ..., \tau_n\}$,

(iii) $\sum_{i=1}^{n} p_i \tau_i > \frac{1}{e}$,

(iv) $\left[\prod_{i=1}^{n} p_i\right] \left(\sum_{i=1}^{n} \tau_i\right) > \frac{1}{e}$,

(v) $\frac{1}{n} \left(\sum_{i=1}^{n} (p_i \tau_i)^{1/2}\right)^2 > \frac{1}{e}$

implies that all solutions of (1) and (1)$'$ oscillate.

Remark 1. ([17,19]) It is noteworthy to observe that when $n = 1$, that is, in the case of a differential equation with one deviating argument, each one of the conditions (i), (ii), (iii), (iv), (v) reduces to

$$p\tau > \frac{1}{e} \tag{15}$$

which is a *necessary and sufficient condition* for all solutions of the retarded

$$x'(t) + px(t - \tau) = 0, \quad p, \tau > 0, \tag{2}$$

and the advanced differential equation

$$x'(t) - px(t + \tau) = 0, \quad p, \tau > 0. \tag{2$'$}$$

to be oscillatory.

We present the proof of this fact in the case of Eq.(2). [The proof in the case of Eq.(2)′ is similar].

Proof. The characteristic equation associated with Eq.(2) is

$$F(\lambda) \equiv \lambda + pe^{-\lambda\tau} = 0.$$

It is easy to compute the critical points of $F(\lambda)$ and evaluate the extreme values. The first derivative $F'(\lambda) = 1 - p\tau e^{-\lambda\tau}$ and therefore the critical point is $\lambda_0 = \frac{1}{\tau}\ln(p\tau)$. The second derivative $F''(\lambda) = p\tau^2 e^{-\lambda\tau} > 0$. Therefore at the critical point λ_0 the function $F(\lambda)$ has a minimum value $F(\lambda_0) = \frac{\ln(p\tau)+1}{\tau}$. The minimum value would be positive if and only if $\ln(p\tau) + 1 > 0$, that is, if and only if $p\tau > \frac{1}{e}$, which completes the proof.

Next we consider neutral differential equations of the retarded and advanced type as well as neutral equations of the mixed type and present explicit sufficient oscillation conditions.

Theorem 9. ([25]) Consider the neutral differential equation with several constant retarded arguments

$$\frac{d}{dt}[x(t) + cx(t-r)] + \sum_{i=1}^{k} p_i x(t-\tau_i) = 0. \tag{16}$$

and the neutral equation with several constant advanced arguments

$$\frac{d}{dt}[x(t) + cx(t+r)] - \sum_{i=1}^{k} p_i x(t+\tau_i) = 0. \tag{16$'$}$$

and the neutral equations of mixed type

$$\frac{d}{dt}[x(t) + cx(t-r)] + \sum_{i=1}^{k} p_i x(t-\tau_i) + \sum_{j=1}^{\ell} q_j x(t+\sigma_j) = 0, \tag{17}$$

and

$$\frac{d}{dt}[x(t) + cx(t+r)] - \sum_{i=1}^{k} p_i x(t+\tau_i) - \sum_{j=1}^{\ell} q_j x(t-\sigma_j) = 0, \tag{17$'$}$$

where c$\in R$, $r \in (0,\infty)$, p_i, $q_j \in (0,\infty)$ and τ_i, $\sigma_j \in [0,\infty)$ for $i = 1,2,...,k$; $j = 1,2,...,\ell$. Then in any of the following cases all solutions of the equations $(16), (16)', (17)$ and $(17)'$ oscillate:

(i) $c = -1$

(ii) $-1 < c$, $r < \tau_1$ or $c < -1$, $r > \tau_k$ and furthermore

$$\frac{1}{1+c}\sum_{i=1}^{k} p_i(\tau_i - r) > \frac{1}{e}$$

or

$$\frac{1}{1+c}\left[\prod_{i=1}^{k} p_i\right]^{1/k}\left(\sum_{i=1}^{k} p_i(\tau_i - r)\right) > \frac{1}{e}$$

is satisfied;

(iii) $-1 < c < 0$ and

$$\sum_{i=1}^{k} p_i \tau_i > \frac{1}{e}$$

or

$$\left[\prod_{i=1}^{k} p_i\right]^{1/k} \left(\sum_{i=1}^{k} \tau_i\right) > \frac{1}{e}$$

is satisfied.

3.2.2 Higher-order equations and inequalities

Consider the n-th order delay differential inequalities

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) \leq 0, \tag{I}$$

and

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) \geq 0, \tag{II}$$

and the delay differential equation

$$x^{(n)}(t) + (-1)^{n+1} p^n x(t - n\tau) = 0, \tag{III}$$

where $p, \tau > 0$ and $n \geq 1$. A *necessary and sufficient condition* for the behavior of the solutions to the above inequalities and equation is given in the following theorem.

Theorem 10. ([18]) The condition

$$p\tau > \frac{1}{e} \tag{15}$$

is necessary and sufficient so that:

(i) When n is odd: (I) has no eventually positive solutions, (II) has no eventually negative solutions, and (III) has only oscillatory solutions.

(ii) When n is even: (I) has no eventually negative bounded solutions, (II) has no eventually positive bounded solutions, and every bounded solution of (III) is oscillatory.

In the general case of the nth-order ($n \geq 1$) differential equation with several retarded arguments

$$x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{k} p_i^n x(t - n\tau_i) = 0, \tag{18}$$

and the nth-order differential equation with several advanced arguments

$$x^{(n)}(t) - \sum_{i=1}^{k} p_i^n x(t + n\tau_i) = 0, \tag{18$'$}$$

where $p_i, \tau_i > 0$, $i = 1, 2, ..., k$, we present the following sufficient oscillation conditions,

Theorem 11. ([20]) Each one of the following conditions

(i) $p_i \, \tau_i > \frac{1}{e}$ for some i, $i = 1, 2, ..., k$,

(ii) $\left(\sum_{i=1}^{k} p_i^n\right)^{1/n} \tau > \frac{1}{e}$, where $\tau = \min\{\tau_1, \tau_2, ..., \tau_k\}$,

implies:

(a) For n odd, every solution of (18) and (18)$'$ oscillates.

(b) For n even every bounded solution of (18) and every unbounded solution of (18)$'$ oscillates.

3.2.3 First-order equations with variable coefficients

In this section we present a generalization of the results of Theorem 8 to differential equations with several variable coefficients of the retarded

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0 \tag{19}$$

and the advanced type

$$x'(t) - \sum_{i=1}^{n} p_i(t)x(t + \tau_i) = 0 \tag*{(19)$'$}$$

where τ_i, $i = 1, 2, ..., n$ are positive constants and $p_i(t)$, $i = 1, 2, ..., n$ are positive and continuous functions.

Theorem 12. ([19]) Consider the differential equations (19) [(19)$'$] and assume that

$$\liminf_{t\to\infty} \int_{t-(\tau_i/2)}^{t} p(s)ds > 0, \quad \left[\liminf_{t\to\infty} \int_{t}^{t+(\tau_i/2)} p(s)ds > 0\right], \quad i = 1, 2, ..., n.$$

Then each one of the following conditions

$$\liminf_{t\to\infty} \int_{t-\tau_i}^{t} p_i(s)ds > \frac{1}{e}, \left[\liminf_{t\to\infty} \int_{t}^{t+\tau_i} p_i(s)ds > \frac{1}{e}\right], \text{ for some } i, \; i = 1, 2, ..., n,$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \sum_{i=1}^{n} p_i(s)ds > \frac{1}{e} \; \left[\liminf_{t\to\infty} \int_{t}^{t+\tau} \sum_{i=1}^{n} p_i(s)ds > \frac{1}{e}\right], \text{where } \tau = \min[\tau_1, ..., \tau_n\},$$

$$\left[\prod_{i=1}^{k}\left(\sum_{j=1}^{n} \liminf_{t\to\infty} \int_{t-\tau_j}^{t} p_i(s)ds\right)\right]^{\frac{1}{n}} > \frac{1}{e} \; \left[\left[\prod_{i=1}^{k}\left(\sum_{j=1}^{n} \liminf_{t\to\infty} \int_{t}^{t+\tau_j} p_i(s)ds\right)\right]^{\frac{1}{n}} > \frac{1}{e}\right]$$

or

$$\frac{1}{n}\sum_{i=1}^{n}\left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_i(s)ds\right) + \frac{2}{n}\sum_{i<j,i,j=1}^{n}\left[\left(\liminf_{t\to\infty}\int_{t-\tau_j}^{t} p_i(s)ds\right) \times \left(\liminf_{t\to\infty}\int_{t-\tau_i}^{t} p_j(s)ds\right)\right]^{1/2} > \frac{1}{e}$$

$$\left[\begin{array}{c} \frac{1}{n}\sum_{i=1}^{n}\left(\liminf_{t\to\infty}\int_t^{t+\tau_i}p_i(s)ds\right)+ \\ \frac{2}{n}\sum_{i<j,i,j=1}^{n}\left[\left(\liminf_{t\to\infty}\int_t^{t+\tau_j}p_i(s)ds\right)\times\left(\liminf_{t\to\infty}\int_t^{t+\tau_i}p_j(s)ds\right)\right]^{1/2}>\frac{1}{e}\end{array}\right]$$

implies that every solution of (19) [(19)′] oscillates.

Next we present a further extension to the following differential equation with variable delay of the form

$$x'(t)+\sum_{i=1}^{n}p_i(t)x(t-\tau_i(t))=0 \tag{20}$$

where τ_i are continuous and positive valued on $[0,\infty)$.

Theorem 13. ([10]) If there is a uniform upper bound τ_0 on the $\tau_i^{,}$ s and

$$\liminf_{t\to\infty}\sum_{i=1}^{n}p_i(t)\tau_i(t)>\frac{1}{e}$$

then all solutions of Eq.(20) oscillate.

4 Results and Discussion

In this survey paper we present necessary and sufficient conditions for the oscillation of solutions of retarded, advanced and neutral differential equations of first and higher order with one or several constant coefficients and constant arguments, in terms of the characteristic equation. Explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions are also presented in the case of one argument only. In the case of nth order equations necessary and sufficient conditions for the oscillation of all solutions are presented when n is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions only are presented when n is even. In this case explicit sufficient conditions for the oscillation of all solutions are presented when n is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when n is even. It is to be pointed out that in the case of several arguments explicit but sufficient conditions only are given. The results are also extended to equations with several variable coefficients where sufficient conditions only are given.

5 Conclusion

We conclude that necessary and sufficient conditions for oscillation of all solutions have been given in the case of differential equations with several constant coefficients and constant arguments in terms of the characteristic equation only. While explicit (in terms of the constant coefficient and constant argument only) necessary and sufficient conditions have been given in the case of one argument only. It is to be pointed out that in the case of nth order equations necessary and sufficient conditions for the oscillation of all solutions are presented when n is odd, while necessary and sufficient conditions for the oscillation of all bounded solutions only are presented when n is even. In this case of nth order equations explicit sufficient conditions for the oscillation of all solutions are presented when n is odd, while explicit sufficient conditions for the oscillation of all bounded solutions for retarded equations and of all unbounded solutions for advanced equations are presented when n is even. Furthermore, in the case of several arguments explicit but sufficient conditions only are given.

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Исследование глобальной асимптотической устойчивости многомерных фазовых систем

Айсагалиев С.А., Казахский национальный университет имени аль-Фараби, г. Алматы, Республика Казахстан, E-mail: Serikbai.Aisagaliev@kaznu.kz
Айсагалиева С.С., Научно-исследовательский институт математики и механики КазНУ имени аль-Фараби, г. Алматы, Республика Казахстан, E-mail: a_sofiya@mail.ru

Создана общая теория глобальной асимптотической устойчивости многомерных динамических систем с цилиндрическим фазовым пространством со счетным положением равновесия. Установлена ограниченность решений многомерных фазовых систем и их производных. Найдены условия при выполнений которых решение и ее производная обладают асимптотическими свойствами. Получены условия глобальной асимптотической устойчивости многомерных фазовых систем с равными нулю в периоде значениями интегралов от компонентов периодических нелинейностей. Получены условия глобальной асимптотической устойчивости фазовых систем с не равными нулю в периоде значениями интегралов от составляющих нелинейных периодических функций. Исследованы асимптотические свойства решений динамических систем со счетным положением равновесия в общем случае, когда часть компонентов нелинейных периодических функции обладают значениями интегралов в периоде равными нулю, а для других компонентов значения интегралов в периоде не равными нулю. Отличительной особенностью предлагаемого метода исследования многомерных фазовых систем от известных методов состоит в том, что он применим для систем любого порядка с любым числом нелинейных периодических функции, и не привлекаются для исследования периодические функции Ляпунова и частотные теоремы. Примечательно то, что предлагаемые условия глобальной асимптотической устойчивости легко проверяемые по сравнению с частотными условиями и условиями полученные с помощью периодических функции Ляпунова.
Ключевые слова: Асимптотические свойства, ограниченность решений, глобальная асимптотическая устойчивость, несобственные интегралы.

Көпөлшемді фазалық жүйелердің глобальді асимптотикалық орнықтылығын зерттеу

Айсағалиев С.Ә., әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қаласы, Қазақстан Республикасы, E-mail: Serikbai.Aisagaliev@kaznu.kz
Айсағалиева С.С., әл-Фараби атындағы Қазақ ұлттық университеті, Математика және механика ғызыми-зерттеу институты, Алматы қаласы, Қазақстан Республикасы, E-mail: a_sofiya@mail.ru

Саналымды тепе-теңдік жағдайымен цилиндрлік фазалық кеңістікте көпөлшемді динамикалық жүйелердің глобальді асимптоталық тұрақтылығының жалпы теориясы құрылған. Көп өлшемді фазалық жүйелер шешімдері мен олардың туындыларының шектеулігі анықталған. Шешімнің және оның туындысының асимптотикалық қасиеттерінің

орындалуы үшін жағдайлар жасалған. Периодты сызықтық емес компоненттерден тәуелді периодта интегралдың мәндері нөлге тең болатын көп өлшемді фазалық жүйелердің глобальдік асимптотикалық тұрақтылық шарттары алынды. Сызықты емес пероидты функциялар компоненттерінің интегралдарының нөлден тыс мәндері бар фазалық жүйелердің глобальдік асимптотикалық тұрақтылығы шарттары алынды. Жалпы жағдайда, сызықтық емес периодты функциялардың кейбір компоненттері нөлге тең болған кезеңде интегралдардың мәндеріне ие болған кезде және басқа компоненттер үшін интегралдардың мәндері нөлге тең болмаған кезде саналымды тепе-теңдік жағдайындағы динамикалық жүйелердің шешімдерінің асимптотикалық қасиеттері зерттерлген. Көпөлшемді фазалық жүйелерді зерттеудің ұсынылатын әдісінің белгілі әдістерден айырықша ерекшелігі – ол сызықтық емес периодты функциялардың кез келген санымен кез келген ретті жүйелер үшін қолданылатыны, сонымен қатар Ляпуновтың периодты функциялары мен жиілік теоремаларын зерттеуге еш қатысы болмауында. Айта кету керек, глобальдік асимптоталық тұрақтылықтың ұсынылған жағдайлары, Ляпуновтың периодты функциялары көмегімен алынған жағдайлармен және жиілік жағдайларымен салыстырғанда оңай тексеріледі.
Түйін сөздер: Асимптотикалық қасиеттер, шешімдердің шектеулілігі, глобальді асимптоталық тұрақтылық, меншіксіз интегралдар.

Investigation of the global asymptotic stability of multidimensional phase systems

Aisagaliev S.A., Al-Farabi Kazakh National university, Almaty, Republic of Kazakhstan,
E-mail: Serikbai.Aisagaliev@kaznu.kz
Aisagalieva S.S., Research Institute of Mathematics and Mechanics of al-Farabi Kazakh National University, E-mail: a_sofiya@mail.ru

A general theory of global asymptotic stability of multidimensional dynamical systems with a cylindrical phase space with a countable equilibrium position is created. The boundedness of solutions of multidimensional phase systems and their derivatives is established. Conditions for the fulfillment of which the solution and its derivative have asymptotic properties are found. Conditions for global asymptotic stability of multidimensional phase systems with values of integrals equal to zero in the period from the components of periodic nonlinearities are obtained. Conditions for global asymptotic stability of phase systems with nonzero values of the integrals of the components of nonlinear periodic functions are obtained. The asymptotic properties of solutions of dynamical systems with a countable equilibrium position are investigated in the general case when some of the components of nonlinear periodic functions have values of the integrals in the period equal to zero, and for other components the values of the integrals in the period are not equal to zero. A distinctive feature of the proposed method for investigating multidimensional phase systems from known methods is that it is applicable to systems of any order with any number of nonlinear periodic functions, and are not involved in research periodic Lyapunov functions and frequency theorems. It is noteworthy, that the proposed conditions for global asymptotic stability, which are easily verified in comparison with the frequency conditions and conditions obtained with the help of periodic Lyapunov functions.
Key words: Asymptotic properties, boundedness of solutions, global asymptotic stability, improper integrals.

1 Введение

Рассмотрим динамическую систему с цилиндрическим фазовым пространством описываемую уравнением следующего вида:

$$\dot{x} = Ax + B\varphi(\sigma), \quad \dot{\sigma} = Cx + R\varphi(x), \quad x(0) = x_0, \quad \sigma(0) = \sigma_0, \quad t \in I = [0, \infty), \tag{1}$$

где A, B, C, R – постоянные матрицы порядков $n\times n$, $n\times m$, $m\times n$, $m\times m$ соответственно, матрица A – гурвицева, т.е. $Re\lambda_j(A) < 0$, $j = \overline{1,n}$, $\lambda_j(A)$ $j = \overline{1,n}$ – собственные значения матрицы A, $|x_0| < \infty$, $|\sigma_0| < \infty$, $\varphi(\sigma) = (\varphi_1(\sigma_1), \ldots, \varphi_m(\sigma_m))$, $\sigma = (\sigma_1, \ldots, \sigma_m)$.

Функция

$$\varphi(\sigma) \in \Phi_0 = \{\varphi(\sigma) = (\varphi_1(\sigma_1), \dots, \varphi_m(\sigma_m)) \in C^1(R^m, R^m)/\mu_{1i} \le \frac{d\varphi_i(\sigma_i)}{d\sigma_i} \le \mu_{2i} \quad (2)$$
$$\varphi_i(\sigma_i) = \varphi_i(\sigma_i + \Delta_i), \ \ \forall\sigma_i, \ \ \sigma_i \in R^1, \ \ i = \overline{1,m}\},$$

где Δ_i – период функции $\varphi_i(\sigma_i)$, μ_{1i}, μ_{2i}, $i = \overline{1,m}$ – заданные числа, $|\mu_1| < \infty$, $|\mu_2| < \infty$, $\mu_1 = (\mu_{11}, \dots, \mu_{1m})$, $\mu_2 = (\mu_{21}, \dots, \mu_{2m})$.

Положение равновесия системы (1), (2) определяется из решения алгебраических уравнений. $Ax_* + B\varphi(\sigma_*) = 0, \ \ Cx_* + R\varphi(\sigma_*) = 0$.

Поскольку $x_* = -A^{-1}B\varphi(\sigma_*)$, $(R - CA^{-1}B)\varphi(\sigma_*) = 0$, то при $R - CA^{-1}B$ – неособая матрица порядка $m \times m$ система (1), (2) имеет стационарное множество.

$$\Lambda = \{(x_*, \sigma_*) \in R^{n+m}/x_* = 0, \ \ \varphi(\sigma_*) = 0\}.$$

Так как $\varphi(\sigma_*) = \varphi(\sigma_* + k\Delta) = 0$, $k = 0, \pm1, \pm2, \dots$, то положение равновесия системы (1), (2) является счетным множеством, $\sigma_* = (\sigma_{1*}, \dots, \sigma_{m*})$, $\Delta = (\Delta_1, \dots, \Delta_m)$.

Определение 1 *Стационарное множество* Λ *системы* (1), (2) *глобально асимптотически устойчиво, если для любой функции* $\varphi(\sigma) \in \Phi_0$ *и любого начального состояния* $(x_0, \sigma_0) \in R^{n+m}$, $|x_0| < \infty$, $|\sigma_0| < \infty$ *решение системы* $x(t) = x(t; 0, x_0, \sigma_0, \varphi)$, $\sigma(t) = \sigma(t; 0, x_0, \sigma_0, \varphi)$, $t \in I$ *обладает свойством* $x(t) \to x_* = 0$, $\sigma(t) \to \sigma_*$ *при* $t \to \infty$, *где* $\varphi(\sigma_*) = 0$.

Определение 2 *Условием глобальной асимптотической устойчивости системы* (1), (2) *называются соотношения, связывающие конструктивные параметры системы* $(A, B, C, R, \mu_1, \mu_2)$, *при выполнении которых множество* Λ *глобально асимптотически устойчиво.*

Необходимо исследовать в отдельности два случая:

1.

$$\int\limits_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i) d\xi_i = 0, \quad \forall\sigma_i, \ \sigma_i \in R^1, \ \ i = \overline{1,m};$$

2.

$$\int\limits_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i) d\xi_i \ne 0, \quad \forall\sigma_i, \ \sigma_i \in R^1, \ \ i = \overline{1,m}.$$

Ставятся следующие задачи:

Задача 1 *Найти оценки несобственных интегралов, вдоль решения системы* (1), (2), *для случаев 1, 2.*

Задача 2 *Найти новое эффективное условие глобальной асимптотической устойчивости стационарного множества* Λ *системы* (1), (2) *для случая, когда*

$$\int_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i)d\xi_i = 0, \quad \forall \sigma_i,\ \sigma_i \in R^1, \ \ i = \overline{1,m};$$

на основе оценки несобственных интегралов для случая 1.

Задача 3 *Найти новое эффективное условие глобальной асимптотической устойчивости стационарного множества* Λ *системы* (1), (2) *для случая, когда*

$$\int_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i)d\xi_i \neq 0, \quad \forall \sigma_i,\ \sigma_i \in R^1, \ \ i = \overline{1,m};$$

на основе оценки несобственных интегралов для случая 2.

Решение задачи 1 приведено в работе [16]. Данная статья является продолжением исследования из [16], в ней приведены решения задач 2, 3.

2 Обзор литературы

Первой работой, посвященной качественно-численным методам исследования фазовых систем, была статья Ф. Трикоми [1]. Применение метода точечных отображений к фазовым системам рассмотрено в работах А.А. Андронова и его последователей [2,3]. Следующим этапом развития качественно-численных методов было применение периодических функций Ляпунова к исследованию фазовых систем. Основы теории периодических функций Ляпунова приведены в работах [4,5]. Методы построения различных периодических функций Ляпунова, обеспечивающих устойчивость в большинстве фазовых систем, можно найти в [2]. Приближенные нелокальные методы исследования фазовых систем изложены в [6].

Оригинальным подходом к исследованию фазовых систем являются частотные условия асимптотической устойчивости, основанные на процедуре Бакаева-Гужа. Такой подход впервые предложен в работе Г.А. Леонова [7]. В последующих работах Леонова и его учеников [8-10] исследованы ограниченности решения фазовых систем, асимптотические свойства решений интегро-дифференциальных уравнений с периодическими нелинейностями, а также устойчивость и колебания фазовых систем. Библиографический обзор научной литературы по фазовым системам можно найти в монографиях [11], а также работах автора, изложенных в [12-16].

3 Материал и методы

Маятниковые системы в механике, навигационные системы в радиотехнике, синхронные машины в энергетике, вибрационные системы в технике являются динамическими системами с цилиндрическим фазовым пространством (или просто фазовыми системами). Математической моделью фазовых систем является класс обыкновенных дифференциальных уравнений имеющий счетное множество положений равновесия с периодическими нелинейностями из заданного множества. Следовательно, уравнения движения фазовых систем относятся к классу уравнений с дифференциальными включениями. Поскольку положение равновесия является счетным множеством, то для устойчивости системы, необходимо, чтобы каждое решение асимптотически стремилось к какому-либо положению равновесия из счетного множества.

В статье предлагается совершенно новый подход к решению глобальной асимптотической устойчивости фазовых систем основанный на априорной оценке несобственных интегралов вдоль решения системы.

3.1 Вспомогательные леммы

Как следует из Лемм 1–3, приведенной в работе [16] уравнение (1) с неособым преобразованием приводится к виду

$$\dot{y} = \overline{A}y + \overline{B}\varphi(\sigma); \quad \dot{\sigma} = \overline{C}y + R\varphi(\sigma), \quad \varphi(\sigma) \in \Phi_0. \tag{3}$$

Из (3) следует, что

$$\begin{gathered} \varphi(\sigma(t)) = H_0\dot{y}(t) - \overline{A}_{11}y(t), \;\; t \in I, \;\; H_1\dot{y}(t) = \overline{A}_{12}y(t), \;\; t \in I, \\ \dot{\sigma}(t) = (\overline{C} - R\overline{A}_{11})y(t) + RH_0\dot{y}(t), \;\; t \in I, \end{gathered} \tag{4}$$

где матрица $\overline{A}$ – гурвицева,

$$\overline{A} = \begin{pmatrix} \overline{A}_{11} \\ \overline{A}_{12} \end{pmatrix}, \quad I_n = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix}, \quad \begin{matrix} H_0 = (I_m, O_{m,n-m}), \\ H_1 = (O_{n-m,m}, I_{n-m}), \end{matrix} \quad H_0\dot{y} = \begin{pmatrix} \dot{y}_1 \\ \cdots \\ \dot{y}_m \end{pmatrix}, \quad H_1\dot{y} = \begin{pmatrix} \dot{y}_{m+1} \\ \cdots \\ \dot{y}_n \end{pmatrix}.$$

Лемма 1 *Пусть выполнены условия лемм 1–3 из [16], матрица $\overline{A}$ – гурвицева, функция $\varphi(\sigma) \in \Phi_0$. Тогда функция $z(t) = (y(t), \dot{y}(t))$, $t \in I$ ограничена, т.е. $|z(t)| \le a$, $t \in I$, непрерывно дифференцируема, причем $|\dot{z}(t)| = |(\dot{y}(t), \ddot{y}(t))| \le c$, $t \in I$, $0 < a < \infty$, $0 < c < \infty$.*

Доказательство. Пусть выполнены условия леммы. Покажем, что производная $\dot{\varphi}(\sigma(t)) = (\dot{\varphi}_1(\sigma_1), \dots, \dot{\varphi}_m(\sigma_m))$, $t \in I$ ограничена. В самом деле,

$$\frac{d\varphi_i(\sigma_i(t))}{dt} = \dot{\varphi}_i(\sigma_i(t)) = \frac{d\varphi_i(\sigma_i(t))}{d\sigma_i} \cdot \frac{d\sigma_i(t)}{dt}, \quad t \in I, \quad i = \overline{1,m}.$$

Так как

$$|\frac{d\varphi_i}{dt}| \le \mu_i, \;\; \mu_i = \max(|\mu_{1i}|, |\mu_{2i}|), \quad |\dot{\sigma}_i(t)| \le c_{4i}, \quad i = \overline{1,m}, \quad t \in I,$$

в силу $\varphi(\sigma) \in \Phi_0$, $|\dot{\sigma}(t)| \le c_4$, $t \in I$ (см. теорему 1 из [16]), то $|\dot{\varphi}(\sigma(t))| \le m_1$, $t \in I$, $0 < m_1 < \infty$. Из первого уравнения тождества (3), имеем $\ddot{y}(t) = \overline{A}\dot{y}(t) + \overline{B}\dot{\varphi}(\sigma(t))$, $t \in I$. Отсюда следует, что $|\ddot{y}(t)| \le \|\overline{A}\||\dot{y}| + \|\overline{B}\||\dot{\varphi}(\sigma(t))| \le \|\overline{A}\| \cdot c_3 + \|\overline{B}\|m_1 = m_2$, $0 < m_2 < \infty$, $\forall t$, $t \in I$. Из оценки $|y(t)| \le c_2$, $|\dot{y}(t)| \le c_3$, $|\ddot{y}(t)| = m_1$, $t \in I$ следует $|z(t)| = |(y(t), \dot{y}(t))| \le a$, $|\dot{z}(t)| = |(\dot{y}(t), \ddot{y}(t))| \le c$, $\forall t$, $t \in I$. Лемма доказана.

Лемма 2 *Пусть выполнены условия леммы 1, и пусть, кроме того:*
1) скалярная непрерывная функция $W(z) > 0$, $\forall z$, $z \in R^{2n}$, $W(0) = 0$;
2) $|z(t)| \le a$, $|\dot{z}(t)| \le c$, $t \in I$;
3) несобственный интеграл $\int\limits_0^\infty W(z(t))dt < \infty$.
Тогда $\lim\limits_{t\to\infty} z(t) = 0$, *где* $z(t) = (y(t), \dot{y}(t))$, $t \in I$.

Доказательство. Пусть выполнены условия 1) – 3) леммы. Покажем, что $\lim\limits_{t\to\infty} z(t) = 0$, $(\lim\limits_{t\to\infty} y(t) = 0, \ \lim\limits_{t\to\infty} \dot{y}(t) = 0)$.

Предположим противное, т.е. $\lim\limits_{t\to\infty} z(t) \ne 0$. Тогда существует последовательность $\{t_k\}$, $t_k > 0$, $t_k \to \infty$ при $k \to \infty$ такая, что $|z(t_k)| \ge \varepsilon > 0$, $k = 1, 2, \ldots$. Выберем $t_{k+1} - t_k \ge \varepsilon_1 > 0$, $k = 1, 2, \ldots$. Поскольку $z(t)$, $t \in I$ непрерывно дифференцируема $|z(t)| \le a$, $|\dot{z}(t)| \le c$, $t \in I$, то $|z(t) - z(t_k)| \le c|t - t_k|$, $\forall t$, $t \in [t_k - \frac{\varepsilon_1}{2}, t_k + \frac{\varepsilon_1}{2}]$, $k = 1, 2, \ldots$. Так как $t_k - \frac{\varepsilon_1}{2} > t_{k-1}$, $t_k + \frac{\varepsilon_1}{2} < t_{k+1}$, $W(z) > 0$, $z \in R^{2n}$, то

$$\int\limits_0^\infty W(z(t))dt \ge \sum_{k=1}^\infty \int\limits_{t_k - \frac{\varepsilon_1}{2}}^{t_k + \frac{\varepsilon_1}{2}} W(z(t))dt,$$

где $|y(t)| = |y(t_k) + y(t) - y(t_k)| \ge |y(t_k)| - |y(t) - y(t_k)| \ge \varepsilon - c\frac{\varepsilon_1}{2} = \varepsilon_0 > 0$, $\forall t$, $t \in [t_k - \frac{\varepsilon_1}{2}, t_k + \frac{\varepsilon_1}{2}]$. Всегда можно выбрать величину $\varepsilon_1 > 0$ так, что величина $\varepsilon_0 > 0$. Итак $|z(t)| \ge \varepsilon_0$, $|z(t)| \le c$, $t \in [t_k - \frac{\varepsilon_1}{2}, t_k + \frac{\varepsilon_1}{2}]$.

Так как функция $W(z)$ непрерывная на компактном множестве $\varepsilon_0 \le |z| \le c$, то существует число $m > 0$ такое, что $\min\limits_{\varepsilon_{01} \le |z| \le c} W(z) = m$. Тогда значение интеграла

$$\int\limits_{t_k - \frac{\varepsilon_1}{2}}^{t_k - \frac{\varepsilon_1}{2}} W[z(t)]dt \ge \varepsilon_1 m, \quad k = 1, 2, \ldots.$$

Следовательно,

$$\int\limits_0^\infty W[z(t)]dt \ge \sum_{k=1}^\infty \int\limits_{t_k - \frac{\varepsilon_1}{2}}^{t_k + \frac{\varepsilon_1}{2}} W[z(t)]dt \ge \lim_{k\to\infty} k(\varepsilon_1 m) = \infty.$$

Это противоречит условию 3) Леммы. Лемма доказана.

3.2 Глобальная асимптотическая устойчивость I

Рассмотрим случай, когда

$$\int\limits_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i)d\xi_i = 0, \quad \forall\sigma_i, \quad \sigma_i \in R^1, \;\; i=\overline{1,m}. \tag{5}$$

Теорема 1 *Пусть выполнены условия Лемм 1–5 и теоремы 1 из [16], и пусть, кроме того, функция* $\varphi(\sigma) \in \Phi_0$, *компоненты вектор функции* $\varphi(\sigma) = (\varphi_1(\sigma_1), \dots, \varphi_m(\sigma_m))$ *удовлетворяют условиям* (5). *Тогда для любых диагональных матриц* $\tau_1 = diag(\tau_{11}, \dots, \tau_{1m}) > 0$, $\tau_2 = diag(\tau_{21}, \dots, \tau_{2m}) > 0$, *вектор строк* $\alpha = (\alpha_1, \dots, \alpha_n) \in R^n$, $\beta = (\beta_1, \dots, \beta_n) \in R^n$, $\gamma = (\gamma_1, \dots, \gamma_n) \in R^n$ *вдоль решения системы* (3) *несобственный интеграл*

$$I_5 = \int\limits_0^\infty [\ddot{y}^*(t)S_1\ddot{y}(t) + \dot{y}^*(t)S_2\ddot{y}(t) + y^*(t)S_3\ddot{y}(t) + \dot{y}^*(t)S_4\dot{y}(t) + y^*(t)S_5\dot{y}(t) +$$
$$+ y^*(t)W_1y(t)]dt \le \int\limits_0^\infty \frac{d}{dt}[\dot{y}^*(t)Q_1\dot{y}(t) + y^*Q_2y(t)]dt - \int\limits_{\sigma(0)}^{\sigma(\infty)} \varphi^*(\sigma)\tau_2 d\sigma < \infty, \tag{6}$$

где $S_1 = H_0^*\tau_1H_0 - \alpha^*\alpha$, $S_2 = \overline{A}_{11}^*\tau_1H_0 + H_0^*R^*\mu_1\tau_1H_0 + H_0^*R^*\tau_1\mu_2H_0 + 2\alpha^*\beta$, $S_3 = -(\overline{C} - R\overline{A}_{11})^*(\mu_1\tau_1 + \tau_1\mu_2)H_0 - 2\alpha^*\gamma$, $S_4 = H_0^*R^*\mu_1\tau_1\overline{A}_{11} + \overline{A}_{11}^*\tau_1\overline{A}_{11} + H_0^*R^*\mu_1\tau_1\mu_2RH_0 + H_0^*R^*\tau_1\mu_2\overline{A}_{11} - \beta^*\beta + H_0^*\tau_2RH_0$, $S_5 = \overline{A}_{11}^*\mu_1\tau_1(\overline{C} - RA_{11}) - 2H_0^*R^*\mu_1\tau_1\mu_2(\overline{C} - R\overline{A}_{11}) - \overline{A}_{11}^*\tau_1\mu_2(\overline{C} - R\overline{A}_{11}) + 2\beta^*\gamma + \overline{A}_{11}^*\tau_2RH_0 - (\overline{C} - R\overline{A}_{11})^*\tau_2H_0$, $W_1 = (\overline{C} - R\overline{A}_{11})^*\mu_1\tau\mu_2(\overline{C} - R\overline{A}_{11}) - \gamma^*\gamma - \overline{A}_{11}^*\tau_2(\overline{C} - R\overline{A}_{11})$, $Q_1 = \overline{A}_{11}^*\tau_1H_0 + H_0^*R^*\mu_1\tau_1H_0 + H_0^*R^*\tau_1\mu_2H_0 + 2\beta^*\alpha$, $Q_2 = (\overline{C} - R\overline{A}_{11})^*\mu_1\tau_1A_{11} - 2(\overline{C} - R\overline{A}_{11})^*\mu_1\tau_1\mu_2RH_0 - (\overline{C} - R\overline{A}_{11})^*\tau_1\mu_2\overline{A}_{11} + 2\gamma^*\beta + \overline{A}_{11}^*\tau_2RH_0 - (\overline{C} - R\overline{A}_{11})^*\tau_2H_0$.

Доказательство. Поскольку выполнены условия лемм 1 – 5 и теоремы 1 из [16], то несобственные интегралы

$$I_{10} = \int\limits_0^\infty [-y^*(t)\Lambda_1\ddot{y}(t) + \dot{y}^*(t)\Lambda_2\ddot{y}(t) + y^*(t)\Lambda_3\dot{y}(t) - \dot{y}^*(t)\Lambda_4\dot{y}(t) - \ddot{y}^*(t)\Lambda_5\ddot{y}(t) -$$
$$- y^*(t)\Lambda_6y(t)]dt \le \int\limits_0^\infty \frac{d}{dt}[\dot{y}^*(t)\Lambda_2\dot{y}(t) + y^*(t)\Lambda_3y(t)]dt < \infty, \tag{7}$$

$$I_{20} = \int\limits_0^\infty [-\ddot{y}^*(t)\alpha^*\alpha\ddot{y}(t) - \dot{y}^*(t)\beta^*\beta\dot{y}(t) - y^*(t)\gamma^*\gamma y(t) + 2\dot{y}^*(t)\alpha^*\beta\ddot{y}(t) -$$
$$- 2y^*(t)\alpha^*\gamma\ddot{y}(t) + 2y^*(t)\beta^*\gamma\dot{y}(t)]dt \le \int_0^\infty \frac{d}{dt}[2\dot{y}^*(t)\beta^*\alpha\dot{y}(t) +$$
$$+ 2y^*(t)\gamma^*\beta y(t)]dt < \infty, \tag{8}$$

$$I_{30} = \int\limits_0^\infty [-y^*(t)T_1^*\dot{y}(t) + y^*(t)T_2 y(t) + \dot{y}^*(t)T_3\dot{y}(t)]dt =$$
$$= \int\limits_{\sigma(0)}^{\sigma\infty} \varphi^*(\sigma)\tau_2 d\sigma - \int\limits_0^\infty \frac{d}{dt}[y^*(t)T_1 y(t)]dt < \infty. \tag{9}$$

Из (7) – (9) следует, что несобственный интеграл

$$I_5 = I_{10} + I_{20} + I_{30} = \int\limits_0^\infty [\ddot{y}^*(t)S_1\ddot{y}(t) + \dot{y}^*(t)S_2\ddot{y}(t) + y^*(t)S_3\ddot{y}(t) + \dot{y}^*(t)S_4\dot{y}(t) +$$
$$+ y^*(t)S_5\dot{y}(t) + y^*(t)W_1 y(t)]dt \le \int\limits_0^\infty \frac{d}{dt}[\dot{y}^*(t)Q_1\dot{y}(t) + y^*(t)Q_2 y(t)]dt + \tag{10}$$
$$+ \int\limits_{\sigma(0)}^{\sigma(\infty)} \varphi^*(\sigma)\tau_2 d\sigma < \infty,$$

где $S_1 = -\Lambda_5 - \alpha^*\alpha$, $S_2 = \Lambda_2 + 2\alpha^*\beta$, $S_3 = -\Lambda_1 - 2\alpha^*\gamma$, $S_4 = -\Lambda_4 - \beta^*\beta + T_3$, $S_5 = \Lambda_3^* + 2\beta^*\gamma - T_1$, $W_1 = -\Lambda_6 - \gamma^*\gamma + T_2$, $Q_1 = \Lambda_2 + 2\beta^*\alpha$, $Q_2 = \Lambda_3 + 2\gamma^*\beta - T_1$.

Так как $|y(t)| \le c_2$, $|\dot{y}(t)| \le c_3$, $\forall t,\ t \in I$, то

$$\int\limits_0^\infty \frac{d}{dt}[\dot{y}^*(t)Q_1\dot{y}(t) + y^*(t)Q_2 y(t)]dt = [\dot{y}^*(t)Q_1\dot{y}(t) + y^*(t)Q_2 y(t)]\Big|_0^\infty < \infty,$$

$$\int\limits_{\sigma(0)}^{\sigma(\infty)} \varphi^*(\sigma)\tau_2 d\sigma < \infty$$

в силу соотношения (5). Следовательно, $I_5 < \infty$. Теорема доказана.

Теорема 2 *Пусть выполнены условия теоремы 1, и пусть, кроме того:*

1) $S_i = 0$, $i = \overline{1,3}$;

2) Матрицы W_{11}, W_{12}, W_{22}, $N = N^*$ *порядков* $n \times n$, $n \times n$, $n \times n$, $(n-m)\times(n-m)$ *соответственно, такие, что* $W_{22} > 0$, $\quad W_{11} - W_{12}W_{22}^{-1}W_{12}^* > 0$, *где* $W_{11} = \frac{1}{2}(W_1 + W_1^*) + \frac{1}{2}(H_1^* N\overline{A}_{12} + \overline{A}_{12}^* N H_1)$, $W_{12} = \frac{1}{2}S_5$, $W_{21} = \frac{1}{2}S_5^*$, $W_{22} = \frac{1}{2}(S_4 + S_4^*)$, $W_{11} = W_{11}^*$, $W_{22} = W_{22}^*$, $N = N^*$.

Тогда стационарное множество Λ *системы* (1), (2) *глобально асимптотически устойчиво.*

Доказательство. Если диагональные матрицы $\tau_1 > 0$, τ_2 и векторы $\alpha \in R^n$, $\beta \in R^n$, $\gamma \in R^n$ выбраны так, чтобы

$$S_1 = H_0^*\tau_1 H_0 - \alpha^*\alpha = 0, \quad S_2 = \overline{A}_{11}^*\tau_1 H_0 + H_0^* R^*\mu_1\tau_1 H_0 + H_0^* R^*\tau_1\mu_2 H_0 + 2\alpha^*\beta = 0,$$

$$S_3 = -(\overline{C} - R\overline{A}_{11})^*(\mu_1\tau_1 + \mu_2\tau_2)H_0 - 2\alpha^*\gamma = 0,$$

то несобственный интеграл (10) запишется в виде

$$I_5 = \int_0^\infty [y^*(t)W_1 y(t) + y^*(t)S_5\dot{y}(t) + \dot{y}^*(t)S_4\dot{y}(t)]dt < \infty. \tag{11}$$

Как следует из тождества (4) для любой симметричной матрицы $N = N^*$ порядка $(n-m)\times(n-m)$ верно неравенство $y^*(t)H_1^*NH_1\dot{y}(t) = y^*(t)H_1^*N\overline{A}_{12}y(t)$, $t \in I$. Тогда несобственный интеграл

$$I_6 = \int_0^\infty y^*(t)H_1^*N\overline{A}_{12}y(t)dt = \int_0^\infty y^*(t)H_1^*NH_1\dot{y}(t)dt = \int_0^\infty \frac{d}{dt}[\frac{1}{2}y^*(t)(H_1^*NH_1 + $$
$$+H_1^*NH_1)y(t)]dt = \frac{1}{2}y^*(t)[H_1^*NH_1 + H_1^*NH_1]y(t)\Big|_0^\infty < \infty, \tag{12}$$

в силу того, что $|y(t)| \le c_2$, $\forall t$, $t \in I$. Отсюда имеем

$$I_6 = \int_0^\infty \frac{1}{2}y^*(t)[H_1^*NH_1 + H_1^*NH_1]y(t) < \infty. \tag{13}$$

Пусть матрица

$$\pi = \begin{pmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{pmatrix},$$

функция $z(t) = (y(t), \dot{y}(t))$, $t \in I$, где матрицы W_{11}, W_{12}, W_{22} определяются соотношениями указанные в условии 2) теоремы. Тогда суммируя несобственные интегралы (11), (12), получим

$$I_5 + I_6 = \int_0^\infty z^*(t)\pi z(t)dt = \int_0^\infty \{y^*(t)[\frac{1}{2}(W_1 + W_1^*) + \frac{1}{2}(A_1^*N\overline{A}_{12} + \overline{A}_{12}^*NH_1)]y(t) +$$

$$+y^*(t)[\frac{1}{2}S_5]\dot{y}(t) + \dot{y}^*[\frac{1}{2}S_5]^*y(t) + \dot{y}^*(t)[\frac{1}{2}(S_4 + S_4^*)]\dot{y}(t)\}dt < \infty.$$

Заметим, что матрица π порядка $2n \times 2n$ положительно определенная, если $W_{22} > 0$, $W_{11} - W_{12}W_{22}^{-1}W_{12}^* > 0$. По условию теоремы данные соотношения выполнены. Тогда $W(z) = z^*\pi z > 0$, $\forall z$, $z \in R^{2n}$, $W(0) = 0$,

$$\int_0^\infty W(z(t))dt = \int_0^\infty z^*(t)\pi z(t)dt < \infty,$$

$|z(t)| \le a$, $|\dot{z}(t)| \le c$, $t \in I$. Следовательно, выполнены все условия Леммы 2.

Тогда $\lim\limits_{t\to\infty} z(t) = 0$. Отсюда имеем $\lim\limits_{t\to\infty} y(t) = 0$, $\lim\limits_{t\to\infty} \dot{y}(t) = 0$. Так как $x(t) = (P^*)^{-1}y(t)$, $t \in I$, то $\lim\limits_{t\to\infty} x(t) = 0$. Как показано в работе [16] системы (1), (2) и

(3) равносильны. Тогда из (4), имеем $\lim\limits_{t\to\infty}\varphi(\sigma(t)) = H_0\lim\limits_{t\to\infty}\dot{y}(t) - \overline{A}_{11}\lim\limits_{t\to\infty}y(t) = 0$. Отсюда следует $\lim\limits_{t\to\infty}\sigma(t) = \sigma_*$, $\lim\limits_{t\to\infty}\varphi(\sigma(t)) = \varphi(\sigma_*) = 0$. Заметим, что $\lim\limits_{t\to\infty}\dot{\sigma}(t) = (\overline{C} - R\overline{A}_{11})\lim\limits_{t\to\infty}y(t) + RH_0\lim\limits_{t\to\infty}\dot{y}(t) = 0$. Следовательно, $\lim\limits_{t\to\infty}\sigma(t) = \sigma_*$, $\varphi(\sigma(t)) \to \varphi(\sigma_*)$. Итак, $x(t) \to x_* = 0$ при $t \to \infty$, $\sigma(t) \to \sigma_*$ при $t \to \infty$.

Легко убедиться в том, что пара $(x_* = 0, \sigma_*) \in \Lambda$. Теорема доказана.

3.3 Глобальная асимптотическая устойчивость II

Рассмотрим случай, когда

$$\int\limits_{\sigma_i}^{\sigma_i+\Delta_i}\varphi_i(\xi_i)d\xi_i = \overline{\alpha}_i \neq 0, \quad \forall\sigma_i, \quad \sigma_i \in R^1, \quad i = \overline{1,m}. \tag{14}$$

Теорема 3 *Пусть выполнены условия лемм 1–5 и теоремы 1 из [16], и пусть, кроме того функция* $\varphi(\sigma) \in \Phi_0$ *удовлетворяет условию* (14)*. Тогда для любых диагональных матриц* $\tau_1 > 0$, τ_3, $\tau_4 > 0$, $\tau_5 > 0$ *таких, что* $4\tau_4\tau_5 - (\nu\tau_3)(\nu\tau_3) > 0$ *и* $\overline{\alpha} - \nu\overline{\beta} = 0$, *вектор строк* $\alpha \in R^n$, $\beta \in R^n$, $\gamma \in R^n$, *вдоль решения системы* (3) *несобственный интеграл*

$$\begin{gathered}I_7 = \int\limits_0^\infty[\ddot{y}^*(t)E_1\ddot{y}(t) + \dot{y}^*(t)E_2\ddot{y}(t) + y^*(t)E_3\ddot{y}(t) + \dot{y}^*(t)E_4\dot{y}(t) + y^*(t)E_5\dot{y}(t)+ \\ +y^*(t)E_6y(t)]dt \leq \int\limits_0^\infty\frac{d}{dt}[\dot{y}^*F_1\dot{y}(t) + y^*(t)F_2y(t)]dt + \sum_{i=1}^m\int\limits_{\sigma_i(0)}^{\sigma_i(\infty)}\Phi_i(\sigma_i)\tau_{3i}d\sigma_i,\end{gathered} \tag{15}$$

где

$$\begin{gathered}E_1 = -\Lambda_5 - \alpha^*\alpha = H_0^*\tau_1H_0 - \alpha^*\alpha, \quad E_2 = S_2, \quad E_3 = S_3, \quad E_4 = S_4 - H_0^*\tau_2RH_0+ \\ +H_0^*\tau_3RH_0 - H_0^*R^*\tau_5RH_0 - H_0^*\tau_4H_0, \quad E_5 = S_5 - \overline{A}_{11}^*\tau_2RH_0 - (\overline{C} - R\overline{A}_{11})^*\tau_2H_0+ \\ +H_0^*R^*\tau_3\overline{A}_{11} - H_0^*\tau_3(\overline{C} - R\overline{A}_{11}) + H_0^*R^*\tau_5(\overline{C} - R\overline{A}_{11}) + (\overline{C} - R\overline{A}_{11})^*\tau_5RH_0- \\ -2\overline{A}_{11}^*\tau_4H_0, \quad E_6 = S_6 + \overline{A}_{11}^*\tau_2(\overline{C} - R\overline{A}_{11}) - \overline{A}_{11}^*\tau_3(\overline{C} - R\overline{A}_{11}) - (\overline{C} - R\overline{A}_{11})^*\tau_5(\overline{C} - R\overline{A}_{11})- \\ -\overline{A}_{11}^*\tau_4\overline{A}_{11}, \quad F_1 = \overline{A}_{11}^*\tau_1H_0 + H_0^*R^*\mu_1\tau_1H_0 + H_0^*R^*\tau_1\mu_2H_0 + 2\beta^*\alpha, \quad F_2 = (\overline{C} - R\overline{A}_{11})^*\mu_1\tau_1\overline{A}_{11}- \\ -2(\overline{C} - R\overline{A}_{11})^*\mu_1\tau_1\mu_2RH_0 - (\overline{C} - R\overline{A}_{11})^*\tau_1\mu_2\overline{A}_{11} + 2\gamma^*\beta + \overline{A}_{11}^*\tau_3RH_0- \\ -(\overline{C} - R\overline{A}_{11})^*\tau_3H_0 + [H_0^*R^*\tau_5(\overline{C} - R\overline{A}_{11})]^* + (\overline{C} - R\overline{A}_{11})^*\tau_5RH_0 - 2\overline{A}_{11}^*\tau_4H_0.\end{gathered}$$

Доказательство. Доказательство теоремы аналогично доказательству теоремы 1. Несобственный интеграл $I_7 = I_{10} + I_{20} + I_{10}$, где

$$\begin{gathered}I_{40} = \int\limits_0^\infty[-y^*(t)\Gamma_1^*\dot{y}(t) + y^*(t)\Gamma_2y(t) + \dot{y}^*(t)\Gamma_3\dot{y}(t)]dt \leq \sum_{i=1}^m\int\limits_{\sigma_i(0)}^{\sigma_i(\infty)}\Phi_i(\sigma_i)\tau_{3i}d\sigma_i- \\ -\int\limits_0^\infty\frac{d}{dt}[\dot{y}^*\Gamma_1y(t)]dt < \infty,\end{gathered} \tag{16}$$

$\Gamma_1 = (\overline{C} - R\overline{A}_{11})^*\tau_3 H_0 - \overline{A}_{11}^*\tau_3 R H_0 - H_0^* R^*\tau_5(\overline{C} - R\overline{A}_{11}) - (\overline{C} - R\overline{A}_{11})^*\tau_5 R H_0 + 2\overline{A}_{11}^*\tau_4 H_0$, $\Gamma_2 = -\overline{A}_{11}^*\tau_3(\overline{C} - R\overline{A}_{11}) - (\overline{C} - R\overline{A}_{11})^*\tau_5(\overline{C} - R\overline{A}_{11}) - \overline{A}_{11}^*\tau_4\overline{A}_{11}$, $\Gamma_3 = H_0^*\tau_3 R H_0 - H_0^* R^*\tau_5 R H_0 - H_0^*\tau_4 H_0$.

Суммируя несобственные интегралы (7), (8), (16) получим (15). Теорема доказана.

Теорема 4 *Пусть выполнены условия теоремы 3, и пусть, кроме того:*

1) $E_i = 0$, $i = 1, 2, 3$;

2) матрицы $V_{11} = V_{11}^*$, V_{12}, $V_{22} = V_{22}^*$, $N = N^*$ *порядков* $n \times n$, $n \times n$, $n \times n$, $(n-m) \times (n-m)$ *соответственно такие, что*

$$V_{22} > 0, \quad V_{11} - V_{12}V_{22}^{-1}V_{12}^* > 0,$$

где $V_{11} = \frac{1}{2}(E_6 + E_6^*) + \frac{1}{2}(H_1^* N\overline{A}_{12} + \overline{A}_{12}^* N H_1)$, $V_{12} = \frac{1}{2}E_5$, $V_{12}^* = \frac{1}{2}E_5^*$, $V_{22} = \frac{1}{2}(E_4 + E_4^*)$.

Тогда стационарное множество Λ *системы (1), (2) глобально асимптотически устойчиво.*

Доказательство. Доказательство теоремы аналогично доказательству теоремы 2. Если выполнено условие 1) теоремы, то несобственный интеграл

$$I_7 = \int_0^\infty [\dot{y}^*(t)E_4\dot{y}(t) + y^*(t)E_5\dot{y}(t) + y^*(t)E_6 y(t)]dt < \infty.$$

Тогда сумма

$$I_7 + I_6 = \int_0^\infty z^*(t)Vz(t)dt < \infty,$$

где

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^* & V_{22} \end{pmatrix}, \quad z(t) = (y(t), \dot{y}(t)), \quad t \in I.$$

Если выполнено условие 2) теоремы, то матрица $V = V^* > 0$. Далее применяя Лемму 2, получим $\lim_{t\to\infty} z(t) = 0$. Следовательно, $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} \dot{y}(t) = 0$.

Далее, повторяя доказательства теоремы 2, можно убедиться в том, что стационарное множество Λ системы (1), (2) глобально асимптотически устойчиво. Теорема доказана.

3.4 Глобальная асимптотическая устойчивость III

Рассмотрим случай, когда

$$\int_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i)d\xi_i = 0, \quad i = \overline{1,s}, \qquad \int_{\sigma_i}^{\sigma_i+\Delta_i} \varphi_i(\xi_i)d\xi_i = \overline{\alpha}_i \neq 0, \quad i = \overline{s+1,m}. \tag{17}$$

Обозначим через $\varphi^{(1)}(\sigma^{(1)}) = (\varphi_1(\sigma_1), \ldots, \varphi_s(\sigma_s))$, $\varphi^{(2)}(\sigma^{(2)}) = (\varphi_{s+1}(\sigma_{s+1}), \ldots, \varphi_m(\sigma_m))$, где $\sigma^{(1)} = (\sigma_1, \ldots, \sigma_s)$, $\sigma^{(2)} = (\sigma_{s+1}, \ldots, \sigma_m)$. Легко убедиться в том, что

$$\varphi^{(1)}(\sigma^{(1)}) = H_{01}\dot{y}(t) - \overline{A}_{11}^{(1)}y(t), \quad \varphi^{(2)}(\sigma^{(2)}) = H_{02}\dot{y} + \overline{A}_{11}^{(2)}y(t), \ \ t \in I,$$
$$\dot{\sigma}^{(1)}(t) = (\overline{C} - R\overline{A}_{11}^{(1)})y(t) + R_1H_{01}\dot{y}(t), \quad \dot{\sigma}^{(2)}(t) = (\overline{C} - R\overline{A}_{11}^{(2)})y(t) + R_2H_{02}\dot{y}(t), \ \ t \in I, \tag{18}$$

где

$$\overline{C} = \begin{pmatrix} \overline{C}_1 \\ \overline{C}_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & O_{s,m-s} \\ O_{m-s,s} & R_2 \end{pmatrix}, \quad H_0 = \begin{pmatrix} H_{01} \\ H_{02} \end{pmatrix}, \quad \overline{A}_{11} = \begin{pmatrix} \overline{A}_{11}^{(1)} \\ \overline{A}_{11}^{(2)} \end{pmatrix}, \quad \begin{matrix} H_{01} = (I_s, O_{s,n-s}), \\ H_{02} = (I_{m-s}, O_{m-s,n-m+s}), \end{matrix}$$

матрицы $\overline{C}_1, \overline{C}_2, R_1, R_2, H_{01}, H_{02}, \overline{A}_{11}^{(1)}, \overline{A}_{11}^{(2)}$ порядков $s\times n$, $(m-s)\times n$, $s\times s$, $(m-s)\times(m-s)$, $s \times n$, $(m-s)\times n$, $s\times n$, $(m-s)\times n$ соответственно.

Теорема 5 *Пусть выполнены условия лемм 1–5 и теоремы 1 из [16], и пусть, кроме того, функция* $\varphi(\sigma) \in \Phi_0$ *удовлетворяет условию* (17). *Тогда для любых диагональных матриц* $\tau_1 = diag(\tau_{11}, \dots, \tau_{1m})$, $\overline{\tau}_2 = diag(\tau_{21}, \dots, \tau_{2s})$, $\overline{\tau}_3 = diag(\tau_{31}, \dots, \tau_{3m-s})$, $\overline{\tau}_4 = diag(\tau_{41}, \dots, \tau_{4m-s}) > 0$, $\tau_5 = diag(\tau_{51}, \dots, \tau_{5m-s}) > 0$, $4\overline{\tau}_4\overline{\tau}_5 - (\nu\overline{\tau}_3)(\nu\overline{\tau}_3) > 0$, $\nu = diag(\nu_1, \dots, \nu_{m-s})$, $\overline{\alpha} - \nu \cdot \overline{\beta} = 0$, $\overline{\alpha} = diag(\overline{\alpha}_1, \dots, \overline{\alpha}_{m-s})$, $\overline{\beta} = diag(\overline{\beta}_1, \dots, \overline{\beta}_{m-s})$ *и векторов* $\alpha \in R^n$, $\beta \in R^n$, $\gamma \in R^n$ *вдоль решения системы* (3) *несобственный интеграл*

$$I_8 = I_{10} + I_{20} + \overline{I}_{30} + \overline{I}_{40} = \int\limits_0^\infty [\ddot{y}^*(t)D_1\ddot{y}(t) + \dot{y}^*(t)D_2\ddot{y}(t) + y^*(t)_3\ddot{y}(t) + \dot{y}^*(t)D_4\dot{y}(t) +$$
$$+ y^*(t)D_5\dot{y}(t) + y^*(t)D_6y(t)]dt \le \int\limits_0^\infty \frac{d}{dt}[\dot{y}^*(t)F_1\dot{y}(t) + y^*(t)(F_2 - T_{11} - \Gamma_{11})y(t)]dt + \tag{19}$$
$$+ \sum_{i=1}^{s} \int\limits_{\sigma_i(0)}^{\sigma_i(\infty)} \varphi_i(\sigma_i)\overline{\tau}_2 d\sigma_i + \sum_{i=s+1}^{m} \int\limits_{\sigma_i(0)}^{\sigma_i(\infty)} \Phi_i(\sigma_i)\overline{\tau}_{3i} d\sigma_i < \infty,$$

где $D_1 = -\Lambda_5 - \alpha^*\alpha$, $D_2 = \Lambda + 2\alpha^*\beta$, $D_3 = -\Lambda_1 - 2\alpha^*\gamma$, $D_4 = -\Lambda_4 - \beta^*\beta + T_{31} + \Gamma_{31}$, $D_5 = \Lambda_3^* + 2\beta^*\gamma - T_{11}^* - \Gamma_{11}^*$, $D_6 = -\Lambda - \gamma^*\gamma + T_{21} + \Gamma_{21}$.

Здесь несобственные интегралы $\overline{I}_{30}$, $\overline{I}_{40}$ *равны:*

$$\overline{I}_{30} = \int\limits_0^\infty [-y^*(t)T_{11}^*\dot{y}(t) + y^*(t)T_{21}y(t) + \dot{y}^*(t)T_{31}\dot{y}(t)]dt =$$
$$= \sum_{i=1}^{s} \int\limits_{\sigma_i(0)}^{\sigma_i(\infty)} \varphi_i(\sigma_i)\overline{\tau}_2 d\sigma_i - \int\limits_0^\infty \frac{d}{dt}[y^*(t)T_{11}y(t)]dt < \infty, \tag{20}$$

где $T_{11} = -[\overline{A}_{11}^{(1)}]^*\overline{\tau}_2R_1H_{01} + (\overline{C}_1 - R_1\overline{A}_{11}^{(1)})^*\overline{\tau}_2H_{01}$, $T_{21} = -[\overline{A}_{11}^{(1)}]^*\overline{\tau}_2(\overline{C}_1 - R_1\overline{A}_{11}^{(1)})$, $T_{31} = H_{01}^*\overline{\tau}_2R_1H_{01}$, $\overline{\tau_2} = diag(\tau_{21}, \dots, \tau_{2s})$;

$$\overline{I}_{40}=\int\limits_0^\infty[-y^*(t)\Gamma_{11}^*\dot{y}(t)+y^*(t)\Gamma_{21}y(t)+\dot{y}^*(t)\Gamma_{31}\dot{y}(t)]dt=$$
$$=\sum_{i=s+1}^m\int\limits_{\sigma_i(0)}^{\sigma_i(\infty)}\Phi_i(\sigma_i)\overline{\tau}_{3i}d\sigma_i-\int\limits_0^\infty\frac{d}{dt}[y^*(t)\Gamma_{11}y(t)]dt<\infty, \tag{21}$$

где $\Gamma_{11}=-[\overline{A}_{11}^{(2)}]^*\overline{\tau}_3R_2H_{02}+(\overline{C}_2-R_2\overline{A}_{11}^{(2)})^*\overline{\tau}_3H_{02}-(\overline{C}_2-R_2\overline{A}_{11}^{(2)})^*\overline{\tau}_5R_2H_{02}-(\overline{C}_2-R_2\overline{A}_{11}^{(2)})^*\overline{\tau}_5R_2H_{02}+2[\overline{A}_{11}^{(2)}]^*\overline{\tau}_4H_{02}$, $\Gamma_{21}=-[\overline{A}_{11}^{(2)}]^*\overline{\tau}_3(\overline{C}_2-R_2\overline{A}_{11}^{(2)})-(\overline{C}_2-R_2\overline{A}_{11}^{(2)})^*\overline{\tau}_5(\overline{C}_2-R_2\overline{A}_{11}^{(2)})^*\overline{\tau}_4A_{11}^{(2)}$, $\Gamma_{31}=H_{02}^*\overline{\tau}_3R_2H_{02}-H_{02}^*R_2^*\overline{\tau}_5R_2H_{02}-H_{02}^*\overline{\tau}_4H_{02}$, $\overline{\alpha}_i-\nu_i\overline{\beta}_i=0$, $i=\overline{s+1,m}$, $\overline{\tau}_3=diag(\tau_{31},\dots,\tau_{3m-s})$, $\overline{\tau}_4=diag(\tau_{41},\dots,\tau_{4m-s})>0$, $\overline{\tau}_5=diag(\tau_{51},\dots,\tau_{5m-s})>0$, $4\overline{\tau}_4\overline{\tau}_5-(\nu\overline{\tau}_3)(\nu\overline{\tau}_3)>0$, $\nu=diag(\nu_1,\dots,\nu_{m-s})$, $\overline{\alpha}=(\overline{\alpha}_1,\dots,\overline{\alpha}_{m-s})$, $\overline{\beta}=(\overline{\beta}_1,\dots,\overline{\beta}_{m-s})$.

Доказательство. Как следует из результатов [16]: несобственный интеграл

$$\overline{I}_3=\int\limits_0^\infty[\varphi^{(1)}(\sigma^{(1)}(t))]^*\overline{\tau}_2\dot{\sigma}^{(1)}(t)dt=\int\limits_0^\infty[H_{01}\dot{y}(t)-\overline{A}_{11}^{(1)}y(t)]^*\overline{\tau}_2[(\overline{C}_1-R_1\overline{A}_{11}^{(1)})y(t)+$$
$$+R_1H_{01}\dot{y}(t)]dt=\sum_{i=1}^s\int\limits_{\sigma_i(0)}^{\sigma_i(\infty)}\varphi_i(\sigma_i)\overline{\tau}_2d\sigma_i<\infty.$$

Отсюда следует, что несобственный интеграл $\overline{I}_{30}$ определяется по формуле (20).

Аналогичным путем, для случая (17) на основе тождеств (18) имеем

$$\overline{I}_4=\int\limits_0^\infty\{[\varphi^{(2)}(\sigma^{(2)}(t))]^*\overline{\tau}_3\dot{\sigma}^{(2)}(t)-[\varphi^{(2)}(\sigma^{(2)}(t))]^*\overline{\tau}_4[\varphi^{(2)}(\sigma^{(2)}(t))]^*-$$
$$-[\dot{\sigma}^{(2)}(t)]^*\overline{\tau}_5[\dot{\sigma}^{(2)}(t)]^*\}dt\le\sum_{i=s+1}^m\int\limits_0^\infty\Phi_i(\sigma_i(t))\overline{\tau}_{3i}d\sigma_i<\infty,$$

где $\Phi_i(\sigma_i)=\varphi_i(\sigma_i)-\nu_i|\varphi_i(\sigma_i)|$, $i=\overline{s+1,m}$. Следовательно, несобственный интеграл $\overline{I}_{40}$ определяется по формуле (21).

Несобственные интегралы I_{10}, I_{20} определяются формулами (7), (8) соответственно. Суммируя несобственные интегралы I_{10}, I_{20}, $\overline{I}_{30}$, $\overline{I}_{40}$ получим оценку (19). Теорема доказана.

Теорема 6 *Пусть выполнены условия теоремы 5, и пусть, кроме того:*

1) $D_i=0$, $i=1,2,3$;

2) Матрицы $G_{11}=G_{11}^*$, G_{12}, $G_{22}=G_{22}^*$, $N=N^*$ *порядков* $n\times n$, $n\times n$, $n\times n$, $(n-m)\times(n-m)$ *соответственно такие, что*

$$G_{22}=\frac{1}{2}(D_4+D_4^*)>0,\quad G_{12}=\frac{1}{2}D_5,\quad G_{11}=\frac{1}{2}(D_6+D_6^*)+\frac{1}{2}(H_1^*N\overline{A}_{12}+$$

$$+\overline{A}_{12}^*NJ_1), \quad G_{11} - G_{12}G_{22}^{-1}G_{12}^* > 0, \quad G_{12}^* = \frac{1}{2}D_5^*.$$

Тогда стационарное множество Λ *системы* (1), (2) *глобально асимптотически устойчиво.*

Доказательство. Доказательство теоремы аналогично доказательству теоремы 4. Если $D_i = 0$, $i = 1, 2, 3$, то несобственный интеграл

$$I_8 = \int_0^\infty [\dot{y}^*(t)D_4\dot{y}(t) + y^*(t)D_5\dot{y}(t) + y^*(t)D_6y(t)]dt < \infty.$$

Тогда несобственный интеграл (см. (12))

$$I_9 = I_8 + I_6 = \int_0^\infty z^*(t)Gz(t)dt < \infty,$$

где матрица

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{12}^* & G_{22} \end{pmatrix}, \quad z(t) = (y(t), \dot{y}(t)), \quad t \in I.$$

Далее, повторяя доказательства теоремы 4, получим $\lim\limits_{t\to\infty} x(t) = 0$, $\lim\limits_{t\to\infty} \sigma(t) = \sigma_*$, $\varphi(\sigma_*) = 0$. Теорема доказана.

3.5 Пример

Уравнения фазовой системы имеют вид

$$\begin{gathered} \dot{x}_1 = x_1 + x_2 - \varphi_2(\sigma_2), \quad \dot{x}_2 = -2x_1 - 1,03x_2 - 0,03x_3 - 0,75\varphi(\sigma_1), \\ \dot{x}_3 = -0,01x_2 - 1,01x_3 - 0,25\varphi_1(\sigma_1) + \varphi_2(\sigma_2), \\ \dot{\sigma}_1 = x_2 + x_3 + \varphi_1(\sigma_1), \quad \dot{\sigma}_2 = -x_1 + x_2 - x_3 + \varphi_2(\sigma_2), \end{gathered} \tag{22}$$

где

$$\begin{gathered} \varphi(\sigma) \in \Phi_0 = \{\varphi(\sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2)) \in C^1(R^21, R^2) \;/\; \mu_{11} \le \frac{d\varphi_1(\sigma_1)}{d\sigma_1} \le \mu_{21}, \\ \mu_{12} \le \frac{d\varphi_2(\sigma_2)}{d\sigma_2} \le \mu_{22}, \quad \varphi_1(\sigma_1) = \varphi_1(\sigma_1 + \Delta_1), \quad \varphi_2(\sigma_2) = \varphi_2(\sigma_2 + \Delta_2), \\ \forall\sigma_1, \;\; \sigma \in R^1, \quad \forall\sigma_2, \quad \sigma_2 \in R^1\}. \end{gathered} \tag{23}$$

В частности, $\varphi_1(\sigma_1) = \sin\sigma_1$, $\varphi_2(\sigma_2) = \sin\sigma_2 + \gamma$, $\gamma \in (0, 1)$, $\Delta_1 = \Delta_2 = 2\pi$. В этом случае $\int\limits_{\sigma_1}^{\sigma_1+2\pi} \varphi_1(\xi_1)d\xi_1 = 0$, $\int\limits_{\sigma_2}^{\sigma_2+2\pi} \varphi_2(\xi_1)d\xi = \overline{\alpha} \ne 0$.

В векторной форме уравнение (22) запишется так

$$\dot{x} = Ax + B_1\varphi(\sigma), \;\; \dot{\sigma} = Cx + R\varphi(\sigma), \tag{24}$$

где

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1,03 & -0,03 \\ 0 & -0,01 & -1,01 \end{pmatrix}, \quad B = (B_1, B_2) = \begin{pmatrix} 0 & -1 \\ -0,75 & 0 \\ -0,25 & 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ -0,75 \\ -0,25 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0, & 1, & 1 \\ -1 & 1 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 1, & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = (0,1,1), \quad C_2 = (-1,1,-1), \quad \varphi(\sigma) = \begin{pmatrix} \varphi_1(\sigma_1) \\ \varphi_2(\sigma_2) \end{pmatrix}.$$

1. Неособое преобразование. Выберем вектор $\theta_1^* = (\theta_{11}, \theta_{12}, \theta_{13})$ так, чтобы $\theta_1^* B_1 = 1$, $\theta_1^* B_2 = 0$. Вектор $\theta_1^* = (1; -5/3; 1)$. Аналогично, определим вектор $\theta_2^* = (\theta_{21}, \theta_{22}, \theta_{23})$ из условия $\theta_2^* B_1 = 0$, $\theta_2^* B_2 = 1$. Вектор $\theta_2^* = (0; -1/3; 1)$. Наконец, вектор $\theta_3^* = (\theta_{31}, \theta_{32}, \theta_{33})$ выберем так, чтобы $\theta_3^* B_1 = 0$, $\theta_3^* B_2 = 0$. Вектор $\theta_3^* = (1; -1/3; 1)$. Определитель

$$\Gamma(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} <\theta_1, \theta_1> & <\theta_1, \theta_2> & <\theta_1, \theta_3> \\ <\theta_2 \theta_1> & <\theta_2, \theta_2> & <\theta_2, \theta_3> \\ <\theta_3, \theta_1> & <\theta_3, \theta_2> & <\theta_3, \theta_3> \end{vmatrix} = \begin{vmatrix} 43/9 & 14/9 & 23/9 \\ 14/9 & 10/9 & 10/9 \\ 23/9 & 10/9 & 19/9 \end{vmatrix} = \frac{16}{9} > 0.$$

Следовательно, векторы $\theta_1, \theta_2, \theta_3$ линейно независимы. Так как векторы $\theta_1^* A = (13/3; 8, 12/3, -0,96)$, $\theta_2^* A = (2/3; 1/3, -1)$, $\theta_3^* A = (5/3; 4/3, -1)$, то

$$\theta_1^* A = -\frac{5,37}{3}\theta_1^* - \frac{15,88}{3}\theta_2^* + \frac{18,37}{3}\theta_3^*; \quad \theta_1^* A x = -\frac{5,37}{3} y_1 - \frac{15,88}{3} y_2 + \frac{18,37}{3} y_3,$$

где $y_1 = \theta_1^* x$, $y_2^* = \theta_2 x$, $y_3 = \theta_3^* x$. Следовательно,

$$\dot{y}_1 = -\frac{5,37}{3} y_1 - \frac{15,88}{3} y_2 + \frac{18,37}{3} y_3 + \varphi_1(\sigma_1).$$

Аналогичным путем, находим

$$\dot{y}_2 = -\frac{5}{3} y_2 + \frac{2}{3} y_3 + \varphi_2(\sigma_2), \quad \dot{y}_3 = -\frac{3}{4} y_1 - \frac{8}{3} y_2 + \frac{29}{13} y_3.$$

Так как $C_1 = (0,1,1) = -1 \cdot \theta_1^* + 1 \cdot \theta_2^* + 1 \cdot \theta_3^*$, то $C_1 x = -y_1 + y_2 + y_3$, $C_2 = (-1,1,-1) = -\frac{1}{2}\theta_1^* - \frac{1}{2}\theta_3^*$, $C_2 x = -\frac{1}{2} y_1 - \frac{1}{2} y_3$.

Из вышеизложенного следует, что уравнение (24) с неособым преобразованием приводится к виду

$$\begin{aligned} &\dot{y}_1 = -\frac{5,37}{3} y_1 - \frac{15,88}{3} y_2 + \frac{18,37}{3} y_3 + \varphi_1(\sigma_1), \\ &\dot{y}_2 = -\frac{5}{3} y_2 + \frac{2}{3} y_3 + \varphi_2(\sigma_2), \quad \dot{y}_3 = -\frac{3}{4} y_1 - \frac{8}{3} y_2 + \frac{29}{12} y_3, \\ &\dot{\sigma}_1 = -y_1 + y_2 + y_3 + \varphi_1(\sigma_1), \quad \dot{\sigma}_2 = -\frac{1}{2} y_1 - \frac{1}{2} y_3 + \varphi_2(\sigma_2), \\ &\varphi(\sigma) = (\varphi_1(\sigma_1), \varphi_2(\sigma_2)) \in \Phi_0. \end{aligned} \tag{25}$$

Матрица

$$P = \|\theta_1, \theta_2, \theta_3\| = \begin{pmatrix} 1 & 0 & 1 \\ -5/3 & -1/3 & -1/3 \\ 1 & 1 & 1 \end{pmatrix}, \quad |P| = -\frac{4}{3} \neq 0.$$

$$K = P^* = \begin{pmatrix} \theta_1^* \\ \theta_2^* \\ \theta_3^* \end{pmatrix} = \begin{pmatrix} 1 & -5/3 & 1 \\ 0 & -1/3 & 1 \\ 1 & -1/3 & 1 \end{pmatrix}, \quad K^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ -3/4 & 0 & 3/4 \\ -1/4 & 1 & 1/4 \end{pmatrix}.$$

Так как

$$\overline{A} = KAK^{-1} = \begin{pmatrix} -1,79 & -15,88/3 & 18,37/3 \\ 0 & -5/3 & 2/3 \\ -3/4 & -8/3 & 29/12 \end{pmatrix}, \quad \overline{B} = KB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\overline{C} = CK^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1/2 & 0 & -1/2 \end{pmatrix}$$

уравнение (25) запишем в векторной форме

$$\dot{y} = \overline{A}y + \overline{B}\varphi(\sigma), \quad \dot{\sigma} = \overline{C}y + R\varphi(\sigma), \quad \varphi(\sigma) \in \Phi_0. \tag{26}$$

2. Свойства решений. Характеристическое уравнение матрицы A равно

$$\Delta(\lambda) = det(\lambda I_3 - A) = |\lambda I_3 - A| = \lambda^3 + 1,04\lambda^2 + \lambda + 0,98 = 0.$$

Так как все коэффициенты характеристического полинома больше нуля и $1,04 > 0,98$, то матрица A – гурвицева. Тогда, как следует из теоремы 1 [16] верны оценки $|x(t)| \le c_0$, $|\dot{x}(t)| \le c_1$, $|y(t)| \le c_2$, $|\dot{y}(t)| \le c_3$, $|\dot{\sigma}(t)| \le c_4$, $t \in I = [0, \infty)$. Поскольку матрицы A, $\overline{A}$ подобны, то $\lambda_j(A) = \lambda_j(\overline{A})$, $j = 1,2,3$. Следовательно, матрица $\overline{A}$ – гурвицева.

Из (25), (26) имеем

$$\begin{aligned} &\varphi(\sigma(t)) = H_0\dot{y}(t) - \overline{A}_{11}y(t), \quad \dot{\sigma}(t) = (\overline{C} - R\overline{A}_{11})y(t) + RH_0\dot{y}(t), \\ &H_1\dot{y}(t) = \overline{A}_{12}y(t), \quad t \in I, \end{aligned} \tag{27}$$

где

$$\varphi(\sigma(t)) = \begin{pmatrix} \varphi_1(\sigma_1(t)) \\ \varphi_2(\sigma_2(t)) \end{pmatrix}, \quad H_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_1 = (0,0,1), \quad \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} = I_3,$$

$$\overline{A} = \begin{pmatrix} \overline{A}_{11} \\ \overline{A}_{12} \end{pmatrix}, \quad \overline{A}_{11} = \begin{pmatrix} -1,79 & -15,88/3 & 18,37/3 \\ 0 & -5/3 & 2/3 \end{pmatrix}, \quad \overline{A}_{12} = (-3/4, -8/3, 29/12).$$

Из (27) следует, что

$$\begin{aligned} &\varphi_1(\sigma_1(t)) = \frac{5,37}{3}y_1 - \frac{15,88}{3}y_2 + \frac{18,37}{3}y_3 + \dot{y}_1, \quad y_i = y_i(t), \quad i = 1,2,3, \quad t \in I; \\ &\varphi_2(\sigma_2(t)) = \frac{5}{3}y_2 - \frac{2}{3}y_3 + \dot{y}_2, \quad t \in I, \quad \dot{y}_i = \dot{y}_i(t), \quad i = 1,2,3; \\ &\dot{\sigma}_1(t) = \frac{2,37}{3}y_1 - \frac{12,88}{3}y_2 + \frac{21,37}{3}y_3 + \dot{y}_1, \quad t \in I; \\ &\dot{\sigma}_2(t) = -\frac{1}{2}y_1 + \frac{5}{3}y_2 - \frac{7}{6}y_3 + \dot{y}_2, \quad t \in I. \end{aligned} \tag{28}$$

На основе тождеств (28) могут быть вычислены несобственные интегралы I_{10}, I_{20}, I_{30}, I_{40}, $\overline{I}_{30}$, $\overline{I}_{40}$, и установлены условия глобальной асимптотической устойчивости для случаев:

$$\text{а)}\qquad \int\limits_{\sigma_1}^{\sigma_1+\Delta_1} \varphi_1(\xi_1)d\xi_1 = 0; \quad \int\limits_{\sigma_2}^{\sigma_2+\Delta_2} \varphi_2(\xi_2)d\xi_2 = 0;$$

$$\text{б)}\qquad \int\limits_{\sigma_1}^{\sigma_1+\Delta_1} \varphi_1(\xi_1)d\xi_1 = \overline{\alpha}_1; \quad \int\limits_{\sigma_2}^{\sigma_2+\Delta_2} \varphi_2(\xi_2)d\xi_2 = \overline{\alpha}_2, \ \ \overline{\alpha}_1 \neq 0, \ \ \overline{\alpha}_2 \neq 0.$$

$$\text{в)}\qquad \int\limits_{\sigma_1}^{\sigma_1+\Delta_1} \varphi_1(\xi_1)d\xi_1 = 0; \quad \int\limits_{\sigma_2}^{\sigma_2+\Delta_2} \varphi_2(\xi_2)d\xi_2 = \overline{\alpha} \neq 0.$$

Для случаев а), б) $\mu_1 = diag(\mu_{11}, \mu_{12})$, $\mu_2 = diag(\mu_{21}, \mu_{22})$, $\tau_1 = diag(\tau_{11}, \tau_{12}) > 0$, $\tau_2 = diag(\tau_{21}, \tau_{22})$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in R^3$, $\beta = (\beta_1, \beta_2, \beta_3) \in R^3$, $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in R^3$, $\tau_3 = diag(\tau_{31}, \tau_{32})$, $\tau_4 = diag(\tau_{41}, \tau_{42}) > 0$, $\tau_5 = diag(\tau_{51}, \tau_{52})$, $\overline{\alpha} = diag(\overline{\alpha}_1, \overline{\alpha}_2)$, $\nu = diag(\nu_1, \nu_2)$, $\overline{\beta} = diag(\overline{\beta}_1, \overline{\beta}_2)$, $4\tau_4\tau_5 - (\nu\tau_3)(\nu\tau_3) > 0$.

В частности, если $\varphi_1(\sigma_1) = \sin\sigma_1$, $\varphi_2(\sigma_2) = \sin\sigma_2 - \overline{\gamma}$, $\overline{\gamma} \in (0, 1)$ величины

$$\overline{\alpha}_2 = \int\limits_{\sigma_2}^{\sigma_2+2\pi} \varphi_2(\xi_2)d\xi_2 = -2\pi\overline{\gamma}, \quad \overline{\beta}_2 = \int\limits_{\sigma_2}^{\sigma_2+2\pi} |\varphi_2(\xi_2)|d\xi_2 = 4[\overline{\gamma}\arcsin\overline{\gamma} + \sqrt{1-\overline{\gamma}^2}],$$

$$\overline{\nu}_2 = \frac{\overline{\alpha}_2}{\overline{\beta}_2} = \frac{-0,5\pi\overline{\gamma}}{\overline{\gamma}\arcsin\overline{\gamma} + \sqrt{1-\overline{\gamma}^2}}.$$

4 Результаты и обсуждение

Создана общая теория глобальной асимптотической устойчивости многомерных фазовых систем со счетным положением равновесия, основанная на априорном оценивании несобственных интегралов вдоль решения системы.

Основными результатами полученных в данной работе являются:

– установлены ограниченность решений многомерных фазовых систем и их производных первого и второго порядков;

– найдены условия при выполнении которых решение динамических систем со счетным положением равновесия и ее производное первого порядка обладают асимптотическими свойствами;

– получены условия глобальной асимптотической устойчивости многомерных фазовых систем с равными нулю в периоде значениями интегралов от компонентов периодических нелинейностей;

– получены условия глобальной асимптотической устойчивости фазовых систем с не равными нулю в периоде значениями интегралов от составляющих нелинейных периодических функций;

– исследованы асимптотические свойства решений динамических систем со счетным положением равновесия в общем случае, когда часть компонентов нелинейных периодических функции обладают значениями интегралов в периоде равными нулю, а для другой части компонентов значения интегралов в периоде не равными нулю.

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О классе потенциалов с тривиальной монодромией

Ишкин Х.К., Башкирский государственный университет,
г. Уфа, Россия, E-mail: Ishkin62@mail.ru
Ахметшина А.Д., Башкирский государственный университет,
г. Уфа, Россия, E-mail: azipuk@mail.ru

Рассматривается задача описания класса $TM(\Omega, A)$ потенциалов, мероморфных в односвязной области Ω, с множеством полюсов A, удовлетворяющих условию тривиальной монодромии: любое решение соответствующего уравнения Штурма–Лиувилля при всех значениях спектрального параметра не имеет точек ветвления ни в одной точке A. Показано, что в случае конечного A линейное (относительно обычного сложения) пространство $TM(\Omega, A)$ имеет конечную размерность по модулю подпространства $TM_0(\Omega, A)$ функций, голоморфных в Ω и имеющих в точках нули заданной кратности (своей для каждой точки). Тем самым при конечном A получено полное описание $TM(\Omega, A, M)$ в терминах любого конечного набора функций – решений интерполяционной задачи с кратными узлами в точках множества A. Полученный результат обобщает известные результаты о классах потенциалов с тривиальной монодромией на всей плоскости, убывающих на бесконечности (J.J. Duistermaat, F.A. Grünbaum) или растущих не быстрее второй (A.A. Обломков) либо шестой (J. Gibbons, A.P. Veselov) степени. В случае, когда множество A счетно и имеет единственную предельную точку, построен достаточно широкий класс функций, удовлетворяющих условию тривиальной монодромии.
Ключевые слова: спектральная неустойчивость, локализация спектра, уравнение Штурма–Лиувилля, тривиальная монодромия.

On the class of potentials with trivial monodromy

Ishkin Kh.K., Bashkir State University,
Ufa, Russia, E-mail: Ishkin62@mail.ru
Akhmetshina A.D., Bashkir State University,
Ufa, Russia, E-mail: azipuk@mail.ru

We consider the problem of describing the class $TM(\Omega, A)$ potentials meromorphic in a simply connected domain Ω with a set of poles A satisfying the trivial monodromy condition: any solution of the corresponding Sturm–Liouville equation for all values of the spectral parameter has no branch points at any point in A. We have shown that in the case of a finite A the linear (with respect to the usual addition) space $TM(\Omega, A)$ has finite dimension modulo the subspace $TM_0(\Omega, A)$ of functions holomorphic in Ω and having at points A, zeros of a given multiplicity (its own for each point). Thus, for a finite A, a complete description of $TM(\Omega, A, M)$ is obtained in terms of any finite set of functions – solutions of an interpolation problem with multiple nodes at points of the set A. The result obtained summarizes the well-known results on classes of potentials with trivial monodromy on the $\mathbb{C}$, decreasing at infinity (J.J. Duistermaat, F.A. Grünbaum) or growing not faster than the second (A. Oblomkov) or the sixth (J.Gibbons, A.P. Veselov) of degree. In the case when the set A is countable and has a unique limit point, a sufficiently wide class of functions that satisfy the condition of trivial monodromy is constructed.
Key words: spectral instability, spectrum localization, Sturm–Liouville equation, trivial monodromy.

1 Введение. Обзор литературы

Методы теории функций комплексной переменной (ТФКП) представляют собой естественное и эффективное средство для решения самых разнообразных задач спектральной теории операторов. Несамосопряженные дифференциальные операторы – тот класс операторов, спектральный анализ которых без привлечения методов ТФКП просто невозможен. Наиболее яркий тому пример — теория регуляризованных следов задач, порожденных обыкновенными дифференциальными выражениями на конечном отрезке: в работах [1, 2] было установлено, что получение формул следов таких задач не связано, вообще говоря, с их операторной трактовкой, а носит чисто теоретико-функциональный характер и сводится к исследованию регуляризованных сумм корней некоторого класса целых функций. Этому классу, в частности, принадлежит характеристический определитель спектральной задачи для системы

$$Y' = \lambda AY,\ x \in [0, 1], \tag{1}$$

где A– кусочно-постоянная невырожденная матрица $n \times n$ [3,4]. Как отмечено в [5], если собственные числа $d_1, \ldots, d_n$ матрицы A^{-1} удовлетворяют известным [6, 7] условиям Биркгофа–Тамаркина $\arg(d_i - d_j) = \mathrm{const},\ i, j = \overline{1, n}$, то указанная задача сводится к спектральной задаче для некоторого оператора, близкого к самосопряженному, то есть представимого в виде относительно компактного возмущения самосопряженного оператора с дискретным спектром. Согласно известной теореме Келдыша [8] операторы, близкие к самосопряженным обладают свойством спектральной устойчивости: при малых возмущениях сохраняются и асимптотика спектра и полнота системы корневых векторов. Если оператор T не близок к самосопряженному, то, как правило, спектрально неустойчив: резольвентная норма $\|(T - \lambda)^{-1}\|$ может быть большой и при λ, далеких от спектра (см. [3, 5, 9–14] и имеющиеся там ссылки). Поэтому в случае, когда T – не близкий к самосопряженному дифференциальный оператор, для исследования спектра приходится накладывать на коэффициенты соответствующего дифференциального выражения дополнительное (гораздо более жесткое по сравнению с близким к самосопряженному случаем) условие голоморфности в некоторой окрестности соответствующего промежутка [13, 15–20]. Система (1) в случае

$$A = \begin{pmatrix} 0 & 1 \\ q & 0 \end{pmatrix},$$

где q – достаточно гладкая на отрезке $[0, 1]$ функция, подстановкой Лиувилля сводится к оператору Штурма–Лиувилля на некоторой гладкой кривой γ (точное определение будет дано ниже в п. 2). В работах [21, 22] одного из авторов показано, что спектр этого оператора локализуется около одного луча тогда и только тогда, когда потенциал допускает мероморфное продолжение в некоторую окрестность Ω кривой γ, удовлетворяющее условию тривиальной монодромии, то есть каждое решение уравнения

$$-y'' + qy = \lambda y,\ z \in \Omega, \tag{2}$$

при каждом $\lambda \in \mathbb{C}$ также мероморфно в области Ω. Условие тривиальной монодромии хорошо известно [23]: уравнение (2) имеет тривиальную монодромию в области Ω тогда и только тогда, когда для любого полюса $a \in \Omega$ функции q найдется ее окрестность U, такая, что

$$q(z) = \frac{m(m+1)}{(z-a)^2} + \sum_{k=0}^{m} c_k (z-a)^{2k} + (z-a)^{2m+1} r(z),\ z \in U \backslash \{a\}, \tag{3}$$

где $m \in \mathbb{N}, c_0 \ldots, c_m$ – некоторые числа, функция r голоморфна в U.

Пусть $A = \{a_k\}_{k=1}^N$ $(N \leq \infty)$ – множество точек Ω, которые (при $N = \infty$) могут скапливаться только к границе Ω. Далее пусть $M = \{m_k \in \mathbb{N}, k = \overline{1, N}\}$. Обозначим через $TM(\Omega, A, M)$ множество функций, голоморфных в $\Omega \backslash A$ и удовлетворяющих в каждой точке a_k условию (3) с $m = m_k$. То, что при конечном N множество $TM(\Omega, A, M)$ не пусто, следует из результатов цитированной выше работы Дюйстермаата и Грюнбаума [23]: любой потенциал, полученный из произвольной голоморфной в области Ω функции с помощью конечного числа преобразований Дарбу (см. ниже п. 2.2 и [24]), принадлежит $TM(\Omega, A, M)$ при некоторых конечных A, M. В этой же работе показано, что при $\Omega = \mathbb{C}$ класс убывающих на бесконечности потенциалов с тривиальной монодромией совпадает с потенциалами, полученными из потенциала q конечным числом преобразований Дарбу. В работе [25]) этот результат был распространен на класс рациональных потенциалов с квадратичным ростом на бесконечности. Но уже для рациональных потенциалов, растущих на бесконечности как z^6, результаты работ [23,25] оказались неверными [26]. Как отмечено в работе [26], задача описания классов $TM(\Omega, A, M)$ вряд ли выполнима. В связи с этим возникает вопрос: при каких Ω, A, M можно получить описание классов $TM(\Omega, A, M)$?

В предлагаемой статье получено полное описание $TM(\Omega, A, M)$ в случае произвольной односвязной области и произвольных конечных A, M. Оказалось, в случае конечных A, M линейное (относительно обычного сложения) пространство $TM(\Omega, A, M)$ имеет конечную размерность по модулю [27, гл. IV, § 14, п. 6] подпространства $TM_0(\Omega, A, M)$ функций, голоморфных в Ω и имеющих в точках a_k нуль m_k-го порядка (Теорема 3). Таким образом, при $N < \infty$ множество $TM(\Omega, A, M)$ допускает полное описание в терминах любого конечного набора функций – решений интерполяционной задачи с кратными узлами в A.

В случае, когда $N = \infty$, в общей ситуации получить такое же полное описание множества $TM(\Omega, A, M)$, по-видимому, невозможно. Чтобы получить какую-то информацию о $TM(\Omega, A, M)$, нужно накладывать дополнительные требования на поведение точек a_k вблизи границы Ω. Так, в предположении, что A имеет единственную предельную точку, удается доказать существование достаточно широкого класса функций из $TM(\Omega, A, M)$ (Теорема 4).

2 Предварительные сведения

2.1 Оператор Штурма–Лиувилля на кривой

Пусть γ – кривая с параметризацией $z(x) = x + is(x)$, $x \in [0, 1]$, где функция s непрерывно дифференцируема, s' не убывает и $s(0) = s(1) = 0, s'(0) < 0 < s'(1)$. Обозначим $\alpha_0 = \operatorname{arctg} s'(0)$, $\alpha_1 = \operatorname{arctg} s'(1)$. Тогда

$$-\pi/2 < \alpha_0 < 0 < \alpha_1 < \pi/2. \tag{4}$$

Пусть функция y абсолютно непрерывна на кривой γ (относительно меры $|dz|$). Функцию

$$y'_\gamma(z) := \lim_{\gamma \ni \zeta \to z} \frac{y(\zeta) - y(z)}{\zeta - z},$$

определенную почти всюду на γ, будем называть *производной вдоль* γ. Аналогично определяем $y''_\gamma(z)$ и т.д. (в предположении что эти объекты существуют). Всюду далее, если не возникает путаницы, значок γ в $y^{(n)}_\gamma$ будем опускать.

Определение 1 *Пусть $q \in L^1(\gamma)$. Оператором Штурма–Лиувилля на кривой γ будем называть оператор L_γ, действующий в пространстве $L^2(\gamma)$ по правилу*

$$\begin{aligned} D(L_\gamma) &= \{y \in L^2(\gamma) :\ y' \in AC(\gamma), -y'' + qy \in L^2(\gamma),\ y(0) = y(1) = 0\}, \\ L_\gamma y &= -y'' + qy,\ y \in D(L_\gamma). \end{aligned}$$

Точно так же, как в случае $\gamma = [0, 1]$ (см., например, [27, § 17, теорема 1]), доказывается, что оператор L_γ плотно определен. Отсюда, поскольку спектр L_γ дискретен [29, лемма 2], то оператор L_γ замкнут. Используя условие (4), интегрированием по частям выражения

$$\int_\gamma (-y''(z) + qy(z))\overline{y(z)}dz$$

легко показать, что

a) оператор L_γ – m-секториален [28, гл. V, § 3, п. 10],

b) за исключением конечного числа все собственные значения оператора L_γ лежат в угле $-2\alpha_1 < \arg\lambda < -2\alpha_0$.

Приведем менее тривиальные факты о спектре оператора L_γ. Пусть $\{\lambda_k^2\}_{k=1}^{\infty}$ ($\mathrm{Re}\lambda_k \geq 0$) — собственные значения L_γ, пронумерованные в порядке возрастания их модулей с учетом их алгебраических кратностей.

Теорема 1 ([29]) *Если существует*

$$\lim_{k\to\infty} \arg(\lambda_k) = \alpha,$$

то $\alpha = 0$ и справедлива асимптотическая формула

$$\lambda_k \sim \pi k, \quad k \to \infty;$$

Определение 2 *Пусть $n(r, \zeta, \theta)$ — число λ_k в секторе $\{\mu : |\mu| < r, \zeta < \arg\mu < \theta\}$. Будем говорить, что спектр оператора L_γ локализован около луча $\arg\lambda = 0$ тогда и только тогда, когда функция*

$$\Delta(\theta) = \lim_{r\to+\infty} \frac{n(r, -\pi/2, \theta)}{r}$$

имеет вид

$$\Delta(\theta) = \begin{cases} 0, & \theta \in (-\pi/2, 0), \\ 1/\pi, & \theta \in (0, \pi/2). \end{cases}$$

Теорема 2 ([22]) *Пусть функция q суммируема на γ. Тогда для того, чтобы спектр оператора L_γ был локализован около луча $\arg\lambda = 0$ необходимо и достаточно, чтобы*

(i) *функция q допускала мероморфное продолжение с кривой γ в область Ω,*

(ii) *каждый полюс q удовлетворял условию тривиальной монодромии (3).*

2.2 Преобразование Дарбу

Рассмотрим дифференциальное выражение $L_0 = -\partial^2 + q_0$, где $\partial = d/dz$, функция q_0 голоморфна в односвязной области Ω. Если f — некоторое решение уравнения Риккати $f' + f^2 = q_0 - \lambda_0$ при некотором $\lambda_0 \in \mathbb{C}$, то выражение L_0 допускает факторизацию: $L_0 = Q^*Q + \lambda_0$, где $Q = -\partial + f,\ Q^* = \partial + f$. Положим $L_1 := QQ^* = -\partial^2 + q_1$, где

$$q_1 = q_0 - 2f' - \lambda_0. \tag{5}$$

Если $f = \varphi_0'/\varphi_0$, то $Q\varphi_0 = 0$, следовательно, $L_0\varphi = \lambda_0\varphi$. Выражение L_1 и соответствующий потенциал $D(q_0) := q_1 = q_0 - 2(\ln\varphi)'' - \lambda_0$ называют *преобразованием Дарбу* [24] выражения L_0 (соответственно потенциала q_0) на уровне φ_0. Поскольку для любого (голоморфного в Ω) решения ψ уравнения $L_0\psi = \mu\psi$ (мероморфная в Ω) функция $\chi = Q\psi$ является решением уравнения $L_1 u = (\mu - \lambda_0)u$, то потенциал q_1 имеет тривиальную монодромию в Ω. Ясно, что то же самое верно и для $D_n(q_0)$ — результата n итераций преобразований Дарбу на некоторых уровнях $\varphi_0, \varphi_1, \dots, \varphi_{n-1}$.

3 Построение потенциалов с тривиальной монодромией

3.1 Случай $N < \infty$

Теорема 3 *Пусть* $A = \{a_1, \dots, a_N\}, M = \{m_1, \dots, m_N\}$. *Функция* $q \in TM\,(\Omega, A, M)$ *тогда и только тогда, когда для* q *справедливо представление*

$$q(z) = \sum_{k=1}^{N} \frac{m_k(m_k+1)}{(z-a_k)^2} + P_0(z) + P_1(z)r(z), \tag{6}$$

где

$$P_1(z) = \prod_{i=1}^{N}(z-a_i)^{2m_i}, \tag{7}$$

$$P_0(z) = \sum_{i=1}^{N}\sum_{j=0}^{m_i} c_{ij}p_{ij}(z), \tag{8}$$

$$c_{ij} = \sum_{\nu\neq i, 1\le\nu\le N} \frac{(2j+1)!m_\nu(m_\nu+1)}{(a_i-a_\nu)^{2j+3}}. \tag{9}$$

Здесь r *– произвольная функция, голоморфная в области* Ω, p_{ij} *– многочлены, удовлетворяющие условиям интерполяции*

$$p_{ij}^{(2s-1)}(a_k) = \delta_{ki}\delta_{sj},\ k, i = \overline{1,m}, s, j = \overline{1,m_k}, \tag{10}$$

δ_{ij} *– символы Кронекера.*

Доказательство. Пусть $A = \{a_k\}_1^N, M = \{m_k\}_1^N$. Согласно (3) функция $q \in TM(\Omega, A, M)$ тогда и только тогда, когда

$$q(z) = \tilde{q} + \sum_{k=1}^{n} \frac{m_k(m_k+1)}{(z-z_k)^2}, \tag{11}$$

где функция $\tilde{q}$ голоморфна в области Ω и должна удовлетворять условиям

$$\tilde{q}^{(2j-1)}(a_i) \;=\; c_{ij}, i = \overline{1,N}, j = \overline{1,m_i}. \tag{12}$$

Легко проверить, что равенства (3) и (11) равносильны равенствам (6) – (9), где p_{ij} – голоморфные в области Ω функции, удовлетворяющие условиям (10). Покажем, что их можно взять в виде многочленов:

$$p_{ij} \;=\; \frac{(z-a_i)^{2j-1}}{(2j-1)!}\psi_i\chi_{ij}, \tag{13}$$

$$\psi_i \;=\; \frac{\prod\limits_{k\neq i}(z-a_k)^{2m_k}}{\prod\limits_{k\neq i}(a_i-a_k)^{2m_k}}, \tag{14}$$

$$\chi_{ij} \;=\; 1+\sum_{\nu=1}^{m_i-j} b_{ij\nu}\frac{(z-a_i)^{2\nu}}{(2\nu)!}, \tag{15}$$

где $b_{ij\nu}(i,j = \overline{1,N}, \nu = \overline{1,m_i-j}$ – некоторые числа. Независимо от этих чисел условия (10) при всех i,k,j,s, кроме $k = i, s = \overline{j+1,m_i}$, выполняются. Подставляя выражения (13) – (15) в (10) при каждом $i = \overline{1,N}, j = \overline{1,m_i}$, получим систему уравнений для чисел $b_{ij\nu}(\nu = \overline{1,m_i-j})$

$$\begin{array}{rcccl} & & b_{ij1} & = & -\psi_i''(a_i),\\ b_{ij2} & + & d_{ij2,m_i-j}b_{ij1} & = & -\psi_i^{(4)}(a_i),\\ & \dots\dots\dots\dots & & & \\ b_{ij,m_i-j} + d_{ij,m_i-j,2}b_{ij,m_i-j-1} & +\dots+ & d_{ij,m_i-j,m_i-j}b_{ij1} & = & -\psi_i^{(2(m_i-j))}(a_i), \end{array}$$

где

$$d_{ij\mu\nu} = \frac{(2\mu)!}{(2\nu)!(2(\mu-\nu)!}\psi^{(2\nu)}(a_i).$$

Эта система разрешима (ее матрица треугольная с определителем 1), откуда и следует утверждение теоремы.

3.2 Случай $n = \infty$

Пусть $A = \{a_k\}_{k=1}^{\infty}, M = \{m_k\}_{k=1}^{\infty}$ и пусть A имеет 1 предельную точку на границе области Ω. Без ограничения общности можно считать, что эта точка 0. Далее, поскольку в области $\Omega' = \Omega\backslash\{|z| < \varepsilon\}$ функция q имеет конечное число полюсов, то вид функции q в Ω' описывается теоремой 3. Поэтому интерес представляет вид q вблизи 0, следовательно, можно считать, что $\Omega = \mathbb{C}\backslash\{0\}$.

Пусть a_k пронумерованы в порядке убывания модулей. Тогда при каждом $N \in \mathbb{N}$ имеет место формула вида (11), где $\tilde{q} = q_N$ голоморфна вне круга $\{|z| < |a_{N+1}\}$. Конечно, такое представление никакой информации о структуре функции q не дает. Более того, отсюда вовсе не следует существование функции, удовлетворяющей условию тривиальной монодромии в бесконечном числе полюсов. Справедлива

Теорема 4 *Для произвольной последовательности натуральных чисел* $\{m_k\}_1^\infty$ *и набора чисел* $\nu_{ij}(i = 1, 2, \ldots \quad j = -2, -1, \ldots, m_i - 1)$ *существует функция* q, *удовлетворяющая следующим условиям:*

(a) q *голоморфна в области* $\Omega = \mathbb{C} \setminus \{0, a_1, a_2, \ldots\}$;

(b) $\forall\ i \in \mathbb{N}$:

$$q(z) = \sum_{s=-2}^{m_i-1} \nu_{is}(z - a_i)^s + O\left((z - a_i)^{m_i}\right), \quad z \to a_i. \tag{16}$$

Доказательство. Рассмотрим каноническое произведение

$$\begin{aligned} F(z) &= \prod_{k=1}^{\infty} \left(1 - \frac{z_k}{z}\right)^{m_k} e^{P_k(z)},\ z \neq 0, \\ P_k(z) &= m_k \sum_{i=1}^{n_k} \frac{1}{i} \left(\frac{z_k}{z}\right)^i, \end{aligned}$$

где числа n_k выбраны таким образом, чтобы бесконечное произведение сходилось равномерно в любом кольце $G_r = \{|z| \geq r > 0\}$. Ясно, что такой выбор существует. Например, можно взять $n_k = k$.

Очевидно, функция F голоморфна в $\mathbb{C}\setminus\{0\}$. Вблизи a_i функция F имеет разложение:

$$F(z) = (z - a_i)^{m_i} \frac{b_i}{a_i^{m_i}} \left[1 + f_{i1}(z - a_i) + \cdots + f_{i,m_i+1} + O\left((z - a_i)^{m_i+2}\right)\right], z \to a_i, \tag{17}$$

где

$$\begin{aligned} b_i &= \prod_{k \neq i} \left(1 - \frac{a_k}{a_i}\right)^{m_k} e^{P_k(a_i)},\ f_{ik} = \frac{f_i^{(k)}(a_i)}{k!}, \\ f_i(z) &= \frac{a_i^{m_i}}{b_i} \prod_{k \neq i} \left(1 - \frac{z_k}{z}\right)^{m_k} e^{P_k(a_i)}. \end{aligned}$$

Функцию q будем искать в виде

$$q(z) = F(z)U(z),$$

где

$$U(z) = \sum_{j=1}^{\infty} \frac{z_j^{m_j}}{b_j} \left(\frac{U_{j0}}{(z - z_j)^{m_j+2}} + \cdots + \frac{U_{j,m_j+1}}{z - z_j} - Q_j(z)\right), \tag{18}$$

$Q_j(z)$ – многочлены относительно $\dfrac{1}{z}$, обеспечивающие сходимость ряда (18) в области Ω.

Из (17) и (18) следует, что функция q удовлетворяет условию (b) тогда и только тогда, когда числа $U_{i0}, \ldots, U_{i,m_i+1}$ – решения (треугольной) системы уравнений:

$$\begin{array}{ccccccc} & & & & U_{i0} & = & \nu_{i0}, \\ & & U_{i1} & + & f_{i2}U_{i0} & = & \nu_{i_1}, \\ & & \ldots & \ldots & \ldots & & \\ U_{i,m_i+1} & + & f_{i1}U_{im_i} & + & \ldots \quad +f_{im_i+1}U_{i0} & = & \nu_{im_i+1}, \end{array} \tag{19}$$

Решая систему (19), найдём $U_{ik},\ k = \overline{0,\, m_i + 1}$.

Укажем теперь выбор $Q_j(z)$. Зафиксируем $r > 0$ и выберем $J_r \in \mathbb{N}$ так, чтобы $|a_j| < \dfrac{r}{2}$ при $j \geq J_r$. Функция

$$U_j(z) = \sum_{k=0}^{m_j+1} \frac{U_{jk}}{(z - a_j)^{m_j+2-k}}$$

голоморфна в области $|z| > |a_j|$. Поэтому

$$U_j(z) = \sum_{s=0}^{\infty} u_{js} z^{-s},\ \ |z| > |a_j|. \tag{20}$$

В силу выбора J_r, при любом $j \geq J_r$ ряд (20) равномерно сходится в области $G_r = \{|z| \geq r\}$. Следовательно, для каждого $j \geq J_r$ найдётся натуральное число n_j, такое, что

$$\left| U_j(z) - \sum_{s=0}^{n_j} u_{js} z^{-s} \right| < \varepsilon_j \frac{|b_j|}{|a_j|^{m_j}}, \qquad z \in G_r, \tag{21}$$

где числа ε_j выбраны так, что ряд $\sum \varepsilon_j$ сходится. Положим

$$Q_j(z) = \sum_{s=0}^{n_j} u_{js} z^{-s}.$$

Тогда ряд

$$W_r(z) = \sum_{j=J_r}^{\infty} \frac{z_j^{m_j}}{b_j} \left(U_j(z) - Q_j(z) \right)$$

сходится равномерно в области G_r. Теорема доказана.

4 Заключение

Рассмотрена задача описания класса потенциалов, мероморфных в односвязной области Ω, удовлетворяющих условию тривиальной монодромии в области Ω. Если $A = \{a_k \in \Omega,\ \ k = \overline{1, N}\}, M = \{m_k \in \mathbb{N},\ \ k = \overline{1, N}\}$ $(N \in \mathbb{N})$, то $TM(\Omega, A, M)$ – линейное (относительно обычного сложения) пространство потенциалов с множеством полюсов A, удовлетворяющих условию (3) в каждой точке a_k с $m = m_k$, имеет конечную размерность по модулю подпространства $TM_0(\Omega, A)$ функций, голоморфных в Ω и имеющих в точках нули кратности m_k. Тем самым при конечном A получено полное описание $TM(\Omega, A, M)$ в терминах любого конечного набора функций – решений интерполяционной задачи с кратными узлами в точках множества A. Полученный результат обобщает известные результаты Дюйстермаата, Грюнбаума [23], Обломкова [25] и Гиббонса, Веселова [?] о классах потенциалов с тривиальной монодромией на всей плоскости, исчезающих на бесконечности или имеющих степенной рост на бесконечности. В случае, когда множество A счетно и имеет единственную предельную точку, построен достаточно широкий класс функций, удовлетворяющих условию тривиальной монодромии.

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2-бөлім Раздел 2 Section 2

Қолданылмалы математика Прикладная математика Applied Mathematics

IRSTI 27.35.29

HFD method for large eddy simulation of MHD turbulence decay

Abdibekova A.U., al-Farabi Kazakh National University,
Almaty, Kazakhstan, +77029299933, E-mail: a.aigerim@gmail.com
Zhakebayev D.B., al-Farabi Kazakh National University,
Almaty, Kazakhstan, +77017537477, E-mail: daurjaz@mail.ru

This work deals with the modelling of the Magnetohydrodynamic (MHD) turbulence decay by hybrid finite-difference method (HFDM) combining two different numerical methods: finite-difference and spectral methods. The numerical algorithm of hybrid method solves the Navier-Stokes equations and equation for magnetic field by a finite-difference method in combination with cyclic penta-diagonal matrix, which yields fourth-order accuracy in space and second-order accuracy in time. The pressure Poisson equation is solved by the spectral method. For validation of the developed algorithm the classical problem of the 3-D Taylor and Green vortex flow is considered without considering the magnetic field, and the simulated time-dependent turbulence characteristics of this flow were found to be in excellent agreement with the corresponding analytical solution valid for short times. We also demonstrate that the developed efficient numerical algorithm can be used to simulate the magnetohydrodynamic turbulence decay at different magnetic Reynolds numbers.
Key words: Magnetohydrodynamics, Taylor-Green vortex problem, hybrid finite difference method, spectral method, turbulence decay.

МГД турбуленттіліктің азғындауын үлкен құйындар әдіспен модельдеу үшін гибридті ақырлы-айырымдылық әдісі

Абдибекова А.У., әл-Фараби атындағы Қазақ ұлттық университеті,
Алматы, Қазахстан, +77029299933, email - a.aigerim@gmail.com
Жакебаев Д.Б., әл-Фараби атындағы Қазақ ұлттық университеті,
Алматы, Казахстан, +77017537477, E-mail: daurjaz@mail.ru

Бұл мақала ақырлы айырымдылық және спектрлік екі сандық әдістерді біріктіретін гибридті ақырлы-айырымдылық әдіспен (ГААӘ) магнитогидродинамикалық (МГД) турбуленттіліктің азғындауын моделдеуіне арналған. Кеңістікте төртінші реттік және уақыт бойынша үшінші реттік дәлдігін беретін бес-диагональды циклдық матрицамен ақырлы –айырымдылық әдіс көмегімен Навье-Стокс теңдеуінің және магнит өріс теңдеуінің шешімдерінің негізінде гибрид әдісінің сандық алгоритмі құрылған. Қысымға арналған Пуассон теңдеуі спектрлік әдіспен шешіледі. Дамытылған алгоритмді тексеру үшін магниттік өрісті ескермейтін Тэйлор және Грин үш өлшемді құйынды ағынның классикалық мәселесін қарастырамыз, және модельдеу арқылы алынған турбулентті сипаттамалары қысқа мерзімді интервалдағы аналитикалық шешімнің нәтижелерімен жақсы келісім береді. Әр түрлі Рейнольдс сандарында магнитогидродинамикалық турбуленттіліктің азғындауын модельдеу үшін дамыған тиімді сандық алгоритм қолданылуы мүмкін.
Түйін сөздер: Магнитогидродинамика, Тейлор-Грин құйындылық мәселесі, соңғы айырымдық гибридті әдіс, спектральдық әдіс, турбуленттіліктің азғындауы.

Метод крупных вихрей для моделирования вырождения МГД турбулентности конечно-разностным гибридным методом

Абдибекова А.У., Казахский национальный университет имени аль-Фараби,
Алматы, Казахстан, +77029299933, email - a.aigerim@gmail.com
Жакебаев Д.Б., Казахский национальный университет имени аль-Фараби,
Алматы, Казахстан, +77017537477, E-mail: daurjaz@mail.ru

Данная работа посвящена моделированию вырождении магнитогидродинамической (МГД) турбулентности конечно-разностным гибридным методом (КРГМ), сочетающейся из двух различных численных методов: конечно-разностный и спектральный. Разработан численный алгоритм гибридного метода на основе решения уравнения Навье-Стокса и уравнения для магнитного поля конечно-разностным методом в сочетании с циклической пятидиагональной матрицей, которая дает точность четвертого порядка по пространству и точность третьего порядка по времени. Уравнение Пуассона для давление решается спектральным методом. Для валидации разработанного алгоритма рассматривается классическая задача трехмерного вихревого потока Тейлора и Грина без учета магнитного поля, и полученные турбулентные характеристики при моделировании имеют отличные согласование с результатами аналитического решения на краткосрочном отрезке времени. Также показано, что разработанный эффективный численный алгоритм может быть использован для моделирования вырождения магнитогидродинамической турбулентности при различных числах Рейнольдса.
Ключевые слова: Магнитогидродинамика, вихревая задача Тейлора-Грина, конечно-разностный гибридный метод, спектральный метод, вырождение турбулентности.

1 Introduction

In the study of turbulent flows of particular interest is the simulation of cascade processes of turbulent energy transmission, large-scale and small-scale vorticity, and various turbulent laws are closely interacting with each other. Cascade processes determine the internal structure of flows and the mechanism of turbulent dissipation. A lot of work was devoted to the study and description of cascade turbulence models [15], [21] So far, cascade models are mainly used for the study of isotropic turbulence, but their capabilities are not limited. Therefore, it is very important to build cascade models and study with their help the properties of such complex turbulent flows as magnetohydrodynamic (MHD) turbulence.

2 Literature review

The problem of the magnetic field influence on turbulent flows was first raised by [2], who provided basic equations and an analytical solution for the movement of an electrically conducting fluid. The first numerical study of magnetohydrodynamic turbulence problem of the first type conducted by [19] at the magnetic number$Re_m << 1$. The numerical experiment of Schumann was the reflection of the idea of [16], who researched a homogeneous isotropic ow influenced by an applied external magnetic field. The modeling outlined in the publications of these scientists is performed using a spectral method, which is used as the basis for presenting a quantitative description of magnetic damping, the emergence of anisotropy, and the dependency of the results on the presence or the absence of a non-linear summand in the Navier-Stokes equation. The low performance of computing machines at that time did not permit the full solution of this problem. Later, a similar problem was researched by [9] and later by [24]. These authors presented the results of direct numerical modeling of large-scale structures in a periodic magnetic field, which reflected a change in the turbulence statistical parameters as a result of an imposed magnetic field influence. The contribution of these scientists in this area of expertise is determined by proving that the behavior of two- and three-dimensional structures varies substantially. A similar result was obtained by [22] in examining locally isotropic structures by the method of large eddies. The process of the magnetic field influence on a developed turbulence was examined by [7],[14], and [14] demonstrated the possibility of using the quasi-stationary approximation for the solution of the second type problem and suggested to use quasi-linear approximations to solve the problem at $Re_m = 20$. The aim of this study is to study MHD turbulence flows that are weakly induced by a homogeneous external magnetic field by adapting the existing finite-difference and spectral methods to this particular problem.

For validation of the developed algorithm the classical problem of the 3-D Taylor and Green vortex flow is considered, and the simulated time-dependent turbulence characteristics of this flow were found to be in excellent agreement with the corresponding analytical solution valid for short times. The classical problem proposed by Taylor and Green [21] who considered a possibility of solving the Navier-Stokes equations analytically by a method for successive approximations, in order to describe three-dimensional turbulence evolution (specifically energy cascade and viscous dissipation) over time, with the resulting flow now known as the Taylor-Green vortex flow. Their work was motivated by the decay of three-dimensional turbulent flow produced in a wind tunnel, a fundamental process in turbulent flow, due to the grinding down of eddies, produced by nonlinearity of the Navier-Stokes equations. In their work the kinetic energy and its dissipation rate were determined analytically.

Taylor and Green's original analytical investigation is rigorous only for short times. To extend the understanding of the 3D Taylor-Green vortex flow, Brachet *et al* [5] solved the Taylor- Green vortex problem by two methods: numerical solution using the spectral method and power-series analysis in time. The resulting average kinetic energy and energy spectra at different flow Reynolds numbers were presented and compared. Later, in [6] three dimensional Navier-Stockes equations were numerically integrated with the periodic Taylor-Green initial condition. In this direct numerical simulation study the slope of energy spectrum was compared with Kolmogorov's $-5/3$ slope in the inertial subrange. Moreover, the compressible Navier-Stokes equations have also been applied to the Taylor-Green vortex problem using large-eddy simulation in [8] at different grid resolutions, and the time evolutions of the kinetic energy and its dissipation rate were compared at different grid resolutions.

3 Materials and methods

To evaluate the MHD turbulence decay is necessary to numerically simulate the change of all physical parameters over time at different magnetic Reynolds number. This work is devoted to study of self-excitation of magnetic field and the motion of the conducting fluid at the same time taking into account acting forces. The idea is to specify in the phase space of initial conditions for the velocity field and magnetic field, which satisfy the condition of continuity [23]. Given initial condition with the phase space is translated into physical space using a Fourier transform. The obtained of velocity field and magnetic field are used as initial conditions for the filtered MHD equations. Further is solved the unsteady three-dimensional equation of magnetohydrodynamics to simulate MHD turbulence decay.

3.1 Statement of the problem

The numerical modeling of MHD turbulence decay based on the large eddy simulation method depending on the conductive properties of the incompressible fluid is reviewed. The numerical modeling of the problem is performed based on solving non-stationary filtered magnetic hydrodynamics equations in conjunction with the continuity equation in the Cartesian coordinate system in a non-dimensional form:

$$\begin{cases} \frac{\partial(\bar{u}_i)}{\partial t} + \frac{\partial(\bar{u}_i\bar{u}_j)}{\partial x_j} = -\frac{\partial(\bar{p})}{\partial x_i} + \frac{1}{Re}\frac{\partial}{\partial x_j}\left(\frac{\partial(\bar{u}_i)}{\partial x_j}\right) - \frac{\partial\left(\tau_{ij}^u\right)}{\partial x_j} + A\frac{\partial}{\partial x_j}\left(\bar{H}_i\bar{H}_j\right), \\ \frac{\partial(\bar{u}_j)}{\partial x_j} = 0, \\ \frac{\partial\left(\bar{H}_i\right)}{\partial t} + \frac{\partial(\bar{u}_j\bar{H}_i)}{\partial x_j} - \frac{\partial(\bar{H}_j\bar{u}_i)}{\partial x_j} = \frac{1}{Re_m}\frac{\partial}{\partial x_j}\left(\frac{\partial\left(\bar{H}_i\right)}{\partial x_j}\right) - \frac{\partial\left(\tau_{ij}^H\right)}{\partial x_j}, \\ \frac{\partial\left(\bar{H}_j\right)}{\partial x_j} = 0, \\ \tau_{ij}^u = ((\overline{u_i u_j}) - (\bar{u}_i\bar{u}_j)) - \left((\overline{H_i H_j}) - (\bar{H}_i\bar{H}_j)\right), \\ \tau_{ij}^H = \left((\overline{u_i H_j}) - (\bar{u}_i\bar{H}_j)\right) - \left((\overline{H_i u_j}) - (\bar{H}_i\bar{u}_j)\right), \end{cases} \tag{1}$$

where $\bar{u}_i$ $(i = 1, 2, 3)$ are the velocity components, $\bar{H}_1$, $\bar{H}_2$, $\bar{H}_3$ are the magnetic field strength components, $A = H^2/(4\pi\rho V^2) = \Pi/Re_m^2$ is the Alfvén number, H is the characteristic value of the magnetic field strength, V is the typical velocity, $\Pi = (V_A L/\nu_m)^2$ is a dimensionless value (on which the value Π depends in the equation for $\bar{H}_i$). If $\Pi << 1$, then $\partial\bar{H}_i/\partial t = 0$. The publication by [11] discussed in detail the physics of phenomena related to the ability to disregard the summand $\partial\bar{H}_i/\partial t$. $(V_A)^2 = H^2/4\pi\rho$ is the Alfvén velocity, $\bar{p} = p + \bar{H}^2 A/2$ is the full pressure, t is the time, $Re = LV/\nu$ is the Reynolds number, $Re_m = VL/\nu_m$ is the magnetic Reynolds number, L is the typical length, ν is the kinematic viscosity coefficient, ν_m is the magnetic viscosity coefficient, ρ is the density of electrically conducting incompressible fluid, and τ_{ij}^u, τ_{ij}^H is the subgrid-scale tensors responsible for small-scale structures to be modeled.

To model a subgrid-scale tensor, a viscosity model is presented as $\tau_{ij}^u = -2\nu_T\bar{S}_{ij}$, where $\nu_T = (C_S\Delta)^2\left(2\bar{S}_{ij}\bar{S}_{ij}\right)^{\frac{1}{2}}$ is the turbulent viscosity, $\bar{S}_{ij} = (\partial\bar{u}_i/\partial x_j + \partial\bar{u}_j/\partial x_i)/2$ is the deformation velocity tensor value. To model a magnetic subgrid-scale tensor, a viscosity model is used: $\tau_{ij}^H = -2\eta_t\bar{J}_{ij}$, where $\eta_t = (D_S\Delta)^2\left(\bar{J}_{ij}\bar{J}_{ij}\right)^{\frac{1}{2}}$ is the turbulent magnetic diffusion, the coefficients C_S, D_S are calculated for each defined time layer, and $\bar{J}_{ij} = (\partial\bar{H}_i/\partial x_j - \partial\bar{H}_j/\partial x_i)/2$ is the magnetic rotation tensor reviewed by [23].

Periodic boundary conditions are selected at all borders of the reviewed area of the velocity components and the magnetic field strength.

The initial values for each velocity component and strength are defined in the form of a function that depends on the wave numbers in the phase space:

$$u_i(k_i, 0) = k_i^{\frac{b-2}{2}} e^{-\frac{b}{4}\left(\frac{k_i}{k_{\max}}\right)^2}; \qquad H_i(k_i, 0) = k_i^{\frac{b-2}{2}} e^{-\frac{b}{4}\left(\frac{k_i}{k_{\max}}\right)^2},$$

where $\bar{u}_i$ is the one-dimensional velocity spectrum, $i = 1$ refers to the longitudinal spectrum, $i = 2$ and $i = 3$ refer to the transverse spectrum, $\bar{H}_i$ is the one-dimensional magnetic field strength spectrum, m is the spectrum power, and k_1, k_2, k_3 are the wave numbers.

For this problem we selected a variational parameter b and the wave number k_{max}, which determine the type of turbulence. In figure 1 the parameter b varies when $k_{max} = 10$. For modeling homogeneous MHD turbulence can be set parameters k_{max} and b, which correspond to the experimental data [20].

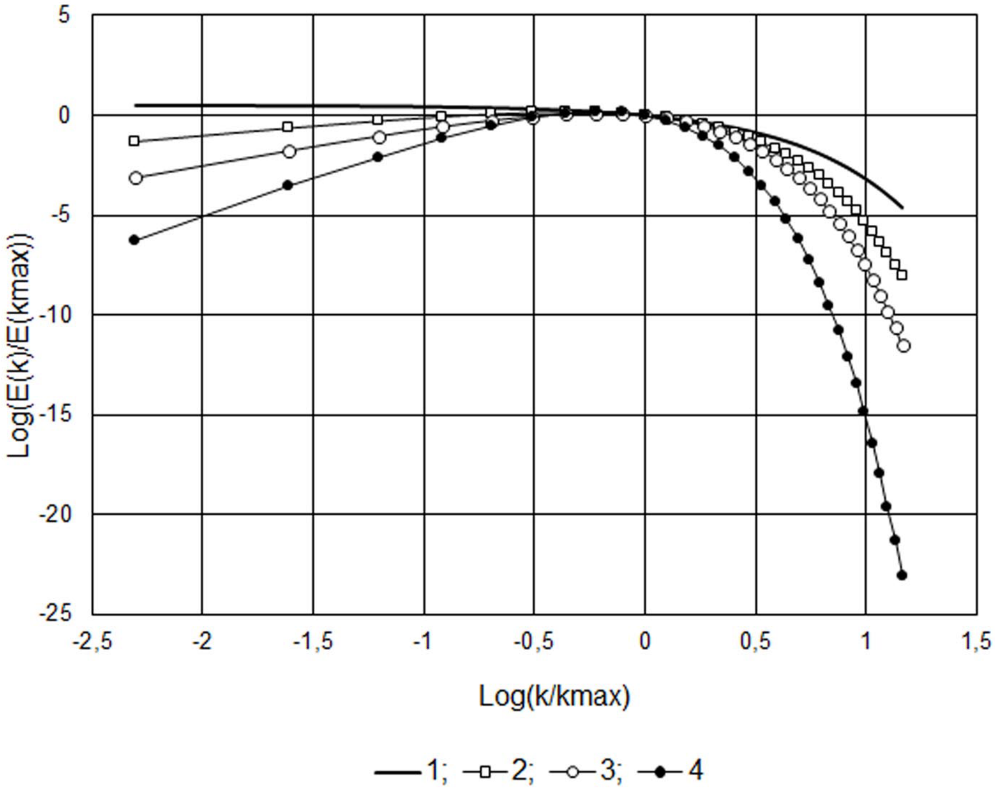

Figure 1: The equation of initial level turbulence, depending on the fixed wave number and the variational parameter b: 1) $b = 2$; 2) $b = 4$; 3) $b = 6$; 4) $b = 8$.

3.2 Numerical method

To solve the problem of homogeneous incompressible MHD turbulence, a scheme of splitting by physical parameters is used:

I. $(\vec{u}^* - \vec{u}^n)/\Delta t = -(\vec{u}^n \nabla)\,\vec{u}^* + A\left(\vec{H}^n \nabla\right)\vec{H}^n + (1/Re)\,(\Delta \vec{u}^*) - \nabla \tau^u,$

II. $\Delta p = \nabla \vec{u}^* / \Delta t,$

III. $(\vec{u}^{n+1} - \vec{u}^*)/\Delta t = -\nabla p.$

IV. $\left(\vec{H}^{n+1} - \vec{H}^n\right)\Big/\Delta t = -rot(\vec{u}^{n+1} \times \vec{H}^{n+1}) + (1/Re_m)\Delta \vec{H}^{n+1} - \nabla \tau^H_.$

During the first stage,, the Navier-Stokes equation is solved without the pressure consideration. for motion is solved, without taking pressure into account. For approximation of the convective and diffusion terms of the intermediate velocity field a finite-difference method in combination with cyclic penta-diagonal matrix is used [4] ,[18], which allowed to increase the order of accuracy in space. The intermediate velocity field is solved by using the Adams-Bashforth scheme in combination with a five-point sweep method. The numerical algorithm for the solution of incompressible MHD turbulence without taking into account large eddy simulation is considered at [1]. Let's consider the velocity component u_1 in the horizontal direction at the spatial location $(i+1/2, j, k)$:

$$\frac{\partial u_1}{\partial t}+\frac{\partial(u_1u_1)}{\partial x_1}+\frac{\partial(u_1u_2)}{\partial x_2}+\frac{\partial(u_1u_3)}{\partial x_3}=A\left(\frac{\partial(H_1H_1)}{\partial x_1}+\frac{\partial(H_1H_2)}{\partial x_2}+\frac{\partial(H_1H_3)}{\partial x_3}\right)+ \\ +\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_1}{\partial x_3^2}\right)-\left(\frac{\partial \tau_{11}^u}{\partial x_1}+\frac{\partial \tau_{12}^u}{\partial x_2}+\frac{\partial \tau_{13}^u}{\partial x_3}\right) \quad (2)$$

When using the explicit Adams-Bachfort scheme for convective terms and the implicit Crank-Nicholson scheme for viscous terms, equation (2) takes the form:

$$\widehat{u_1}_{i+\frac{1}{2},j,k}^{n+1}-u_{1}{}_{i+\frac{1}{2},j,k}^{n}=-\frac{3\Delta t}{2}[hx]_{i+\frac{1}{2},j,k}^{n}+\frac{\Delta t}{2}[hx]_{i+\frac{1}{2},j,k}^{n-1}+\frac{\Delta t}{2}[ax]_{i+\frac{1}{2},j,k}^{n}+ \\ +\frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\cdot\left[\left(\frac{\partial^2\widehat{u_1}}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k}^{n+1}+\left(\frac{\partial^2\widehat{u_1}}{\partial x_2^2}\right)_{i+\frac{1}{2},j,k}^{n+1}+\left(\frac{\partial^2\widehat{u_1}}{\partial x_3^2}\right)_{i+\frac{1}{2},j,k}^{n+1}\right]+ \\ +\frac{3\Delta t}{2}[bx]_{i+\frac{1}{2},j,k}^{n}-\frac{\Delta t}{2}[bx]_{i+\frac{1}{2},j,k}^{n-1}-\frac{3\Delta t}{2}[\tau x]_{i+\frac{1}{2},j,k}^{n}+\frac{\Delta t}{2}[\tau x]_{i+\frac{1}{2},j,k}^{n-1}, \quad (3)$$

where

$$[hx]_{i+\frac{1}{2},j,k}^{n}=\left(\frac{\partial u_1u_1}{\partial x_1}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial u_1u_2}{\partial x_2}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial u_1u_3}{\partial x_3}\right)_{i+\frac{1}{2},j,k}^{n},$$

$$[ax]_{i+\frac{1}{2}jk}^{n}=\frac{1}{\mathrm{Re}}\cdot\left[\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial^2 u_1}{\partial x_3^2}\right)_{i+\frac{1}{2},j,k}^{n}\right]$$

$$[bx]_{i+\frac{1}{2}jk}^{n}=A\cdot\left[\left(\frac{\partial(H_1H_1)}{\partial x_1}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial(H_1H_2)}{\partial x_2}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial(H_1H_3)}{\partial x_3}\right)_{i+\frac{1}{2},j,k}^{n}\right]$$

$$[\tau x]_{i+\frac{1}{2}jk}^{n}=\left(\frac{\partial \tau_{11}^u}{\partial x_1}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial \tau_{12}^u}{\partial x_2}\right)_{i+\frac{1}{2},j,k}^{n}+\left(\frac{\partial \tau_{13}^u}{\partial x_3}\right)_{i+\frac{1}{2},j,k}^{n}$$

Discretization of convective terms look as [12]:

$$\left(\frac{\partial u_1 u_1}{\partial x_1}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(u_1^2)_{i+2,j,k} + 27(u_1^2)_{i+1,j,k} - 27(u_1^2)_{i,j,k} + (u_1^2)_{i-1,j,k}}{24\Delta x_1};$$

$$\left(\frac{\partial u_1 u_2}{\partial x_2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(u_1u_2)_{i+\frac{1}{2},j-\frac{3}{2},k} - 27(u_1u_2)_{i+\frac{1}{2},j-\frac{1}{2},k}}{24\Delta x_2} + \\ + \frac{27(u_1u_2)_{i+\frac{1}{2},j+\frac{1}{2},k} - (u_1u_2)_{i+\frac{1}{2},j+\frac{3}{2},k}}{24\Delta x_2};$$

$$\left(\frac{\partial u_1 u_3}{\partial x_3}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(u_1u_3)_{i+\frac{1}{2},j,k-\frac{3}{2}} - 27(u_1u_3)_{i+\frac{1}{2},j,k-\frac{1}{2}}}{24\Delta x_3} + \\ + \frac{27(u_1u_3)_{i+\frac{1}{2},j,k+\frac{1}{2}} - (u_1u_3)_{i+\frac{1}{2},j,k+\frac{3}{2}}}{24\Delta x_3};$$

Discretization of diffusion terms look as:

$$\left(\frac{\partial^2 u_1}{\partial x_1^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(u_1)_{i+\frac{5}{2},j,k} + 16(u_1)_{i+\frac{3}{2},j,k} - 30(u_1)_{i+\frac{1}{2},j,k}}{12\Delta x_1^2} + \\ + \frac{16(u_1)_{i-\frac{1}{2},j,k} - (u_1)_{i-\frac{3}{2},j,k}}{12\Delta x_1^2};$$

$$\left(\frac{\partial^2 u_1}{\partial x_2^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(u_1)_{i+\frac{1}{2},j+2,k} + 16(u_1)_{i+\frac{1}{2},j+1,k} - 30(u_1)_{i+\frac{1}{2},j,k}}{12\Delta x_2^2} + \\ + \frac{16(u_1)_{i+\frac{1}{2},j-1,k} - (u_1)_{i+\frac{1}{2},j-2,k}}{12\Delta x_2^2};$$

$$\left(\frac{\partial^2 u_1}{\partial x_3^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(u_1)_{i+\frac{1}{2},j,k+2} + 16(u_1)_{i+\frac{1}{2},j,k+1} - 30(u_1)_{i+\frac{1}{2},j,k}}{12\Delta x_3^2} + \\ + \frac{16(u_1)_{i+\frac{1}{2},j,k-1} - (u_1)_{i+\frac{1}{2},j,k-2}}{12\Delta x_3^2};$$

where

$$(u_1u_1)_{i,j,k} = \left(\frac{-u_{1_{i+\frac{3}{2},j,k}} + 9u_{1_{i+\frac{1}{2},j,k}} + 9u_{1_{i-\frac{1}{2},j,k}} - u_{1_{i-\frac{3}{2},j,k}}}{16}\right)^2;$$

$$(u_1u_2)_{i+\frac{1}{2},j+\frac{1}{2},k} = \left(\frac{-u_{1_{i+\frac{1}{2},j+2,k}} + 9u_{1_{i+\frac{1}{2},j+1,k}} + 9u_{1_{i+\frac{1}{2},j,k}} - u_{1_{i+\frac{1}{2},j-1,k}}}{16}\right) \cdot \\ \cdot \left(\frac{-u_{2_{i+2,j+\frac{1}{2},k}} + 9u_{2_{i+1,j+\frac{1}{2},k}} + 9_{u_2}i, j + \frac{1}{2}, k - u_{2_{i-1,j+\frac{1}{2},k}}}{16}\right);$$

$$(u_1u_3)_{i+\frac{1}{2},j,k+\frac{1}{2}} = \left(\frac{-u_{1_{i+\frac{1}{2},j,k+2}} + 9u_{1_{i+\frac{1}{2},j,k+1}} + 9u_{1_{i+\frac{1}{2},j,k}} - u_{1_{i+\frac{1}{2},j,k-1}}}{16}\right) \cdot \\ \cdot \left(\frac{-u_{3_{i+2,j,k+\frac{1}{2}}} + 9u_{3_{i+1,j,k+\frac{1}{2}}} + 9u_{3_{i,j,k+\frac{1}{2}}} - u_{3_{i-1,j,k+\frac{1}{2}}}}{16}\right);$$

Discretization of magnetic field terms look as:

$$\left.\left(\frac{\partial(H_1H_1)}{\partial x_1}\right)\right|_{i+\frac{1}{2},j,k} = \frac{-(H_1^2)_{i+2,j,k} + 27(H_1^2)_{i+1,j,k}}{24\Delta x_1} + \\ + \frac{-27(H_1^2)_{i,j,k} + (H_1^2)_{i-1,j,k}}{24\Delta x_1};$$

$$\left.\left(\frac{\partial(H_1H_2)}{\partial x_1}\right)\right|_{i+\frac{1}{2},j,k} = \frac{(H_1H_2)_{i+\frac{1}{2},j-\frac{3}{2},k} - 27(H_1H_2)_{i+\frac{1}{2},j-\frac{1}{2},k}}{24\Delta x_2} + \\ + \frac{27(H_1H_2)_{i+\frac{1}{2},j+\frac{1}{2},k} - (H_1H_2)_{i+\frac{1}{2},j+\frac{3}{2},k}}{24\Delta x_2};$$

$$\left.\left(\frac{\partial(H_1H_3)}{\partial x_3}\right)\right|_{i+\frac{1}{2},j,k} = \frac{(H_1H_3)_{i+\frac{1}{2},j,k-\frac{3}{2}} - 27(H_1H_3)_{i+\frac{1}{2},j,k-\frac{1}{2}}}{24\Delta x_3} + \\ + \frac{27(H_1H_3)_{i+\frac{1}{2},j,k+\frac{1}{2}} - (H_1H_3)_{i+\frac{1}{2},j,k+\frac{3}{2}}}{24\Delta x_3};$$

The viscosity model and the subgrid-scale tensor are, respectively,

$$\tau_{11}^u = -2\nu_T \cdot S_{11}, \qquad S_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = 0,$$

$$\tau_{12}^u = -2\nu_T \cdot S_{12}, \qquad S_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right),$$

$$\tau_{13}^u = -2\nu_T \cdot S_{13}, \qquad S_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right),$$

Discretization of the strength tensor terms look as:

$$\left.\left(\frac{\partial(-\tau_{11}^u)}{\partial x_1}\right)\right|_{i+\frac{1}{2},j,k} = \frac{\partial}{\partial x_1}\left(2\nu_T \cdot S_{11}\right) = \frac{2}{\Delta x_1}\left[(\nu_T)_{i+\frac{1}{2},j,k} \cdot \left[\frac{(u_1)_{i+1,j,k} - (u_1)_{i,j,k}}{\Delta x_1}\right] + \\ + (\nu_T)_{i-\frac{1}{2},j,k} \cdot \left[\frac{(u_1)_{i,j,k} - (u_1)_{i-1,j,k}}{\Delta x_1}\right]\right] = 0,$$

$$\left(\frac{\partial(-\tau_{12}^{u})}{\partial x_2}\right)\Bigg|_{i+\frac{1}{2},j,k} = \frac{\partial}{\partial x_2}\left(2\nu_T \cdot S_{12}\right) =$$

$$= \frac{2}{2\cdot\Delta x_2}\left[(\nu_T)_{i,j+\frac{1}{2},k}\cdot\left[\frac{(u_1)_{i,j+1,k}-(u_1)_{i,j,k}}{\Delta x_2} - \frac{(u_2)_{i+1,j,k}-(u_2)_{i,j,k}}{\Delta x_1}\right]+\right.$$

$$\left. + (\nu_T)_{i,j-\frac{1}{2},k}\cdot\left[\frac{(u_1)_{i,j,k}-(u_1)_{i,j-1,k}}{\Delta x_2} - \frac{(u_2)_{i,j,k}-(u_2)_{i-1,j,k}}{\Delta x_1}\right]\right],$$

$$\left(\frac{\partial(-\tau_{13}^{u})}{\partial x_3}\right)\Bigg|_{i+\frac{1}{2},j,k} = \frac{\partial}{\partial x_3}\left(2\nu_T \cdot S_{13}\right) =$$

$$= \frac{2}{2\cdot\Delta x_3}\left[(\nu_T)_{i,j,k+\frac{1}{2}}\cdot\left[\frac{(u_1)_{i,j,k+1}-(u_1)_{i,j,k}}{\Delta x_3} - \frac{(u_3)_{i+1,j,k}-(u_3)_{i,j,k}}{\Delta x_1}\right]+\right.$$

$$\left. + (\nu_T)_{i,j,k-\frac{1}{2}}\cdot\left[\frac{(u_1)_{i,j,k}-(u_1)_{i,j,k-1}}{\Delta x_3} - \frac{(u_3)_{i,j,k}-(u_3)_{i-1,j,k}}{\Delta x_1}\right]\right],$$

Then the left hand side of equation (3) is denoted by $q_{i+\frac{1}{2}jk}$

$$q_{i+\frac{1}{2}jk} \equiv \widehat{u_1}^{n+1}_{i+\frac{1}{2},j,k} - {u_1}^{n}_{i+\frac{1}{2},j,k}. \tag{4}$$

We find $\widehat{u_1}^{n+1}_{i+\frac{1}{2}jk}$ from equation (4)

$$\widehat{u_1}^{n+1}_{i+\frac{1}{2},j,k} = q_{i+\frac{1}{2}jk} + {u_1}^{n}_{i+\frac{1}{2},j,k}.$$

Replacing all $\widehat{u_1}^{n+1}_{i+\frac{1}{2},j,k}$ from the equations (3) we obtain

$$\begin{aligned} q_{i+\frac{1}{2}jk} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 q}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 q}{\partial x_2^2}\right)_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 q}{\partial x_3^2}\right)_{i+\frac{1}{2},j,k} = \\ = -\frac{3\Delta t}{2}\left[hx\right]^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t}{2}\left[hx\right]^{n-1}_{i+\frac{1}{2},j,k} + \Delta t\left[ax\right]^{n}_{i+\frac{1}{2},j,k} + \\ + \frac{3\Delta t}{2}\left[bx\right]^{n}_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\left[bx\right]^{n-1}_{i+\frac{1}{2},j,k} - \frac{3\Delta t}{2}\left[\tau x\right]^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t}{2}\left[\tau x\right]^{n-1}_{i+\frac{1}{2},j,k}, \end{aligned} \tag{5}$$

We can re-write equation (5) as

$$\left[1 - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_1^2} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_2^2} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_3^2}\right] q_{i+\frac{1}{2}jk} = d_{i+\frac{1}{2},j,k}, \tag{6}$$

where

$$d_{i+\frac{1}{2}jk} = -\frac{3\Delta t}{2}\left[hx\right]^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t}{2}\left[hx\right]^{n-1}_{i+\frac{1}{2},j,k} + \Delta t\left[ax\right]^{n}_{i+\frac{1}{2},j,k} +$$
$$+\frac{3\Delta t}{2}\left[bx\right]^{n}_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\left[bx\right]^{n-1}_{i+\frac{1}{2},j,k} - \frac{3\Delta t}{2}\left[\tau x\right]^{n}_{i+\frac{1}{2},j,k} + \frac{\Delta t}{2}\left[\tau x\right]^{n-1}_{i+\frac{1}{2},j,k},$$

Assuming that equation (6) has the second-order accuracy in time, we may solve the following equation instead:

$$\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x_1^2}\right]\left[1-\frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_2^2}\right]\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial x_3^2}\right]q^{*}_{i+\frac{1}{2},j,k} = d_{i+\frac{1}{2},j,k}. \quad (7)$$

We can show that Equation (7) is an $O(\Delta t^4)$ approximation to equation (6) [13].

Equation (7) is a factorization approximation to equation (6), which allows each spatial direction to be treated sequentially. If we denote the solution to Equation (7) as $q^{*}_{i+\frac{1}{2}jk}$, by expanding Equation (7), subtracting equation (6) from it, and noting that $q_{i+\frac{1}{2}jk} \sim \mathcal{O}\left(\Delta t^2\right)$, we obtain $\left(q^{*}_{i+\frac{1}{2}jk} - q_{i+\frac{1}{2}jk}\right) \sim \mathcal{O}\left(\Delta t^4\right)$. Therefore, Equation (7) is actually an order $\mathcal{O}\left(\Delta t^4\right)$ approximation to equation (6), rather than an order $\mathcal{O}\left(\Delta t^3\right)$ approximation as stated in [13] without proof. Since the difference between $q^{*}_{i+\frac{1}{2}jk}$ and $q_{i+\frac{1}{2}jk}$ is of higher order, we shall return to the same notation and just use $q_{i+\frac{1}{2}jk}$.

To determine $q_{i+\frac{1}{2}jk}$, equation (7) is solved in 3 stages in sequence as follows:

$$\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_1^2}\right]A_{i+\frac{1}{2},j,k} = d_{i+\frac{1}{2},j,k}; \quad (8)$$

$$\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_2^2}\right]B_{i+\frac{1}{2},j,k} = A_{i+\frac{1}{2},j,k}; \quad (9)$$

$$\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_3^2}\right]q_{i+\frac{1}{2},j,k} = B_{i+\frac{1}{2}jk}. \quad (10)$$

At the first stage, $A_{i+\frac{1}{2},j,k}$ is sought in the coordinate direction x_1:

$$\left[1-\frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\frac{\partial^2}{\partial x_1^2}\right]A_{i+\frac{1}{2},j,k} = d_{i+\frac{1}{2},j,k},$$

$$A_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\cdot\frac{1}{\mathrm{Re}}\cdot\left(\frac{\partial^2 A}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k} = d_{i+\frac{1}{2},j,k},$$

$$A_{i+\frac{1}{2},j,k} - \frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\frac{-A_{i+\frac{5}{2},j,k} + 16A_{i+\frac{3}{2},j,k} - 30A_{i+\frac{1}{2},j,k}}{12\Delta x_1^2} + \\ + \frac{16A_{i-\frac{1}{2},j,k} - A_{i-\frac{3}{2},j,k}}{12\Delta x_1^2} = d_{i+\frac{1}{2},j,k}, \quad (11)$$

$$s_1 A_{i+\frac{5}{2},j,k} - 16s_1 A_{i+\frac{3}{2},j,k} + (1+30s_1)\, A_{i+\frac{1}{2},j,k} - \\ - 16s_1 A_{i-\frac{1}{2},j,k} + s_1 A_{i-\frac{3}{2},j,k} = d_{i+\frac{1}{2},j,k}, \quad (12)$$

where $s_1 = \frac{\Delta t}{24\cdot\mathrm{Re}\cdot\Delta x_1^2}$.

This equation (12) is solved by the cyclic penta-diagonal matrix method, which yields $A_{i+\frac{1}{2}jk}$.

The same procedure is repeated next for the x_2 directions in the second stage, namely, $B_{i+\frac{1}{2}jk}$ is obtained by solving equation (9), with the solution from the first stage as the coefficient on the right hand and the coefficient s_1 in the penta-diagonal matrix replaced by $s_2 = \frac{\Delta t}{24\cdot\mathrm{Re}\cdot\Delta x_2^2}$. Finally, in the third stage, $q_{i+\frac{1}{2}jk}$ is solved through the similar penta-diagonal system shown in equation (10).

Once we have determined the value of $q_{i+\frac{1}{2},j,k}$, we find $\widehat{u_1}^{n+1}_{i+\frac{1}{2},j,k}$

$$\widehat{u_1}^{n+1}_{i+\frac{1}{2},j,k} = q_{i+\frac{1}{2},j,k} + u_1{}^{n}_{i+\frac{1}{2},j,k}.$$

The velocity components $\widehat{u_2}^{n+1}_{i,j+\frac{1}{2},k}$ and $\widehat{u_3}^{n+1}_{i,j,k+\frac{1}{2}}$ are solved in a similar manner.

3.3 Algorithm of solving the Poisson equation

In the second step, the pressure Poisson equation is solved, which ensures that the continuity equation is satisfied. The Poisson equation is transformed from the physical space into the spectral space by using a Fourier transform. The resulting intermediate velocity field does not satisfy the continuity equation. The final velocity field is obtained by adding to the intermediate field the term corresponding to the pressure gradient:

$$u_1^{n+1} = \widehat{u_1}^{n+1} - \Delta t\frac{\partial p}{\partial x_1};$$
$$u_2^{n+1} = \widehat{u_2}^{n+1} - \Delta t\frac{\partial p}{\partial x_2};$$
$$u_3^{n+1} = \widehat{u_3}^{n+1} - \Delta t\frac{\partial p}{\partial x_3}.$$

Substituting the continuity equation, we obtain the Poisson equation for the pressure field:

$$\frac{\partial^2 p}{\partial x_1^2}+\frac{\partial^2 p}{\partial x_2^2}+\frac{\partial^2 p}{\partial x_3^2}=\Delta t\left(\frac{\partial \widehat{u}_1^{n+1}}{\partial x_1}+\frac{\partial \widehat{u}_2^{n+1}}{\partial x_2}+\frac{\partial \widehat{u}_3^{n+1}}{\partial x_3}\right)\equiv F_{i,j,k},$$

where $F_{i,j,k}$ denotes the known right hand side of the Poisson equation, with each term approximated by an order $O(\Delta x^4)$ finite-difference approximation. For example, the first term in $F_{i,j,k}$ is approximated as

$$\Delta t\frac{\partial \widehat{u}_1^{n+1}}{\partial x_1}\bigg|_{i+\frac{1}{2},j,k}=\Delta t\frac{\widehat{u}_{1\,i-\frac{3}{2},j,k}^{\,n+1}-8\widehat{u}_{1\,i-\frac{1}{2},j,k}^{\,n+1}+8\widehat{u}_{1\,i+\frac{3}{2},j,k}^{\,n+1}-\widehat{u}_{1\,i+\frac{5}{2},j,k}^{\,n+1}}{12\Delta x_1}.$$

To be consistent with the spatial accuracy in the first step, the left hand side of the above Poisson equation is discretized using 5-point scheme of $O(\Delta x^4)$ accuracy, as follows:

$$\left[\frac{-P_{i+2,j,k}+16P_{i+1,j,k}-30P_{i,j,k}+16P_{i-1,j,k}-P_{i-2,j,k}}{12\Delta x_1^2}\right]+$$
$$+\left[\frac{-P_{i,j+2,k}+16P_{i,j+1,k}-30P_{i,j,k}+16P_{i,j-1,k}-P_{i,j-2,k}}{12\Delta x_2^2}\right]+$$
$$+\left[\frac{-P_{i,j,k+2}+16P_{i,j,k+1}-30P_{i,j,k}+16P_{i,j,k-1}-P_{i,j,k-2}}{12\Delta x_3^2}\right]=F_{i,j,k}. \quad (13)$$

Now we apply the three dimensional Fourier transform

$$P_{i,j,k}=\frac{1}{N}\sum_{m=0}^{N_1-1}\sum_{n=0}^{N_2-1}\sum_{s=0}^{N_3-1}V_1^{im}V_2^{jn}V_3^{sk}\cdot\widehat{p}_{m,n,s};$$
$$F_{i,j,k}=\frac{1}{N}\sum_{m=0}^{N_1-1}\sum_{n=0}^{N_2-1}\sum_{s=0}^{N_3-1}V_1^{im}V_2^{jn}V_3^{sk}\cdot\widehat{f}_{m,n,s}. \quad (14)$$

The inverse transforms are:

$$\widehat{p}_{m,n,s}=\frac{1}{N}\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\sum_{k=0}^{N_3-1}V_1^{-im}V_2^{-jn}V_3^{-sk}\cdot P_{i,j,k};$$
$$\widehat{f}_{m,n,s}=\frac{1}{N}\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\sum_{k=0}^{N_3-1}V_1^{-im}V_2^{-jn}V_3^{-sk}\cdot F_{i,j,k}. \quad (15)$$

where $N=N_1\cdot N_2\cdot N_3$, $V_1=e^{\iota\left(\frac{2\pi}{N_1}\right)}$, $V_2=e^{\iota\left(\frac{2\pi}{N_2}\right)}$, and $V_3=e^{\iota\left(\frac{2\pi}{N_3}\right)}$.

Substituting equation (15) into equation (14), we obtain quickly the solution for the pressure field in the spectral space as

$$\widehat{p}_{m,n,s}=\frac{12\widehat{f}_{m,n,s}}{Q_1+Q_2+Q_3} \quad (16)$$

where

$$Q_1 = \frac{1}{\Delta x_1^2}\left[-2\cos\left(\frac{4\pi m}{N_1}\right) + 32\cos\left(\frac{2\pi m}{N_1}\right) - 30\right],$$
$$Q_2 = \frac{1}{\Delta x_2^2}\left[-2\cos\left(\frac{4\pi n}{N_2}\right) + 32\cos\left(\frac{2\pi n}{N_2}\right) - 30\right],$$
$$Q_3 = \frac{1}{\Delta x_3^2}\left[-2\cos\left(\frac{4\pi s}{N_3}\right) + 32\cos\left(\frac{2\pi s}{N_3}\right) - 30\right].$$

An inverse Fourier transform is then performed to obtain the pressure $P_{i,j,k}$ in the physical space. The obtained pressure field is then used at the third step to determine the final velocity field.

At the third stage, it is assumed that the transfer is carried out only by the pressure gradient, where the final velocity field is recalculated.

$$\left(\vec{u}^{n+1} - \vec{u}^{*}\right)/\Delta t = -\nabla p.$$

3.4 Algorithm for solving the equation of the magnetic field strength

Let us review equation (1) for the first component of the magnetic field strength in the horizontal direction at the spatial location $(i+1/2, j, k)$:

$$\begin{gathered}\frac{\partial H_1}{\partial t} + \frac{\partial}{\partial x_2}\left(u_2 H_1 - H_2 u_1\right) + \frac{\partial}{\partial x_3}\left(u_3 H_1 - H_3 u_1\right) - \\ -\frac{1}{Re_m}\left[\frac{\partial^2 H_1}{\partial x_1^2} + \frac{\partial^2 H_1}{\partial x_2^2} + \frac{\partial^2 H_1}{\partial x_3^2}\right] = -\left(\frac{\partial \tau_{11}^H}{\partial x_1} + \frac{\partial \tau_{12}^H}{\partial x_2} + \frac{\partial \tau_{13}^H}{\partial x_3}\right).\end{gathered} \tag{17}$$

The strength of the magnetic field is found using the explicit Adams-Bachfort scheme for magnetic convective terms and the implicit Crank-Nicholson scheme for viscous terms, equation (17) takes the form:

$$\begin{aligned}\widehat{H}_{1\,i+\frac{1}{2},j,k}^{\,n+1} - H_{1\,i+\frac{1}{2},j,k}^{\,n} = -\frac{3\Delta t}{2}\left[Hx\right]_{i+\frac{1}{2},j,k}^{n} + \frac{\Delta t}{2}\left[Hx\right]_{i+\frac{1}{2},j,k}^{n-1} + \frac{\Delta t}{2}\left[aHx\right]_{i+\frac{1}{2},j,k}^{n} + \\ +\frac{\Delta t}{2}\frac{1}{\mathrm{Re}}\cdot\left[\left(\frac{\partial^2 \widehat{H}_1}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k}^{n+1} + \left(\frac{\partial^2 \widehat{H}_1}{\partial x_2^2}\right)_{i+\frac{1}{2},j,k}^{n+1} + \left(\frac{\partial^2 \widehat{H}_1}{\partial x_3^2}\right)_{i+\frac{1}{2},j,k}^{n+1}\right] - \\ -\frac{3\Delta t}{2}\left[\tau Hx\right]_{i+\frac{1}{2},j,k}^{n} + \frac{\Delta t}{2}\left[\tau Hx\right]_{i+\frac{1}{2},j,k}^{n-1},\end{aligned} \tag{18}$$

where

$$\left[Hx\right]_{i+\frac{1}{2},j,k}^{n} = \left[\frac{\partial}{\partial x_2}(u_2 H_1 - H_2 u_1)\right]_{i+\frac{1}{2},j,k}^{n} + \left[\frac{\partial}{\partial x_3}(u_3 H_1 - H_3 u_1)\right]_{i+\frac{1}{2},j,k}^{n}$$

$$\left[aHx\right]_{i+\frac{1}{2}jk}^{n} = \frac{1}{\mathrm{Re_m}}\cdot\left[\left(\frac{\partial^2 H_1}{\partial x_1^2}\right)_{i+\frac{1}{2},j,k}^{n} + \left(\frac{\partial^2 H_1}{\partial x_2^2}\right)_{i+\frac{1}{2},j,k}^{n} + \left(\frac{\partial^2 H_1}{\partial x_3^2}\right)_{i+\frac{1}{2},j,k}^{n}\right]$$

$$[\tau Hx]^n_{i+\frac{1}{2}jk} = \left(\frac{\partial \tau^u_{11}}{\partial x_1}\right)^n_{i+\frac{1}{2},j,k} + \left(\frac{\partial \tau^u_{12}}{\partial x_2}\right)^n_{i+\frac{1}{2},j,k} + \left(\frac{\partial \tau^u_{13}}{\partial x_3}\right)^n_{i+\frac{1}{2},j,k}$$

Discretization of magnetic convective terms look as:

$$\left(\frac{\partial u_2 H_1}{\partial x_2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(u_2H_1)_{i+\frac{1}{2},j-\frac{3}{2},k} - 27(u_2H_1)_{i+\frac{1}{2},j-\frac{1}{2},k}}{24\Delta x_2} + \\ + \frac{27(u_2H_1)_{i+\frac{1}{2},j+\frac{1}{2},k} - (u_2H_1)_{i+\frac{1}{2},j+\frac{3}{2},k}}{24\Delta x_2};$$

$$\left(\frac{\partial H_2 u_1}{\partial x_2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(H_2u_1)_{i+\frac{1}{2},j-\frac{3}{2},k} - 27(H_2u_1)_{i+\frac{1}{2},j-\frac{1}{2},k}}{24\Delta x_2} + \\ + \frac{27(H_2u_1)_{i+\frac{1}{2},j+\frac{1}{2},k} - (H_2u_1)_{i+\frac{1}{2},j+\frac{3}{2},k}}{24\Delta x_2};$$

$$\left(\frac{\partial H_3 u_1}{\partial x_3}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(H_3u_1)_{i+\frac{1}{2},j,k-\frac{3}{2}} - 27(H_3u_1)_{i+\frac{1}{2},j,k-\frac{1}{2}}}{24\Delta x_3} + \\ + \frac{27(H_3u_1)_{i+\frac{1}{2},j,k+\frac{1}{2}} - (H_3u_1)_{i+\frac{1}{2},j,k+\frac{3}{2}}}{24\Delta x_3};$$

$$\left(\frac{\partial u_3 H_1}{\partial x_3}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{(u_3H_1)_{i+\frac{1}{2},j,k-\frac{3}{2}} - 27(u_3H_1)_{i+\frac{1}{2},j,k-\frac{1}{2}}}{24\Delta x_3} + \\ + \frac{27(u_3H_1)_{i+\frac{1}{2},j,k+\frac{1}{2}} - (u_3H_1)_{i+\frac{1}{2},j,k+\frac{3}{2}}}{24\Delta x_3};$$

Discretization of magnetic diffusion terms look as:

$$\left(\frac{\partial^2 H_1}{\partial x_1^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(H_1)_{i+\frac{5}{2},j,k} + 16(H_1)_{i+\frac{3}{2},j,k} - 30(H_1)_{i+\frac{1}{2},j,k}}{12\Delta x_1^2} + \\ + \frac{16(H_1)_{i-\frac{1}{2},j,k} - (H_1)_{i-\frac{3}{2},j,k}}{12\Delta x_1^2};$$

$$\left(\frac{\partial^2 H_1}{\partial x_2^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(H_1)_{i+\frac{1}{2},j+2,k} + 16(H_1)_{i+\frac{1}{2},j+1,k} - 30(H_1)_{i+\frac{1}{2},j,k}}{12\Delta x_2^2} + \\ + \frac{16(H_1)_{i+\frac{1}{2},j-1,k} - (H_1)_{i+\frac{1}{2},j-2,k}}{12\Delta x_2^2};$$

$$\left(\frac{\partial^2 H_1}{\partial x_3^2}\right)\bigg|_{i+\frac{1}{2},j,k} = \frac{-(H_1)_{i+\frac{1}{2},j,k+2} + 16(H_1)_{i+\frac{1}{2},j,k+1} - 30(H_1)_{i+\frac{1}{2},j,k}}{12\Delta x_3^2} + \\ + \frac{16(H_1)_{i+\frac{1}{2},j,k-1} - (H_1)_{i+\frac{1}{2},j,k-2}}{12\Delta x_3^2},$$

where

$$(u_2H_1)_{i+\frac{1}{2},j+\frac{1}{2},k} = \left(\frac{-u_{2_{i+2,j+\frac{1}{2},k}} + 9u_{2_{i+1,j+\frac{1}{2},k}} + 9u_{2_{i,j+\frac{1}{2},k}} - u_{2_{i-1,j+\frac{1}{2},k}}}{16}\right)\cdot$$
$$\cdot\left(\frac{-H_{1_{i+\frac{1}{2},j+2,k}} + 9H_{1_{i+\frac{1}{2},j+1,k}} + 9H_{1_{i+\frac{1}{2},j,k}} - H_{1_{i+\frac{1}{2},j-1,k}}}{16}\right);$$
$$(H_2u_1)_{i+\frac{1}{2},j+\frac{1}{2},k} = \left(\frac{-H_{2_{i+2,j+\frac{1}{2},k}} + 9H_{2_{i+1,j+\frac{1}{2},k}} + 9H_{2_{i,j+\frac{1}{2},k}} - H_{2_{i-1,j+\frac{1}{2},k}}}{16}\right)\cdot$$
$$\cdot\left(\frac{-u_{1_{i+\frac{1}{2},j+2,k}} + 9u_{1_{i+\frac{1}{2},j+1,k}} + 9u_{1_{i+\frac{1}{2},j,k}} - u_{1_{i+\frac{1}{2},j-1,k}}}{16}\right);$$

$$(u_3H_1)_{i+\frac{1}{2},j,k+\frac{1}{2}} = \left(\frac{-u_{3_{i+2,j,k+\frac{1}{2}}} + 9u_{3_{i+1,j,k+\frac{1}{2}}} + 9u_{3_{i,j,k+\frac{1}{2}}} - u_{3_{i-1,j,k+\frac{1}{2}}}}{16}\right)\cdot$$
$$\cdot\left(\frac{-H_{1_{i+\frac{1}{2},j,k+2}} + 9H_{1_{i+\frac{1}{2},j,k+1}} + 9H_{1_{i+\frac{1}{2},j,k}} - H_{1_{i+\frac{1}{2},j,k-1}}}{16}\right);$$
$$(H_3u_1)_{i+\frac{1}{2},j,k+\frac{1}{2}} = \left(\frac{-H_{3_{i+2,j,k+\frac{1}{2}}} + 9H_{3_{i+1,j,k+\frac{1}{2}}} + 9H_{3_{i,j,k+\frac{1}{2}}} - H_{3_{i-1,j,k+\frac{1}{2}}}}{16}\right)\cdot$$
$$\cdot\left(\frac{-u_{1_{i+\frac{1}{2},j,k+2}} + 9u_{1_{i+\frac{1}{2},j,k+1}} + 9u_{1_{i+\frac{1}{2},j,k}} - u_{1_{i+\frac{1}{2},j,k-1}}}{16}\right);$$

The viscosity model and the magnetic rotation tensor are, respectively,

$$\tau_{11}^H = -2\eta_t \cdot J_{11}, \qquad J_{11} = \frac{1}{2}\left(\frac{\partial H_1}{\partial x_1} - \frac{\partial H_1}{\partial x_1}\right) = 0,$$
$$\tau_{12}^H = -2\eta_t \cdot J_{12}, \qquad J_{12} = \frac{1}{2}\left(\frac{\partial H_1}{\partial x_2} - \frac{\partial H_2}{\partial x_1}\right),$$
$$\tau_{13}^H = -2\eta_t \cdot J_{13}, \qquad J_{13} = \frac{1}{2}\left(\frac{\partial H_1}{\partial x_3} - \frac{\partial H_3}{\partial x_1}\right),$$

The discretization of the magnetic rotation tensor terms look as:

$$\frac{\partial}{\partial x_1}\left(-\tau_{11}^H\right) = 0,$$

$$\frac{\partial}{\partial x_2}\left(-\tau_{12}^H\right) = \frac{\partial}{\partial x_2}\left(2\eta_t \cdot J_{12}\right) =$$
$$= \frac{2}{2\cdot\Delta x_2}\left[(\eta_t)_{i,j+\frac{1}{2},k}\cdot\left[\frac{(H_1)_{i,j+1,k}-(H_1)_{i,j,k}}{\Delta x_2} - \frac{(H_2)_{i+1,j,k}-(H_2)_{i,j,k}}{\Delta x_1}\right]-\right.$$
$$\left.-(\eta_t)_{i,j-\frac{1}{2},k}\cdot\left[\frac{(H_1)_{i,j,k}-(H_1)_{i,j-1,k}}{\Delta x_2} - \frac{(H_2)_{i,j,k}-(H_2)_{i-1,j,k}}{\Delta x_1}\right]\right],$$

$$\frac{\partial}{\partial x_3}\left(-\tau_{13}^H\right) = \frac{\partial}{\partial x_3}\left(2\eta_t \cdot J_{13}\right) =$$

$$= \frac{2}{2\cdot\Delta x_3}\left[\left(\eta_t\right)_{i,j,k+\frac{1}{2}} \cdot \left[\frac{(H_1)_{i,j,k+1}-(H_1)_{i,j,k}}{\Delta x_3} - \frac{(H_3)_{i+1,j,k}-(H_3)_{i,j,k}}{\Delta x_1}\right] -\right.$$

$$\left. - \left(\eta_t\right)_{i,j,k-\frac{1}{2}} \cdot \left[\frac{(H_1)_{i,j,k}-(H_1)_{i,j,k-1}}{\Delta x_3} - \frac{(H_3)_{i,j,k}-(H_3)_{i-1,j,k}}{\Delta x_1}\right]\right],$$

The equation is solved by he similar penta-diagonal system shown in section II and is found to be $(H_1)_{i,j,k}^{n+\frac{1}{3}}$.

$(H_1)_{i,j,k}^{n+\frac{2}{3}}$, $(H_1)_{i,j,k}^{n+1}$ components of the magnetic field strength are defined in a similar way.
Thus, all the components of the magnetic field strength determined this way.

3.5 Definition of homogeneous MHD turbulence characteristics

To identify turbulent characteristics in the physical space, it is necessary to average different values in volume. The averaged values will be used to find the turbulent characteristics. The procedure for calculating the turbulent characteristics is similar to the one specified in papers by [17] and [3]. The value averaged along the entire calculated area is calculated by the following formula:

$$\langle u_i \rangle = \frac{1}{N_1 N_2 N_3} \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \sum_{q=1}^{N_3} \left(\bar{u}_i\right)_{n,m,q}.$$

$$\langle H_i \rangle = \frac{1}{N_1 N_2 N_3} \sum_{n=1}^{N_1} \sum_{m=1}^{N_2} \sum_{q=1}^{N_3} \left(\bar{H}_i\right)_{n,m,q}$$

$$\left\langle u_1^2 \right\rangle = \left\langle u_1(x,y,z,t) \cdot u_1(x,y,z,t) \right\rangle,$$

$$\left\langle u_2^2 \right\rangle = \left\langle u_2(x,y,z,t) \cdot u_2(x,y,z,t) \right\rangle,$$

$$\left\langle u_3^2 \right\rangle = \left\langle u_3(x,y,z,t) \cdot u_3(x,y,z,t) \right\rangle.$$

The microscale length is determined by the following ratio:

$$\lambda_f = \left\{\frac{2}{-f''(0)}\right\}^{1/2}, \quad \lambda_g = \left\{\frac{2}{g''(0)}\right\}^{1/2}.$$

The integral scale is expressed as

$$\Lambda_f(t) = \int_0^{Ł/2} f\,(r,t)\,dr, \quad \Lambda_g(t) = \int_0^{Ł/2} g\,(r,t)\,dr.$$

The dissipation rate is calculated by the following formula:

$$\epsilon =< 2vS_{ij}S_{ij} >= 2v\left[\left\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_2}{\partial x_2}\right)^2\right\rangle + \left\langle\left(\frac{\partial u_3}{\partial x_3}\right)^2\right\rangle\right] +$$
$$+ 2v\left[\frac{1}{2}\left\langle\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2\right\rangle + \frac{1}{2}\left\langle\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)^2\right\rangle + \frac{1}{2}\left\langle\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2\right\rangle\right]$$

The turbulent kinematic energy is found in the following way: The turbulent kinetic and magnetic energy are, respectively,

$$E_{ku} = \frac{1}{2}\left(\langle u_1\rangle^2 + \langle u_2\rangle^2 + \langle u_3\rangle^2\right) = \frac{3}{2}\left\langle u_1^2\right\rangle,$$

$$E_{kh} = \frac{1}{2}\left(\langle H_1\rangle^2 + \langle H_2\rangle^2 + \langle H_3\rangle^2\right) = \frac{3}{2}\left\langle H_1^2\right\rangle.$$

Velocity derivative skewness is defined in the following form:

$$S(t) = \frac{\left\langle \frac{1}{3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^3 + \left(\frac{\partial u_2}{\partial x_2}\right)^3 + \left(\frac{\partial u_3}{\partial x_3}\right)^3\right]\right\rangle}{\left(\left\langle \frac{1}{3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\right\rangle\right)^{3/2}}$$

Flatness is defined in the following form:

$$F(t) = \frac{\left\langle \frac{1}{3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^4 + \left(\frac{\partial u_2}{\partial x_2}\right)^4 + \left(\frac{\partial u_3}{\partial x_3}\right)^4\right]\right\rangle}{\left(\left\langle \frac{1}{3}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_3}\right)^2\right]\right\rangle\right)^{2}}$$

3.6 Analytical solution of the Taylor-Green vortex problem

For validation of the developed algorithm the classical problem of the 3-D Taylor and Green vortex flow is considered without considering the magnetic field, and the simulated time-dependent turbulence characteristics of this flow were found to be in excellent agreement with the corresponding analytical solution valid for short times.

We duplicate the classical example proposed in [21] in order to validate the numerical simulation of increasing order of accuracy in time and in space $O(\Delta t^2, h^4)$, with efficient acceleration for sequential algorithm. Starting from a simple incompressible three-dimensional initial condition of the form.

$$\begin{cases} u_1(x_1, x_2, x_3, t=0) = \cos(ax_1)\sin(ax_2)\sin(ax_3), \\ u_2(x_1, x_2, x_3, t=0) = -\sin(ax_1)\cos(ax_2)\sin(ax_3), \\ u_3(x_1, x_2, x_3, t=0) = 0. \end{cases} \quad (19)$$

and assuming periodic conditions in a cubic domain: $0 \le x_1 \le 2\pi, 0 \le x_2 \le 2\pi, 0 \le x_3 \le 2\pi$ with $a = 1$, the three-dimensional filtered Navier-Stokes equation

$$\begin{cases} \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u_i}{\partial x_i \partial x_j}, \\ \\ \frac{\partial u_i}{\partial x_i} = 0. \end{cases} \quad (20)$$

can be solved analytically at small times, using perturbation expansion. In (1) all quantities have been properly normalized by the initial maximum velocity magnitude U_0 in the x_1 or x_2 direction, and $L/2\pi$, where L is the physical domain size, u_i -velocity at $i = 1, 2, 3$, corresponding to x_1, x_2, x_3 directions, $\mathrm{Re} = LU_0/\nu$ is the Reynolds number of flow, U_0 - the characteristic velocity, $T = aU_0t, a = 2\pi/L$. The pressure p has been normalized by ρU_0^2 . Taylor and Green obtained a perturbation expansion of the velocity field, up to $O(t^5)$. The resulting average kinetic energy is:

$$E_k = \frac{U_0^2}{8}u'^2 \quad (21)$$

where

$$u'^2 = 1 - \frac{6T}{\mathrm{Re}} + \frac{18T^2}{\mathrm{Re}^2} - \left(\frac{5}{24} + \frac{36}{\mathrm{Re}^2}\right)\frac{T^3}{\mathrm{Re}} + \left(\frac{5}{2\mathrm{Re}^2} + \frac{54}{\mathrm{Re}^4}\right)T^4 - \\ - \left(\frac{5}{44.12} + \frac{367}{24\mathrm{Re}^2} + \frac{4.81}{5\mathrm{Re}^4}\right)\frac{T^5}{\mathrm{Re}} + \left(\frac{361}{44.32} + \frac{761}{12\mathrm{Re}^2} + \frac{324}{5\mathrm{Re}^4}\right)\frac{T^6}{\mathrm{Re}^2}. \quad (22)$$

The dissipation rate is written in the following form:

$$W = \mu\frac{3U_0^2a^2}{4}W' \quad (23)$$

where

$$W' = 1 - \frac{6T}{\mathrm{Re}} + \left(\frac{5}{48} + \frac{18T^2}{\mathrm{Re}^2}\right)T^2 - \left(\frac{5}{3} + \frac{36}{\mathrm{Re}^2}\right)\frac{T^3}{\mathrm{Re}} + \\ + \left(\frac{50}{99.64} + \frac{1835}{9.16\mathrm{Re}^2} + \frac{54}{\mathrm{Re}^4}\right)T^4 - \left(\frac{361}{44.32} + \frac{761}{12\mathrm{Re}^2} + \frac{324}{5\mathrm{Re}^4}\right)\frac{T^5}{\mathrm{Re}}. \quad (24)$$

Simulation at different Reynolds numbers was compared with the analytical solution of the Taylor-Green vortex problem from the point of view of: the average kinetic energy and the average dissipation rate of the turbulent flow. Figure 2 compares the average turbulent kinetic energy obtained in this paper with the analytical solution of the Taylor-Green vortex problem for different Reynolds numbers. The results obtained by analytical solution of short-time theory of TG, spectral methods at 256^3 grid resolution and hybrid finite difference method at 128^3 grid resolution show a satisfactory agreement till $T = 3$ at $Re = 100$, and till $T = 4$ at $Re = 300$ and $Re = 600$ for the average turbulent kinetic energy. The error between analytical and numerical solutions for the average kinetic energy was defined as: $Error(E_k) = |E_k^{HFDM} - E_k^{TG}| = 10^{-4}$.

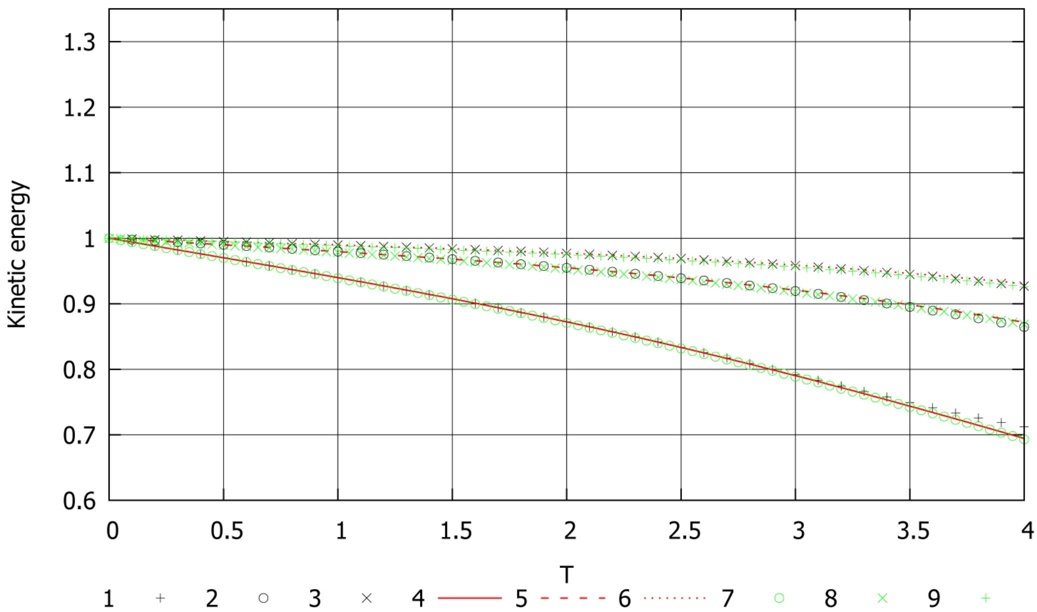

Figure 2: Comparative results of modeling the evolution of the average kinetic energy in time, spectral and hybrid methods of modeling the Taylor-Green vortex of: TG short-time theory at: 1) Re=100; 2)Re=300; 3)Re=600; Spectral method, 256^3 at: 4)Re=100; 5)Re=300; 6)Re=600; HFD method at: 7)Re=100; 8)Re=300; 9)Re=600.

Figure 3 compares the results of average rate of dissipation of the turbulence decay with respect to time of the numerical simulation, and the analytical solution of the Taylor-Green vortex problem at different Reynolds number. It can be seen from Figure 3 that the short-term theoretical results and numerical simulation results are in good agreement till $T = 2.5$ for $Re = 100$, and $T = 2$ for $Re = 300; Re = 600$. It is difficult to compare the analytical solution with numerical simulation, since the analytical solution valid only for short-term time, and the numerical solution can provide good results for long term, so it is worthwhile to compare simulation results of spectral method and HFD method for long term. The rate of dissipation increases sharply due to the formation of small-scale flow structures and reaches a maximum at $T = 3$, for short time theory of TG at $Re = 100$, and at $T = 4$ for other case, and then the rate of dissipation shows a decrease in the tendency for result of analytical solution of TG at $Re = 100$ because of the decrease in the total Reynolds number of the stream. In the simulation results, the error between analytical and numerical solutions for the average dissipation rate is: $Error(\epsilon) = |\epsilon^{HFDM} - \epsilon^{TG}| = 10^{-2}$.

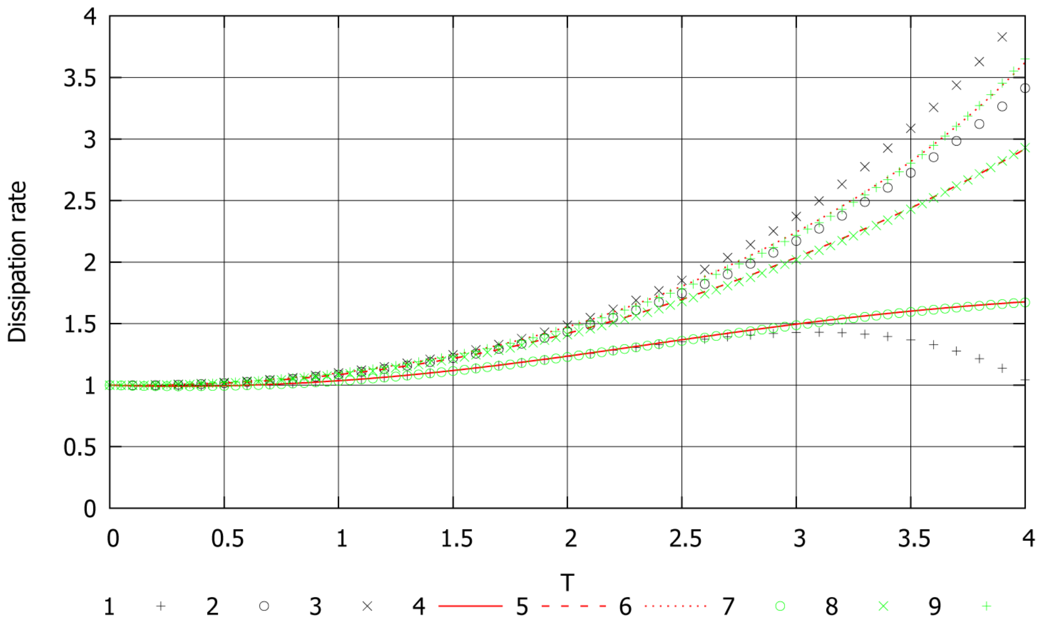

Figure 3: Comparative results of modeling the evolution of the average rate of dissipation of the decay of turbulence in time, the spectral and hybrid methods of modeling the Taylor-Green vortexof: TG short-time theory at: 1) Re=100; 2)Re=300; 3)Re=600; Spectral method, 256^3 at: 4)Re=100; 5)Re=300; 6)Re=600; HFD method at: 7)Re=100; 8)Re=300; 9)Re=600.

Figure 4 shows that with the increase in the resolution of the computational grid, the results of skewness of the turbulence of hybrid method tends gently to the exponential results of the pseudospectral method for the computational grid 256x256x256.

Figure 5 shows the results of modeling the evolution of flatness, spectral and hybrid methods for modeling the Taylor-Green vortex at $Re = 300$.

4 Results and discussion

Numerical model allows to describe the homogeneous magnetohydrodynamic turbulence decay based on large eddy simulation. For this task, the kinematic viscosity $\nu = 10^{-4}$ was taken constant and the magnetic viscosity were set in the range of $\nu_m = 10^{-3} \div 10^{-4}$. The characteristic values of the velocity, length, magnetic field strength were taken equal to: $U_{CH} = 1$, $L_{CH} = 1$, $H_{CH} = 1$ respectively. Reynolds number is $Re = 10^4$, the magnetic Reynolds number varied depending on the magnetic viscosity coefficient. The Alfven number characterizing the motion of conductive fluid for various numbers of magnetic Reynolds: $A = Ha^2/Re_m$, where Hartmann number is $Ha = 1$. For the calculations used grid size 128x128x128. The time step was taken equal $\Delta\tau = 0.001$.

As result of simulation at different magnetic Reynolds numbers were obtained the following turbulence characteristics: integral scale and Taylor scale.

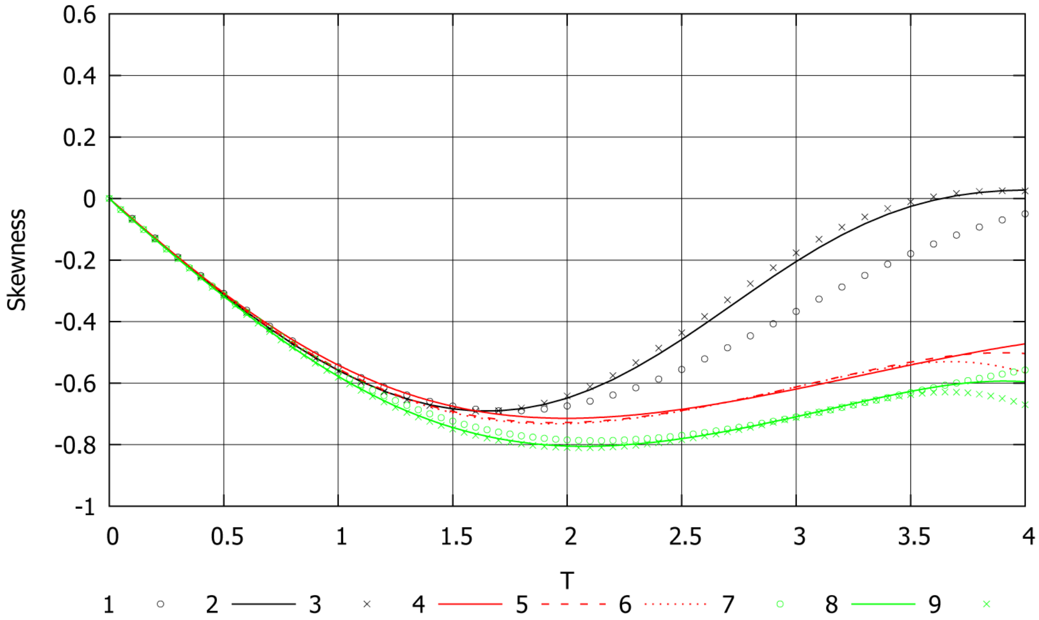

Figure 4: Comparison of the results of modeling the evolution of skewness, spectral and hybrid methods for modeling the Taylor-Green vortex of: TG short-time theory at: 1) Re=100; 2)Re=300; 3)Re=600; Spectral method, 256^3 at: 4)Re=100; 5)Re=300; 6)Re=600; HFD method at: 7)Re=100; 8)Re=300; 9)Re=600.

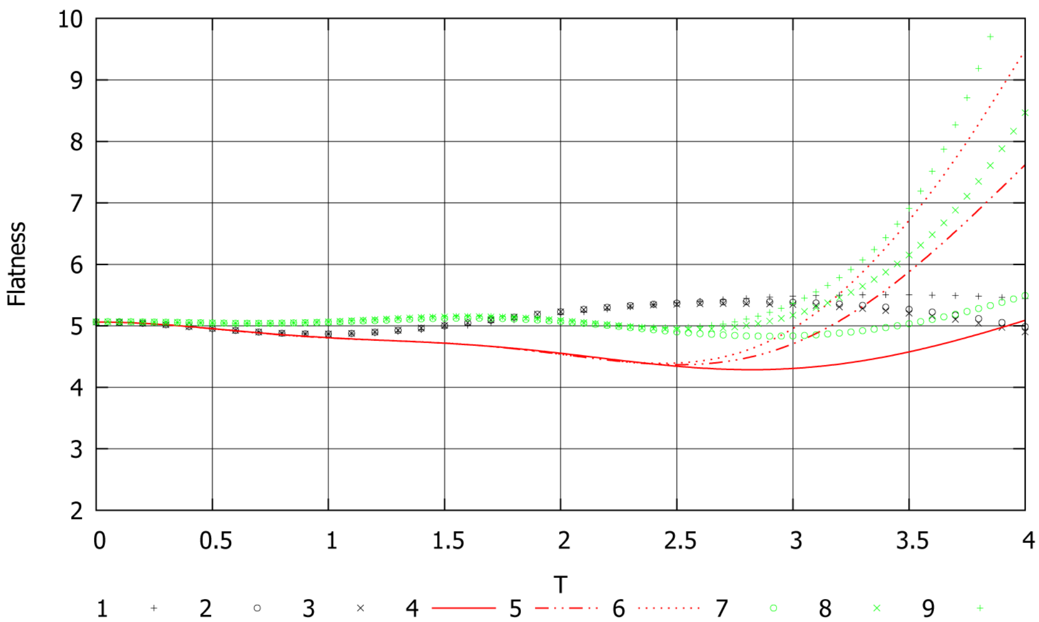

Figure 5: Comparison of the results of modeling the evolution of flatness, spectral and hybrid methods for modeling the Taylor-Green vortex of: TG short-time theory at: 1) Re=100; 2)Re=300; 3)Re=600; Spectral method, 256^3 at: 4)Re=100; 5)Re=300; 6)Re=600; HFD method at: 7)Re=100; 8)Re=300; 9)Re=600.

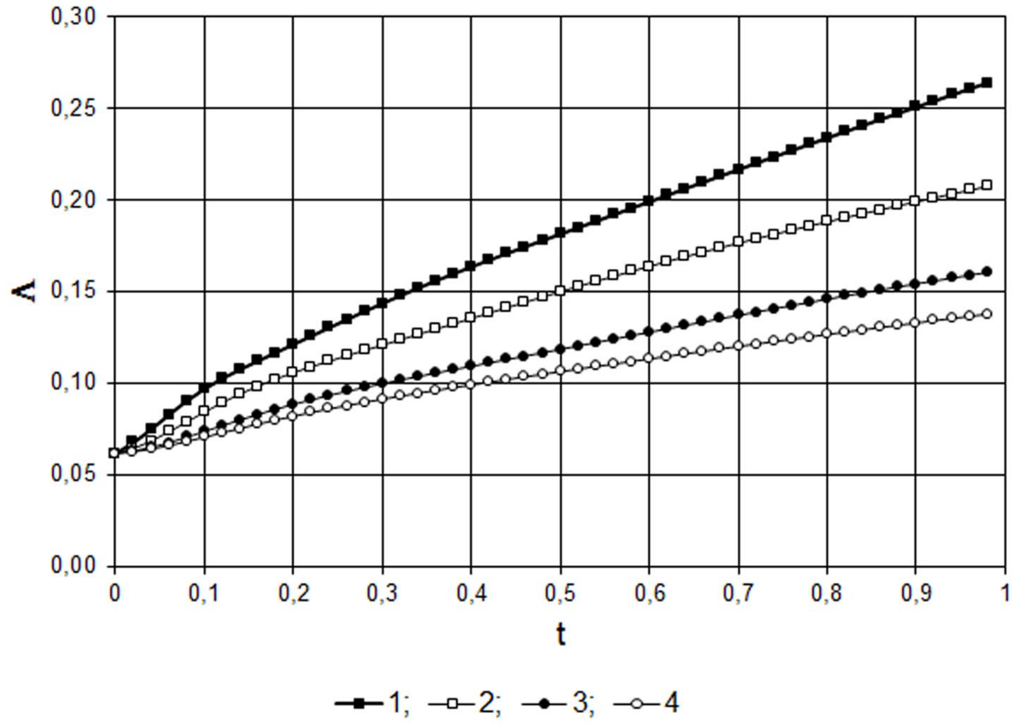

Figure 6: Change of the integral turbulence scale calculated at different magnetic Reynolds numbers: 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$.

According to semi-empirical theory of turbulence integral scale should grow with time. The results presented in Figure 6 illustrates the effect of magnetic viscosity on the internal structure of the MHD turbulence. Variation of the coefficient of magnetic viscosity leads to a proportional change in the integral scale. Figure 6 shows that the size of large eddies rapidly increases at small number of magnetic Reynolds $Re_m = 10^3$, than in the case, when $Re_m = 10^4$ which leads to fast energy dissipation.

Figure 7 shows the change in the micro scale - calculated at different numbers of magnetic Reynolds 1) $Re_m = 10^3$; 2) $Re_m = 2 \cdot 10^3$; 3) $Re_m = 5 \cdot 10^3$; 4) $Re_m = 10^4$. Figure 7 shows the change of the Taylor microscale at different magnetic Reynolds numbers. It can be seen that in the case $Re_m = 10^3$ when the magnetic viscosity coefficient is large then the dissipation rate increases. In the case when the magnetic viscosity coefficient is smaller then the scale gradually increases, and the small scale structure of the turbulence tends to slowly isotropy. This also indicates that with small numbers Re_m the decay of isotropic turbulence occurs faster than in the case when Re_m is high.

From the figures it is seen that in the case of high medium conductivity at $Re_m = 10^3$ the frictional force increases and the flow rate is reduced faster than, at $Re_m = 10^4$, that corresponds to the low conductivity of the medium, in this version, the frictional force have minimal impact on the flow velocity. Based on the study of the results determined that the first part of the turbulent kinetic energy is used for turbulent mixing, the second part - at creating magnetic field and the third part - on the forces of resistance between the components of the velocity and magnetic tension.

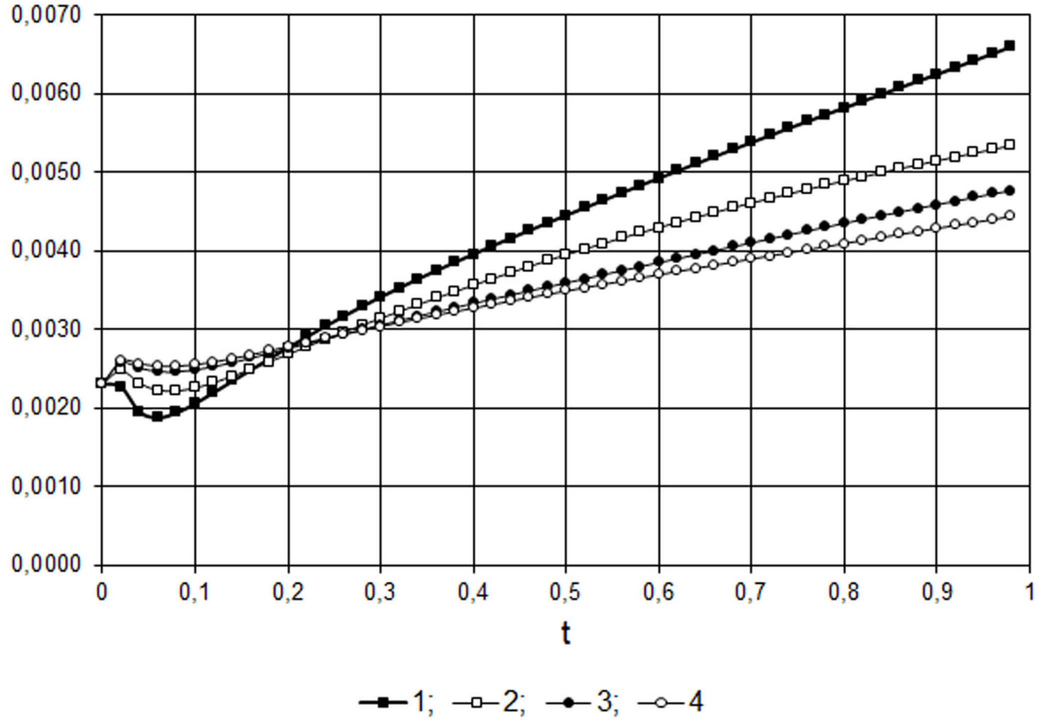

Figure 7: Change of Taylor-scale calculated at different magnetic Reynolds numbers: 1)$Re_m = 10^3$; 2)$Re_m = 2 \cdot 10^3$; 3)$Re_m = 5 \cdot 10^3$; 4)$Re_m = 10^4$.

5 Conclusion

Based on the method large-eddy simulation was produced the numerical modelling of influence magnetic viscosity to decay of magnetohydrodynamic turbulence, analyzing simulation results it is possible to make the following conclusion: the magnetic viscosity of the flow has a significant influence on the MHD turbulence. Obtained results allow sufficiently accurately calculate the change characteristics of magnetohydrodynamic turbulence over time at different magnetic Reynolds numbers. To simulate the turbulence energy degeneration, a numerical algorithm for solving the unsteady three-dimensional Navier-Stokes equations based on the hybrid method was developed. The numerical algorithm is a hybrid method combining finite difference and spectral methods. It is also computationally efficient. The finite-difference method combined with the cyclic Penta-diagonal matrix for the solution of the Navier-Stokes equations allowed to achieve the accuracy of the fourth order in space and the accuracy of the second order in time. The spectral method for solving the Poisson equation has a high computational efficiency by using a fast Fourier transform library.

To check the adequacy of the developed algorithm, the classical Taylor and green problem with the same initial flow conditions, for modeling the degeneracy of the kinetic energy of the flow and the time evolution of viscous dissipation is considered. Average normalized errors between analytical and numerical solutions for mean kinetic energy and mean dissipation rate were established as $Error(E_k) = |E_k^{FDM} - E_k^{TG}| = 10^{-4}$, $Error(\epsilon) = |\epsilon^{FDM} - \epsilon^{TG}| = 10^{-2}$. respectively. Thus, the results of numerical simulation of turbulence characteristics show very good agreements with the analytical solution. Thus, the numerical algorithm was developed for solving unsteady three-dimensional magnetohydrodynamic equations, and makes it possible to simulate the MHD turbulence decay at different magnetic Reynolds numbers.

6 Acknowledgment

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On positive solutions of Liouville-Gelfand problem

Kolosova S. V., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: lanakol@ukr.net
Lukhanin V. S., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: lukhanin.volodymyr@gmail.com
Sidorov M. V., Kharkiv National University of Radio Electronics,
Kharkiv, Ukraine, E-mail: maxim.sidorov@nure.ua

Modern science is highly interested in processes in nonlinear media. Mathematical models of these processes are often described by boundary-value problems for nonlinear elliptic equations. And the construction of two-sided approximations to the desired function is a perspective direction of solving such problems. The purpose of this work is to consider the existence and uniqueness of a regular positive solution to the Liouville-Gelfand problem and justify the possibility of constructing two-sided approximations to a solution. The two-sided approximations monotonically approximate the desired solution from above and below, and therefore have such an important advantage over other approximate methods that they provide an opportunity to obtain a convenient a posteriori estimate of the error of the calculations. The study of the Liouville-Gelfand problem is carried out by methods of the operator equations theory in partially ordered spaces. The mathematical model of the problem under consideration is the Dirichlet problem for a nonlinear elliptic equation with a positive parameter. The established properties of the corresponding nonlinear operator equation have given us an opportunity to obtain a condition for an input parameter, which guarantees the existence and uniqueness of the regular positive solution, as well as the possibility of constructing two-sided approximations, regardless of the domain geometry in which the problem is considered. The corresponding Liouville-Gelfand problem of the operator equation contains the Green's function for the Laplace operator of the first boundary value problem, and therefore the condition that the input parameter satisfies also contains it. Since the Green's function is known for a small number of relatively simple domains, Green's quasifunction method is used to solve the problem in domains of complex geometry. We note that the Green's quasifunction can be constructed practically for a domain of any geometry. The proposed approach allows us: a) to obtain a formula, which the parameter in the problem statement must satisfy, regardless of the domain geometry; b) for the first time, construct two-sided approximations to a solution to the Liouville-Gelfand problem; c) for the first time to obtain an a priori estimate of the solution depending on the selected value of the parameter in the problem statement. The proposed method has advantages over other approximate methods in relative simplicity of the algorithm implementation. The proposed method can be used for solving applied problems with mathematical models that are described by boundary value problems for nonlinear elliptic equations. In cases when the Green's function is unknown or has a complex form, the application of the Green's quasifunction method is proposed.
Key words: Green's function, Green's quasifunction, two-sided approximations, invariant cone segment, monotone operator.

О положительных решениях задачи Лиувилля-Гельфанда

Колосова С. В., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: lanakol@ukr.net
Луханин В. С., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: lukhanin.volodymyr@gmail.com
Сидоров М. В., Харьковский национальный университет радиоэлектроники,
Харьков, Украина, E-mail: maxim.sidorov@nure.ua

В современной науке наблюдается большой интерес к процессам, происходящим в нелинейных средах. Математическими моделями таких процессов зачастую являются краевые задачи для нелинейных эллиптических уравнений. Перспективными направлениями для решения таких задач есть построение двусторонних приближений к искомой функции. Целью данной работы является рассмотрение вопросов существования и единственности регулярного положительного решения у задачи Лиувилля-Гельфанда, а также обоснование возможности построения двусторонних приближений к решению. Двусторонние приближения монотонно сверху и снизу аппроксимируют искомое решение, и поэтому обладают тем важным преимуществом по сравнению с другими приближенными методами, что они дают возможность получить удобную апостериорную оценку погрешности вычислений. Исследование задачи Лиувилля-Гельфанда проводится методами теории операторных уравнений в полуупорядоченных пространствах. Математической моделью рассматриваемой задачи является задача Дирихле для нелинейного эллиптического уравнения с положительным параметром. Установленные свойства соответствующего нелинейного операторного уравнения дали возможность получить условие для входящего в постановку задачи параметра, которое гарантирует существование и единственность регулярного положительного решения, а также возможность построения двусторонних приближений независимо от геометрии области, в которой рассматривается задача. Соответствующее задачи Лиувилля-Гельфанда операторное уравнение содержит функцию Грина оператора Лапласа первой краевой задачи, а поэтому и условие, которому удовлетворяет параметр, также ее содержит. Так как функция Грина известна для небольшого числа достаточно простых областей, для решения задачи в областях сложной геометрии в работе применяется метод квазифункций Грина. Заметим, что квазифункцию Грина можно построить практически для области любой геометрии. Использованный в работе подход позволил: а) получить формулу, которой должен удовлетворять входящий в постановку задачи параметр, независимо от геометрии области; б) впервые для задачи Лиувилля-Гельфанда построить двусторонние приближения к решению; в) впервые получить априорную оценку решения в зависимости от выбранного значения параметра, который входит в постановку задачи. Предложенный метод решения имеет преимущества в сравнении с другими приближенными методами относительной простотой реализации алгоритма. Предлагаемый метод может быть использован при решении прикладных задач, математическими моделями которых являются краевые задачи для нелинейных эллиптических уравнений. В ситуациях, когда функция Грина неизвестна или имеет сложный вид, предложено применение метода квазифункций Грина.
Ключевые слова: функция Грина, квазифункция Грина, двусторонние приближения, инвариантный конусный отрезок, монотонный оператор.

1 Introduction

Modern science is highly interested in processes that take place in nonlinear environments. Mathematical models of these processes typically are represented by nonlinear boundary value problems of mathematical physics of the following form

$$-\Delta u = f(x,u) \quad \forall x \in \Omega \subset R^N,$$

$$u > 0, \quad u|_{\partial\Omega} = 0.$$

It is important to identify among the analytical methods ones that provide specific ways of constructing the sought solution. These methods include iterative ones which are simpler than the others and can be implemented on a computer. Among the iterative methods we highlight a class of two-sided processes that approximate the sought solutions monotonically from above and below. They have such an important advantage in comparison with other approximate methods that they place the sought solution in a "plug"at each step of the iterative process which makes it possible to obtain a posteriori error estimate of the calculations.

The aim of this paper is to prove the existence and uniqueness of the regular positive solution of the Liouville-Gelfand problem and the possibility of constructing two-sided approximations to it.

In this work we consider the boundary-value problem for the nonlinear elliptic equation in the bounded domain $\Omega \subset R^N$

$$\begin{aligned} -\Delta u = \lambda e^u \quad \forall x \in \Omega, \quad u > 0, \\ u|_{\partial\Omega} = 0 \quad (\lambda > 0)\,. \end{aligned} \tag{1}$$

The equation of (1) is the stationary equation of the thermal-ignition theory at constant thermal conductivity, $u\,(x)$ is the temperature at the point x, the parameter λ represents all the quantities that are essential for problems of the thermal-ignition theory [1].

2 Literature review

The formulation of this problem belongs to Frank-Kamenetskii [1] and Zeldovich [2]. The same problem arises in the study of prescribed curvature problems [3, 4].

If the domain Ω is the unit ball in R^N, then by the classical result of Gidas, Ni and Nirenberg [5], all positive solutions of (1) are radially symmetric, reducing (1) to the boundary value problem

$$\begin{aligned} u'' + (N-1)\,/ru' + \lambda e^u = 0, \quad r \in (0,1)\,, \\ u'(0) = u(1) = 0, \quad u(r) = u(|x|)\,. \end{aligned} \tag{2}$$

For $N = 1$ this equation was first solved by Liouville in 1853 [6], using reduction of order methods. In 1914, Bratu [7] found an explicit solution of (2) when $N = 2$. For $N = 3$ numerical progress was made in 1934 by both Frank-Kamenetskii [1] in his study of combustion theory and Chandrasekhar [8] in his study of isothermal gas stars. In 1963, Gelfand published a comprehensive paper [9] hat included a review of (2) for $N = 1, 2, 3$. Approximately ten years later Joseph and Lundgren [10] determined the multiplicity of solutions for all N.

The problem (1) also attracted the attention of many other authors [11, 12, 13, 14]. However, they often considered (1) in fairly simple domains and found the exact solutions in cases where this was possible. In this paper we investigate a nonlinear operator equation that is equivalent to (1). The investigation is based on methods in nonlinear operator equations theory in half-ordered spaces [15, 16, 17]. This approach allows us to obtain theoretical results for almost any domain and justify the method two-sided approximations. Moreover, we impose a condition on the numerical parameter of the problem λ and on the introduced parameter β which is an a priori estimate of the sought solution.

3 Material and methods

The problem (1) is a particular case of a more general problem

$$\begin{aligned} -\Delta u = f\,(\lambda, x, u) \quad \forall x \in \Omega \subset R^N, \\ u > 0, \quad u|_{\partial\Omega} = 0. \end{aligned} \tag{3}$$

We assume that $f(\lambda, x, u) \geq 0$ in $\bar{\Omega}$, $\lambda > 0$. It is known [15, 16, 17] that (3) is equivalent to the operator equation in the class of continuous functions in Ω

$$u(x) = \int\limits_{\Omega} G(x, s) f(\lambda, s, u(s)) ds, \quad (4)$$

where $G(x, s)$ is a Green's function of the operator Δ of the Dirichlet problem in the domain Ω, $x = (x_1, \ldots, x_N)$, $s = (s_1, \ldots, s_N)$.

Let $A(\lambda, u)$ be an operator with the domain $D(A) = K$

$$A(\lambda, u) = \int\limits_{\Omega} G(x, s) f(\lambda, s, u(s)) ds,$$

where K is a cone of nonnegative functions in the space $C(\bar{\Omega})$.

We will investigate questions related to the positive solutions of (1) and hence the equivalent operator equation (4) using methods in nonlinear operator equations theory in half-ordered spaces. Let us give some definitions and main conclusions of this theory [15, 16, 17].

Definition 1 *Let E be a real Banach space. A convex closed set $K \subset E$ is called a cone if $au \in K$ $(a \geq 0)$ and $-u \notin K$ follows from $u \in K$, $u \neq 0$.*

Using the cone K in E we introduce a half-order as follows:

$$u < v, \text{ if } v - u \in K, \quad u, v \in E.$$

Definition 2 *The cone K is called normal if there exists an $N(K)$ such that $\|u\| \leq N(K)\|v\|$ for $0 < u < v$.*

It is known [15] that the cone of non-negative functions is normal in the space $C(\Omega)$.

Definition 3 *An operator A is positive if $AK \subset K$.*

Definition 4 *An operator A is monotone on the set $T \subset E$ if $Au \leq Av$ follows from $u \leq v$ $(u, v \in T)$.*

Definition 5 *A positive operator in K is called concave if there exists a fixed non-zero element $u_0 \in K$ such that for any non-zero $u \in K$*

$$B_1(u) u_0 \leq Au \leq B_2(u) u_0$$

where $B_1 > 0$, $B_2 > 0$, and also $\forall t \in (0, 1)$

$$A(tu) \geq tAu. \quad (5)$$

Definition 6 *A concave operator A is called u_0-concave if (5) is replaced by a stronger condition: $\forall t \in (0, 1)$ there exists an $\eta(u, t) > 0$ such that*

$$A(tu) \geq (1 + \eta)\ t(Au).$$

Definition 7 *A collection of elements* $\langle v_0, w_0\rangle = \{u : v_0 \le u \le w_0\}$ *is called the conical interval.*

Definition 8 *A conical interval* $\langle v_0, w_0\rangle$ *is called invariant for a monotone operator* A *if* A *transforms* $\langle v_0, w_0\rangle$ *into itself, that is* $Av_0 \ge v_0$, $Aw_0 \le w_0$.

The following theorems hold.

Theorem 1 *[15, Theorem 4.1]. It suffices for the existence, for the monotone operator* A*, of at least one fixed point that there exists an invariant conical interval and that the cone* K *is normal and the operator* A *is completely continuous.*

Theorem 2 *[15, Theorem 4.4]. Let* A *be a monotone operator on the invariant conical interval* $\langle v_0, w_0\rangle$ *and has the unique fixed point* u^* *in* $\langle v_0, w_0\rangle$*. Let* K *be a normal cone and the operator* A *be completely continuous. Then successive approximations*

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \ldots, \tag{6}$$

converge in the norm of the space $C\left(\bar{\Omega}\right)$ *to the exact solution* u^* *of* (3)*, whatever the initial approximation* $\tilde{u} \in \langle v_0, w_0\rangle$ *is.*

Remark 1 *From the uniqueness of the fixed point it follows that the limits of* (6) *coincide.*

If $A(\lambda, u) = \lambda Au$, the following theorem holds.

Theorem 3 *[15, Theorem 6.3]. If the operator* A *is* u_0*-concave and monotone, then the equation* $u = \lambda Au$ *does not have two distinct non-zero solutions in the cone* K *for any value of the parameter* λ*.*

Let us investigate the properties of the operator that corresponds to (1)

$$A(\lambda, u) = \lambda \int\limits_{\Omega} G(x, s)\, e^{u(s)} ds, \quad D(A) = K. \tag{7}$$

It is obvious that the operator A is monotone, since $u_1 \le u_2$ is followed by $Au_1 \le Au_2$. In addition, the operator A is completely continuous in the cone K [16, 17].

Let us build the invariant conical interval $\langle v_0, w_0\rangle \subset K$. We put $u = v_0 = 0$ in (7) and build the element $v_1 = \lambda \int\limits_{\Omega} G(x, s)\, e^{v_0(s)} ds = \lambda \int\limits_{\Omega} G(x, s)\, ds \ge v_0 = 0$. Having v_1, we build the element $v_2(x) = \lambda \int\limits_{\Omega} G(x, s)\, e^{v_1(s)} ds \ge v_1$. Continuing this process, we obtain the relations $0 = v_0 \le v_1 \le v_2 \le \cdots \le v_n$. If we put $u = w_0 = \beta = const > 0$ in (7) we obtain the element $w_1 = \lambda \int\limits_{\Omega} G(x, s)\, e^{w_0(s)} ds = \lambda e^{\beta} \int\limits_{\Omega} G(x, s)\, ds$. The parameters λ and β are chosen in such a way that $w_1 \le w_0 = \beta$ which leads to the condition $\lambda e^{\beta} \int\limits_{\Omega} G(x, s)\, ds \le \beta\ \forall x \in \Omega$. It now follows that

$$\max_{x \in \bar{\Omega}} \int\limits_{\Omega} G(x, s)\, ds \le \frac{1}{\lambda} \beta e^{-\beta}. \tag{8}$$

Building the elements w_i is similar to the process for v_i. We obtain the inequalities

$$0 = v_0 \le v_1 \le \dots \le v_n \le \dots \le w_n \le \dots \le w_1 \le w_0 = \beta,$$

therefore the conical interval $\langle v_0, w_0 \rangle = \langle 0, \beta \rangle$ is invariant for the operator $A(\lambda, u)$.

In order to prove the concavity of the operator A we use Definition 5. We compose

$$A(\lambda, tu) - tA(\lambda, u) = \lambda \int_\Omega G(x,s) \left(e^{tu(s)} - te^{u(s)} \right) ds.$$

It suffices for this difference to be nonnegative that $e^{tu} \ge te^u$ $\forall t \in (0,1)$, $u > 0$, whence $tu \ge \ln t + u$, or $u(t-1) \ge \ln t$, or since $t \in (0,1)$, $u \le \frac{\ln t}{t-1}$. Let φ denote the function $\varphi(t) = \frac{\ln t}{t-1}$, $0 < t < 1$. Since $\varphi(+0) = \lim\limits_{t \to +0} \varphi(t) = +\infty$, $\varphi(1-0) = \lim\limits_{t \to 1-0} \varphi(t) = \lim\limits_{t \to 1-0} \frac{\ln t}{t-1} = 1$, it follows that the sought solution $u^*(\lambda, x)$ of (1) satisfies the condition $0 < u^*(\lambda, x) < 1$, which coincides with the results of Frank-Kamenetskii [1]. Let u_0 be $u_0(x) = \int\limits_\Omega G(x,s)\, ds$. Then since $u \in \langle v_0, w_0 \rangle$ it follows that the inequalities (5) are satisfied.

In order to prove the u_0-concavity of the operator A, where $u_0(x) = \int\limits_\Omega G(x,s)\, ds$, we compose the difference

$$A(\lambda, tu) - (1+\eta)\, tA(\lambda, u) = \lambda \int_\Omega G(x,s) \left(e^{tu(s)} - (1+\eta)\, te^{u(s)} \right) ds.$$

It suffices for this difference to be nonnegative that $e^{tu} - (1+\eta)\, te^u \ge 0$ $\forall t \in (0,1)$, $u > 0$, whence it follows that $0 < \eta(u,t) \le \frac{e^{tu} - te^u}{te^u}$, which proves the u_0-concavity of the operator A. Thus, we have just proved the following theorem.

Theorem 4 *The problem* (1) *has the unique nonnegative regular solution* $u^* \in C(\bar{\Omega})$ *in the cone segment* $\langle v_0, w_0 \rangle$, $v_0 = 0$, $w_0 = \beta$ *which can be constructed with two-sided approximations according to the scheme*

$$\begin{aligned} v_n(x) &= \lambda \int_\Omega G(x,s)\, e^{v_{n-1}(s)} ds, \quad n = 1, 2, \dots, \\ w_n(x) &= \lambda \int_\Omega G(x,s)\, e^{w_{n-1}(s)} ds, \quad n = 1, 2, \dots, \end{aligned} \tag{9}$$

which converge uniformly to the sought solution if λ *and* β *satisfy* (8).

Remark 2 *It follows from Theorem 3 that* (1) *does not have two distinct nonnegative regular solutions for any value of the parameter* λ *in the cone* K.

Now we prove the following theorem which has a direct relation to (1) using the technique of proving a similar theorem in [18].

Theorem 5 *Let operator* $A(\lambda, u)$ *be monotone and concave for each* $\lambda > 0$ *and monotonically increasing for each* $u \in K$ *with respect to* λ *and satisfy the condition*

$$A(t\lambda, u) \le \frac{1}{t} A(\lambda, u), \quad t \in (0,1]. \tag{10}$$

Let u_1 *and* u_2 *be positive solutions of the equation* $u = A(\lambda, u)$ *which correspond to two distinct values* λ_1 *and* λ_2, $\lambda_1 < \lambda_2$. *Then* $u_1 < u_2$.

Proof. Suppose that it follows from $\lambda_1 < \lambda_2$ that $u_1 > u_2$. Let τ_0 be the maximum constant such that $\tau_0 u_1 \leq u_2$ and $tu_1 \geq u_2$ if $t > \tau_0$, $t \in (0,1]$. Obviously, $\tau_0 \in (0,1]$. According to the statement of the theorem we have $u_2 = A(\lambda_2, u_2) \geq$ [since $u_2 \geq \tau_0 u_1$] $\geq A(\lambda_2, \tau_0 u_1) \geq *$. It follows from the operator A concavity in the variable u that the inequality (5) can be rewritten as: $A(\lambda_2, \tau_0 u_1) \geq \tau_0 A(\lambda_2, u_1)$ since $\tau_0 \in (0,1)$, and therefore we have

$$* \geq \tau_0 A(\lambda_2, u_1) = \tau_0 A\left(\frac{\lambda_2}{\lambda_1}\lambda_1, u_1\right) \geq [(10)] \geq \tau_0 \frac{\lambda_2}{\lambda_1} A(\lambda_1, u_1) = \tau_0 \frac{\lambda_2}{\lambda_1} u_1.$$

Thus, we have obtained that $u_2 \geq \tau_0 \frac{\lambda_2}{\lambda_1} u_1$. Further, it follows from the maximality of the constant τ_0 that $\frac{\lambda_2}{\lambda_1} \leq 1$ or $\lambda_2 \leq \lambda_1$ which contradicts the assumption $\lambda_1 < \lambda_2$. This completes the proof of the theorem.

Now we show that all conditions of Theorem 5 are satisfied with respect to (1). The monotonicity and concavity of the operator $A(\lambda, u)$ of the form (7) are shown at the beginning of this section. Assume that $\lambda_1 < \lambda_2$, it follows that $A(\lambda_1, u) - A(\lambda_2, u) = (\lambda_1 - \lambda_2) \int\limits_\Omega G(x,s) e^{u(s)} ds < 0$, that is, the operator A is increasing in the variable λ $\forall u \in K$. Next, we compose the difference $A(t\lambda, u) - \frac{1}{t} A(\lambda, u) = \frac{\lambda(t^2-1)}{t} \int\limits_\Omega G(x,s) e^{u(s)} ds \leq 0$ $\forall t \in (0,1]$, which proves (10). Thus, two different values λ_1 and λ_2, $\lambda_1 < \lambda_2$, correspond to two positive solutions u_1 and u_2, having $u_1 < u_2$.

4 Results and discussion

Computational experiments for (1) are conducted in four domains for different values of the parameter λ and the corresponding values of the parameter β.

For the domain $\Omega_1 = \{(x_1, x_2) | 1 - x_1^2 - x_2^2 > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 1.47151$, the corresponding value of β is $\beta = 0.99999$.

For the domain $\Omega_2 = \{(x_1, x_2) | x_2 (1 - x_1^2 - x_2^2) > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 3.79257$ with $\beta = 0.99999$. Table 1 lists the values of $w_{11}(x)$ (in the numerator) and $v_{11}(x)$ (in the denominator) at the points of Ω_2 with the polar coordinates (ρ_i, φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i = \overline{0,5}$, $j = \overline{0,5}$ (the values in the other quarter are symmetric). Figure 1 and 2 show the surface and the level lines of the approximate solution $w_{11}(x)$ respectively and Figure 3 shows the graphs of $w_n(0, x_2)$ (solid line) and $v_n(0, x_2)$ (dashed line) for $n = \overline{0,5}$.

For the domain $\Omega_3 = \{(x_1, x_2) | (1 - x_1^2)(1 - x_2^2) > 0\}$ the maximum value of λ^* which satisfies (8) is $\lambda^* = 1.24704$ with $\beta = 0.99999$. Table 2 lists the values of $w_{11}(x)$ (in the numerator) and $v_{11}(x)$ (in the denominator) at the points of Ω_3 with coordinates $(-1 + 0.2i, -1 + 0.2j)$, where $i = \overline{0,5}$, $j = \overline{0,5}$ (the values in the other quarters are symmetrical).

The dependency of the norm $\|u_n\|$ in the space $C(\Omega_i)$, $i = \overline{1,3}$ from λ is shown in Figure 4 in the form of graphs for Ω_1 (solid line), Ω_2 (dashed line) and Ω_3 (dotted line), where $u_n = \frac{v_n + w_n}{2}$.

Hence it follows that if λ tends to zero then the desired solution $u(x)$ tends to zero too.

Since the Green's function is known for several fairly simple domains we apply the Green's quasifunction method for the solution of (3) in the regions Ω_2 and Ω_3 and compare the results

Table 1: The values of $w_{11}(x)$ and $v_{11}(x)$ at the points of Ω_2

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0	0	0	0	0	0
	0	0	0	0	0	0
0.2	0	0.12494	0.22831	0.30383	0.34954	0.36482
	0	0.12493	0.22828	0.30378	0.34949	0.36477
0.4	0	0.21457	0.36929	0.47301	0.53271	0.55213
	0	0.21455	0.36924	0.47294	0.53262	0.55204
0.6	0	0.23401	0.37849	0.46710	0.51620	0.53179
	0	0.23399	0.37845	0.46703	0.51612	0.53170
0.8	0	0.16504	0.24750	0.29450	0.31992	0.32787
	0	0.16503	0.24748	0.29446	0.31987	0.32782
1	0	0	0	0	0	0
	0	0	0	0	0	0

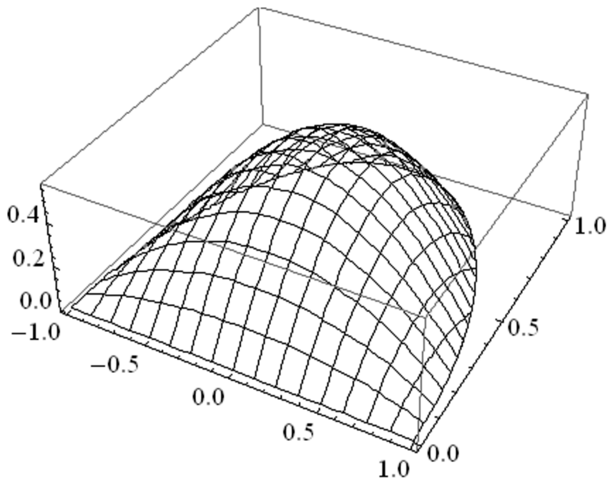

Figure 1: The surface of $w_{11}(x)$

with those obtained according to the scheme (9).

The essence of the Green's quasifunction method in Rvachev's interpretation [19] (for linear partial differential equations) with our adjustments for nonlinear partial differential equations [20, 21, 22] consists in the transition from the boundary value problem (1) to the equivalent nonlinear integral equation

$$u(x) = \int\limits_{\Omega} G_q(x,s)\,\lambda e^{u(s)} ds + \int\limits_{\Omega} u(s)\,K(x,s)\,ds, \tag{11}$$

where

$$G_q(x,s) = \frac{1}{2\pi}\left(\ln\frac{1}{r} - \zeta(x,s)\right), \quad \zeta(x,s) = -\frac{1}{2}\ln\left(r^2 + 4\omega(x)\,\omega(s)\right),$$

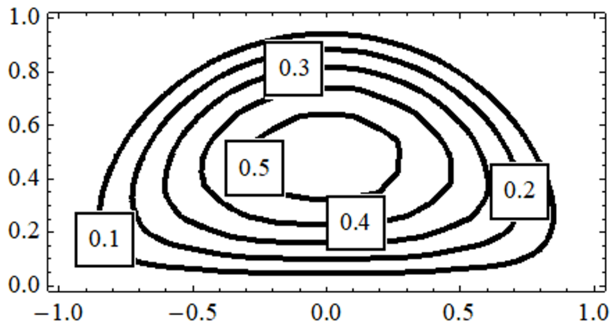

Figure 2: The level lines of $w_{11}(x)$

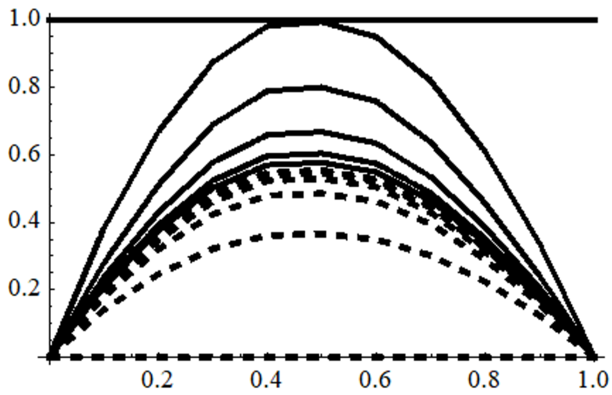

Figure 3: The graphs of $w_n(0, x_2)$ (solid line) and $v_n(0, x_2)$ (dashed line) for $n = \overline{0, 5}$

Table 2: The values of $w_{11}(x)$ and $v_{11}(x)$ at the points of Ω_2

x_1	x_2					
	-1	-0.8	-0.6	-0.4	-0.2	0
-1	0	0	0	0	0	0
	0	0	0	0	0	0
-0.8	0	0.08458	0.13846	0.17550	0.19564	0.20311
	0	0.08457	0.13845	0.17549	0.19562	0.20310
-0.6	0	0.13846	0.23664	0.30471	0.34332	0.35683
	0	0.13845	0.23662	0.30468	0.34328	0.35679
-0.4	0	0.17550	0.30471	0.39602	0.44870	0.46700
	0	0.17549	0.30468	0.39598	0.44865	0.46695
-0.2	0	0.19564	0.34332	0.44870	0.51022	0.53146
	0	0.19562	0.34328	0.44865	0.51016	0.53140
0	0	0.20311	0.35683	0.46700	0.53146	0.55376
	0	0.20310	0.35679	0.46695	0.53140	0.55370

$$K(x, s) = -\frac{1}{2\pi}\Delta_s \zeta(x, s),$$

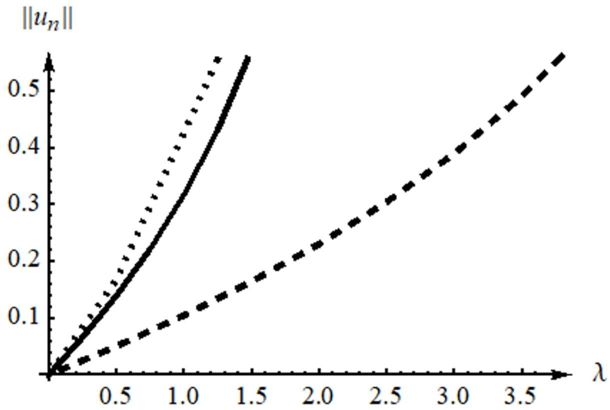

Figure 4: The dependency of the norm $\|u_n\|$ from λ for Ω_1 (solid line), Ω_2 (dashed line) and Ω_3 (dotted line)

for $\Omega \subset R^2$ and

$$G_q(x,s) = \frac{1}{4\pi}\left(\frac{1}{r} - \zeta(x,s)\right), \quad \zeta(x,s) = \left(r^2 + 4\omega(x)\omega(s)\right)^{-\frac{1}{2}},$$

$$K(x,s) = -\frac{1}{4\pi}\Delta_s \zeta(x,s),$$

for $\Omega \subset R^3$. Also in both cases

$$r = |x-s|; \quad \Delta_s = \sum_{i=1}^{N} \frac{\partial^2}{\partial s_i^2}, \quad s \in \Omega \subset R^N; \quad \omega(x) = \begin{cases} > 0 & \forall x \in \Omega, \\ 0 & \forall x \in \partial\Omega. \end{cases}$$

We use the method of successive approximations in Svirsky's interpretation [23] to construct an approximate solution of (11) which leads us to a sequence of linear integral equations

$$u_{n+1}(x) - \int\limits_{\Omega} u_{n+1}(s) K(x,s)\, ds = \int\limits_{\Omega} G_q(x,s) \lambda e^{u_n(s)} ds, \quad n = 1, 2, \ldots,$$

where we put $u_1(x) = 0$.

Each of these equations can be solved by the Bubnov-Galerkin method [23]. We obtain the following sequence of approximate solutions

$$u_n(x) = \sum_{i=1}^{k} c_{n,i} \phi_i(x), \quad n = 1, 2, \ldots,$$

where $\{\phi_i(x)\}_{i=1}^{k}$ is a coordinate sequence, $c_{n,i}$ $(i = \overline{1,k},\ n = 2, 3, \ldots)$ is a solution of a system of linear algebraic equations

$$\sum_{i=1}^{k} c_{2,i} \left[\int\limits_{\Omega} \phi_i(x) \phi_j(x)\, dx - \int\limits_{\Omega} \int\limits_{\Omega} K(x,s) \phi_i(s) \phi_j(x)\, ds dx \right] =$$

$$= \int_\Omega \int_\Omega G_q(x,s)\,\lambda e^{u_1(s)} \phi_j(x)\,dsdx, \quad j = \overline{1,k},$$

$$\sum_{i=1}^{k} c_{n,i} \left[\int_\Omega \phi_i(x)\,\phi_j(x)\,dx - \int_\Omega \int_\Omega K(x,s)\,\phi_i(s)\,\phi_j(x)\,dsdx \right] =$$

$$= \int_\Omega \int_\Omega G_q(x,s)\,\lambda e^{u_{n-1}(s)} \phi_j(x)\,dsdx, \quad j = \overline{1,k}, \quad n = 3, 4, \ldots.$$

We use the Legendre polynomials which are orthogonal on the segment $[-1,1]$ to construct the coordinate sequence

$$P_i(z) = \frac{1}{2^i i!} \frac{d^i}{dz^i} \left(z^2 - 1\right)^i, \quad z \in R.$$

For the domain $\Omega_2 = \{(x_1,x_2)\,|x_2(1 - x_1^2 - x_2^2) > 0\}$, $\lambda^* = 3.79257$ and $\beta = 0.99999$ Table 3 lists the values of $u_n(x)$ for $n = 10$ at the points of Ω_2 with the polar coordinates (ρ_i, φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i = \overline{0,5}$, $j = \overline{0,5}$ (the values in the other quarter are symmetric).

Table 3: The values of $u_n(x)$ for $n = 10$ at the points of Ω_2

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0	0	0	0	0	0
0.2	0	0.12847	0.23407	0.31131	0.35798	0.37355
0.4	0	0.21253	0.37045	0.47570	0.53492	0.55393
0.6	0	0.22795	0.37949	0.47028	0.51744	0.53199
0.8	0	0.15910	0.25238	0.30189	0.32573	0.33292
1	0	0	0	0	0	0

For the domain $\Omega_3 = \{(x_1,x_2)\,|\,(1 - x_1^2)(1 - x_2^2) > 0\}$, $\lambda^* = 1.24704$ and $\beta = 0.99999$ Table 4 lists the values of $u_n(x)$ for $n = 10$ at the points of Ω_3 with coordinates $(-1 + 0.2i, -1 + 0.2j)$, where $i = \overline{0,5}$, $j = \overline{0,5}$ (the values in the other quarters are symmetric).

Now we apply the Green's quasifunction method to (3) for the domain $\Omega_4 = \{(x_1,x_2)\,|1 - x_1^8 - x_2^8 > 0\}$. We use the inequality $\lambda < \frac{\beta e^{-\beta}}{\max_{x\in\Omega_3} \int_{\Omega_3} G(x,s)ds}$ to select the values of the parameter λ, where Ω_3 is the smallest square containing Ω_4. Hence we have $\lambda^* = 1.24704$, $\beta = 0.99999$. Table 5 lists the values of $u_n(x)$ for $n = 9$ at the points of Ω_4 with polar coordinates (ρ_i, φ_j), where $\rho_i = 0.2i$, $\varphi_j = \frac{\pi j}{10}$, $i = \overline{0,5}$, $j = \overline{0,5}$ (the values in the other quarters are symmetric).

In contrast to the authors who solved the Liouville-Gelfand problem in some rather simple domains and for the most part found solutions in cases where the equations of the problem could be reduced to an ordinary differential equation, in our work we propose a technique for finding a regular solution in almost any domain. However, it should be noted that we have not considered the solutions multiplicity, but proved the existence and uniqueness of a regular solution of (1).

Table 4: The values of $u_n(x)$ for $n = 10$ at the points of Ω_3

x_1	x_2					
	-1	-0.8	-0.6	-0.4	-0.2	0
-1	0	0	0	0	0	0
-1	0	0	0	0	0	0
-0.8	0	0.07651	0.13407	0.17415	0.19778	0.20559
-0.6	0	0.13406	0.23487	0.30502	0.34638	0.36004
-0.4	0	0.17411	0.30500	0.39605	0.44971	0.46744
-0.2	0	0.19772	0.34632	0.44969	0.51060	0.53071
0	0	0.20553	0.35998	0.46741	0.53071	0.55161

Table 5: The values of $u_n(x)$ for $n = 9$ at points of Ω_4

ρ	φ					
	0	$\frac{\pi}{10}$	$\frac{\pi}{5}$	$\frac{3\pi}{10}$	$\frac{2\pi}{5}$	$\frac{\pi}{2}$
0	0.50295	0.50295	0.50295	0.50295	0.50295	0.50295
0.2	0.48881	0.48883	0.48884	0.48885	0.48885	0.48885
0.4	0.44608	0.44621	0.44636	0.44635	0.44621	0.44611
0.6	0.36934	0.37146	0.37444	0.37440	0.37140	0.36930
0.8	0.23022	0.24560	0.26746	0.26739	0.24544	0.23008
1	—	0.04938	0.11977	0.11966	0.04927	—

5 Conclusion

In this paper we have proven the possibility of constructing of two-sided approximations to regular positive solutions of the Liouville-Gelfand problem. We have obtained the conditions that guarantee the convergence of the two-sided iterative process. Constructing the cone segment $\langle v_0, w_0 \rangle$, we have obtained an a priori estimate of the sought solution u^*, since $v_0 \leq u^* \leq w_0$. The obtained two-sided approximations to the solution of the problem makes it possible to make a posteriori conclusions.

One of the advantages of the applied method in comparison with others is the relatively simple algorithm in terms of implementation.

We note that for the first time we have constructed two-sided approximations for the Liouville-Gelfand problem in certain domains for which Green's function of the problem is known. We propose to use Green's quasifunction method in case of complex domains where Green's function is unknown. We have improved the method to solve boundary value problems for nonlinear elliptic equations. The above-mentioned represents the scientific novelty of the results.

The practical value lies in the fact that this approach can be used to find solutions to applied problems with mathematical models represented by boundary value problems for nonlinear elliptic equations.

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3-бөлім Раздел 3 Section 3

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Three dimensional visualization of models and physical characteristics of oil and gas reservoir for virtual reality systems

Akhmed-Zaki D.Zh., University of International Business
Almaty, Kazakhstan, E-mail: Darhan.Ahmed-Zaki@kaznu.kz
Turar O.N., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: Olzhas.Turar@kaznu.ru
Rakhymova A.R., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: Aktumar@mail.ru

The paper describes three-dimensional visualization of grid models of oil and gas reservoir for virtual reality systems. It was implemented in a C ++ programming language, for visualization of the model using the OpenGL library and in the virtual environment of the OpenVR library, which needs use of the SteamVR utility. Created module of visualization requires connection of special equipment for operations with the virtual environment, such as headset with its own display, base stations and controllers. As input data for drawing of model geometrical data and physical characteristics of oil field in .GRDECL format provided by Shchlumberger Eclipse are offered. Files of this format store data describing three-dimensional models consisting of $N_x \times N_y \times N_z$ of cells on Ox, Oy and Oz, which represent the distorted parallelepipeds. The advantage of using virtual reality in visualization is that for the observer visual perceptions considerably improves, and immersion in a virtual environment is accompanied by the effect of presence. In the VR display the quality of drawing of an object significantly differs from what can be watched on a flat screen monitor.

Key words: computer graphics, computer animation, machine graphics, virtual reality, OpenGL, OpenVR, shader, visualization, grid model visualization.

Трехмерная визуализация модели и физических характеристик нефтегазового пласта для систем виртуальной реальности

Ахмед-Заки Д.Ж., Университет Международного Бизнеса
Алматы, Казахстан, E-mail: Darhan.Ahmed-Zaki@kaznu.kz
Турар О.Н., Казахский национальный университет имени Аль-Фараби
Алматы, Казахстан, E-mail: Olzhas.Turar@kaznu.ru
Рахымова А.Р., Казахский национальный университет имени Аль-Фараби
Алматы, Казахстан, E-mail: Aktumar@mail.ru

В работе описаны основные действия для трехмерной визуализаций сеточных моделей нефтяных и газовых месторождений для систем виртуальной реальности. Работа была реализована на языке программирования C ++, для визуализации модели была использована библиотека OpenGL и для визуализации модели в виртуальной среде использовалась библиотека OpenVR в дополнении с программой SteamVR. Созданный модуль визуализации требует подключения специальных оборудований для работы с виртуальной средой, таких как шлем виртуальной реальности, базовые станции и контроллеры. В качестве входных данных для прорисовки модели предложены геометрические данные и физические характеристики модели в формате .GRDECL. Данный формат создан фирмой Shchlumberge

Eclipse и используется для описания моделирования нефтяного месторождения. Файлы такого формата хранят данные описывающие трехмерные модели, состоящие из $N_x \times N_y \times N_z$ ячеек по Ox, Oy и Oz, которые представляют собой искаженные параллелепипеды. Преимущество применения виртуальной реальности при визуализации состоит в том, что для наблюдателя визуальное восприятия значительно улучшается, также погружение в виртуальную среду сопровождается эффектом присутствия. В виртуальных очках качество прорисовки объекта существенно отличается от того, что можно наблюдать на плоском экране монитора.
Ключевые слова: компьютерная графика, компьютерная анимация, машинная графика, виртуальная реальность, OpenGL, OpenVR, шейдер, визуализация, визуализация сеточной модели.

Виртуалды шындық жүйелеріне арналған мұнай және газ қабатының моделін және физикалық сипаттамаларын үш өлшемді визуализациялау

Ахмед-Заки Д.Ж., Халықаралық Бизнес Университеті
Алматы қ., Қазақстан, E-mail: Darhan.Ahmed-Zaki@kaznu.kz
Тұрар О.Н., Әл-Фараби атындағы Қазақ Ұлттық Университеті,
Алматы қ., Қазақстан, E-mail: Olzhas.Turar@kaznu.ru
Рахымова А.Р., Әл-Фараби атындағы Қазақ Ұлттық Университеті,
Алматы қ., Қазақстан, E-mail: Aktumar@mail.ru

Мақалада виртуалды шындық жүйесінде мұнай және газ қабаты моделінің торлы үлгілерін үш өлшемді визуализациялаудың негізгі әрекеттері сипатталады. Жұмыс С ++ бағдарламалау тілінде іске асырылды, модельді бейнелеу үшін OpenGL кітапханасы және виртуалды ортада модельді визуализациялау үшін OpenVR кітапханасы және қосымша SteamVR бағдарламасы пайдаланылды. құрылған визуализация модулі виртуалды ортада жұмыс істеу үшін виртуалды шындық көзәйнегі, базалық станциялар және контроллерлер секілді арнайы жабдықтарды қосуды талап етеді. Модельді визуализациялау үшін қолданылатын деректер ретінде .GRDECL форматындағы геометриялық деректер және модельдің физикалық сипаттамалары ұсынылады. Schlumberge Eclipse фирмасымен құрылған .GRDECL форматы мұнай кен орнының моделін сипаттау үшін қолданылады. Бұл форматтағы файлдар Ox, Oy және Oz бойында $N_x \times N_y \times N_z$ ұяшықтарынан тұратын үш өлшемді модельді сипаттайтын деректерді сақтайды. Виртуалды шындықты пайдаланудың артықшылығы – бақылаушының көрнекті қабылдауы айтарлықтай жақсарады және бақылаушы өзге виртуалды ортада қатысу әсеріне ие болады. Виртуалды шындық көзәйнегінде объектіні сызу сапасы тегіс экранды монитордағы бейнелеуден айтарлықтай ерекшеленеді.
Түйін сөздер: компьютерлік графика, компьютерлік анимация, машиналық графика, виртуалды шындық, OpenGL, OpenVR, шейдер, визуализация, торлы модельдің визуализациясы.

1 Introduction

Visualization is an integral part of science, which represents evident display of big arrays of numerical and other information, which is an obviously possible thanks to computer graphics. Currently, the computer graphics has a wide application, both in scientific activities and in everyday life. In scientific activities the computer graphics helps to build the virtual three-dimensional objects for the analysis of simulation results, presentation of work, etc. And also this branch found the application in a pattern of computer games, for creating animated films and special effects for movies.

There visualization in a format of the virtual reality, which demonstrates projects in head mounted display (HMD) or in special rooms of virtual reality, is described. Today the virtual reality founds the application in many spheres. For example, in science the virtual reality allows to plunge into the environment and in details to research different models. In architecture the virtual environment considerably reduces expenses by replacing construction of expensive products with an illustration of the virtual model, which also allows to research a product and even to test different technical characteristics. The virtual environment, created for trainings and preliminary training in spheres of education: piloting, driving or for military tests, considerably reduces risk of different injuries and unnecessary expenses. Also, virtual reality has found the application in medicine.

Virtual reality objects are as close as possible to similar objects in the real world. They have a texture, material, behavior, close to reality, in case of collision with other objects and it can be noted in case of interaction with an object, since in a virtual environment there is the possibility of installing physics.

The paper describes a three-dimensional visualization of oil and gas reservoir model, which is performed using OpenGL [1,2] library and with the known data in a format .GRDECL in virtual reality system using OpenVR [3] library. For this purpose it is necessary to study the geometrical basic data provided in a format .GRDECL and to create the tool to read the following data: the volume, model coordinates, activity of the cells, physical characteristics of model. Then it is necessary to consider the main possibilities of OpenGL library, of shading programming language (GLSL) and to define spheres of their application and draw a three-dimensional model within the developed program with computation on GPU. In conclusion it is necessary to connect HMD, base stations, controllers and to make drawing of oil and gas reservoir model with effect of presence.

2 Literature review

Nowadays there is a research laboratory "The Collaboration Centre which works on solving problems of modeling and visualization [4, 5]. For this purpose the laboratory uses tools to support applications for the virtual and augmented reality, which work at different platforms. These tools include rooms and the head systems of virtual reality, sensor desktops, different systems of tracking, etc. The research laboratory "The Collaboration Centre"works in different directions; scientific programming, visualization, virtual reality consulting, display and smart space consulting, etc.

"TechViz" company also works in the field of virtual reality [6]. The company is engaged in different decisions for 3D – visualization, where the latest technologies are used. Also they are engaged in development of technologies in the field of computer architecture, cluster computing, etc.

The article [7] "A Collaborative Virtual Reality Oil & Gas Workflow"describes the research in oil and gas branch, in particular marine engineering. This research considers the visualization of marine engineering projects with use of the virtual reality technology. Also the specialized web interface was created for users, where they could work jointly with other users, imitating workflow in virtual reality environment. As a result of the research, the authors proposed their own version of the problem solution.

VR is used in many different science fields to improve human interaction with computations [8, 9, 10, 11].

3 Materials and methods

3.1 Input data for three-dimensional visualization of the oil and gas reservoir model and physical characteristics

There is a simulator called ECLIPSE, which is used to create hydrodynamic oil and gas field's models [12]. Source data of ECLIPSE is a plain text file, so they can be read out and used for designated purpose by means of library intended for files. ECLIPSE has two formats of a grid – .GRDECL and .EGRID, where the grid geometry is given in binary format. Format .GRDECL, created by Schlumberge Eclipse, describes modeling of the three-dimensional oil and gas reservoir consisting of $N_x \times N_y \times N_z$ of cells. The advantage of this format is to minimize the volume of the used random access memory.

To draw a grid model the main files with geometrical data, which store arrays called COORD, ZCORN, ACTNUM and an array that stores data about physical characteristics of model - NTG are used.

- The COORD file provides the array of X, Y, Z coordinates of directing vectors for cell's edges. The array of the size $(N_x + 1) \times (N_y + 1) \times 6$ contains $(N_x + 1) \times (N_y + 1)$ of vertical or oblique needles, which in turn have 2 points $(X1, Y1, Z1, X2, Y2, Z2)$. On needles, which are directing vectors, are located the vertices of cells.
- The ZCORN file provides the array Z coordinates of cell's vertices. An array of size $2N_x \times 2N_y \times 2N_z$ contains values of coordinates on an Oz axis of eight peaks of cells.
- The ACTNUM file provides the array that determines the activity of cells. The array of the $N_x \times N_y \times N_z$ size contains values 0 and 1, respectively mean inactivity and activity of cells.
- The fourth file, named according to the title of the physical parameter, provides the array, which defines color of each cell. The array of the $N_x \times N_y \times N_z$ size contains data, which are transformed to values on an interval [0, 1].

The used input data for testing of the developed program were obtained from real fields, such as the model of East Moldabek section of the Kenbay field from JSC KazMunaiGas Exploration Production [13], open data of the project "SAIGUP". Also the test model "Sample" and MATLAB Reservoir Simulation Toolbox Models: Project Data Geological Storage of CO2: Mathematical Modelling and Risk Analysis (MatMoRA) [14] and project data of "Sensitivity Analysis of the Impact of Geological Uncertainties on Production"[15] were used.

3.2 Three-dimensional visualization of model and physical characteristics of oil and gas reservoir by means of OpenGL library and the OpenGL Shading Language

The used OpenGL library [16] is the low level, hardware-independent program interface, which makes visualization [17]. In other words, it is possible to define an object by specifying the corresponding coordinates of all vertices, set color, interact with an object (rotate, reduce, increase, move), determine the location of an object or position of the observer in three-dimensional space. This interface has many sets of functions for defining operations and commands necessary for visualization of three-dimensional objects.

At the time of an application window creation the basic frame buffer is created, that is area of graphic memory. OpenGL was designed as a state machine to update the contents of the frame buffer. In fact, OpenGL is the finite state machine having different statuses. In OpenGL states can be modified by controlling buffers, various options, and then to draw on a certain context.

Using only OpenGL library leads to the fact that all work on a draw of frames is performed by the central processor. Therefore the graphic accelerator was used for implementation of the program, which gives an opportunity to write programs for computation the pixel‘s color on the screen. To use a graphics accelerator, a program called a shader is used. Applying shaders, it is possible to use most effectively all computing power of the modern graphic chips. Shaders are programmed in the C programming language similar GLSL [18] language, which is a high-level programming language.

Each shader must perform its mandatory work, that is, write some data and transfer them further on the graphic pipeline. It is a small program consisting of vertex, fragment and many other shaders and running on the GPU. There are several types of a shader:

- Vertex shader - performs transformation associated with vertex data, such as multiplying vertices and normals by a projection and modeling matrix [19], setting vertex colors, etc. The compulsory work for a vertex shader is to record the vertex position in the built-in variable gl_Position.

- Geometrical shader - a shader that can handle not only one vertex, but also a whole primitive. It can either drop the primitives, or create new ones, that is, the geometry shader is able to generate primitives.

- Fragment shader - processes each received fragment at the previous stages of the graphic pipeline. Processing can include such stages as: obtaining data from a texture, rendering light, mixing miscalculation. Mandatory operation for a fragmentary shader is to record fragment color in the built-in gl_FragColor variable.

All objects in OpenGL are presented in the form of a set of graphic primitives: in the form of points, lines and triangles. With use of these geometrical primitives and the subsequent mathematical processing of basic data, as a result, it is possible to construct difficult screen objects. Therefore for visualization of required model, it is necessary to create the instrument to identify the coordinates of all vertices of triangles.

3.3 Virtual Reality System

Virtual reality (VR) represents the three-dimensional environment generated by means of the computer, where the user can fully or partially immerse into this environment and interact with it. The probable virtual reality supports the user's sense of the reality of what is happening. VR providing interaction with the environment is called interactive. Also there is computer-generated VR and VR, available to a study, giving an opportunity to research the big detailed world.

Today there are several types of the VR. One of them has effect of complete dipping in probable simulation of the world with a high level of detailing. It uses a high-performance computer capable of recognizing user actions and responding to them in real time.

In addition special equipment connected to the computer is used, which create, output the image and provide an immersive effect in the process of the environment research. During operation HMD, base stations and controllers of the HTC VIVE model were used (fig. 1). For interaction with the virtual reality was used the program interface OpenVR [20]. It was developed for support of SteamVR technology and virtual reality equipment, where SteamVR is the environment for performing virtual reality [21].

Figure 1: Headset - HMD, base station and controller of HTC VIVE

The used HMD consists of two small screens located opposite to each eye to which images for the left and right eye are displayed; the system monitoring orientation of the device in space; the blinders preventing hit of external light. Screens show the stereoscopic images which are slightly offset from each other, providing realistic three-dimensional perception. HMD also contain the built-in accelerometers and position sensors. The most important thing in the system of this type – the accuracy of tracking operation, when tracing turns of the head for correct output of the corresponding image to displays.

There are special devices for interaction with the virtual environment - controllers. They contain the built-in position and motion sensors and also buttons and scrolling wheels, as at a computer mouse. With OpenVR library it is possible to connect controllers for interaction with objects in the virtual space. For convenient use of controllers by developers was entered an object controller, which creates even more realistic effect of presence by drawing the identical controller in the environment. In case of connection of these devices on the screen and HMD it is possible to observe how this object moves to identically real movements of controllers.

The realized program as a result gives three-dimensional model of oil and gas reservoir in virtual reality system (fig. 2 - 3). A cube $(3 \times 3 \times 3)$ was chosen as the initial test model.

After visualization, it is possible to add various transformation functions for three-dimensional objects. In graphic programming these functions of interaction are used quite often. They allow examining in detail the drawn object from all sides by moving, increasing or rotating. The idea of this method consists that all transformations are presented in the form of matrixes, where they are multiplied among themselves and then coordinates of vertices are multiplied by a final result. In the program these conversions are carried out in parts of the visualization with use of projection matrixes for the left and right eye, tracking of the position of HMD and controllers. To interact with drawn model it is necessary to use the following formula, where current position of HMD, controllers and a projection of the left

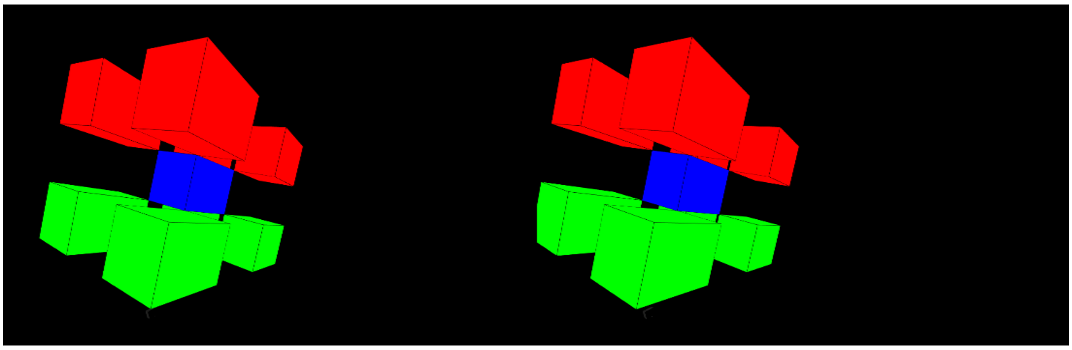

Figure 2: The constructed cube (3 × 3 × 3) in the virtual reality system. Projection of the image from the virtual reality glasses on the display. The display shows two frame buffers for the left and right eyes with different angular displacement of the image

Figure 3: The constructed cube (3 × 3 × 3) in the virtual reality system. Example of display of model in the head mounted display

and right eye can be received by means of special functions in OpenVR library.

$$M = Projection \times View \times Model \tag{1}$$

M – transformation matrix; $Projection = GetProjectionMatrix(nEye, nearClip, farClip)$ – projection matrix for specified eye; $View = HMDPosition^{-1}$ – inverse HMD position matrix; $Model = Model_{new}$ – matrix transformer, which is solved using the formula (5). Formula (5) is calculated using a well-known mathematical expression. To obtain a new transformed matrix, the transformer matrix T and the current matrix are used (2, 3).

$$Model_{new} = T \times Model_{old} \tag{2}$$

$$ControllerPosition_{new} = T \times ControllerPosition_{old} \tag{3}$$

$ControllerPosition_{old}$ – current controller position matrix; $ControllerPosition_{new}$ – new controller position matrix. Next, to find an unknown matrix $Model_{new}$, using equation (4), we derive formula (5).

$$T = ControllerPosition_{new} \times ControllerPosition_{old}^{-1} \tag{4}$$

$$Model_{new} = ControllerPosition_{new} \times ControllerPosition_{old}^{-1} \times Model_{old} \tag{5}$$

It is known that for transformation (rotation, displacement, scaling) it is necessary to multiply the resulting matrix by the starting point, where it is changed to the vector (x, y, z, w), adding the new parameter w. If $w = 0$, then this direction, if $w = 1$, then this vector is a position in space (fig. 4).

$$X_{new} = MX_{old} \quad (6)$$

X_{old} – current position vector of the model; X_{new} – new position vector of the model.

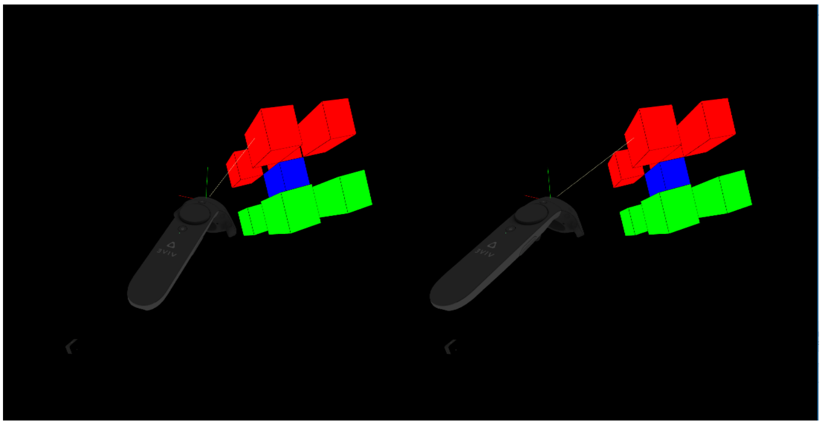

Figure 4: Connection of controllers

4 Results and discussion

For testing of the realized program different reservoir models were used. Figures 5 – 13 provided visualization of models with use of OpenGL in virtual reality system:

- Model of East Moldabek section of the field Kenbay with JSC "KazMunaiGas Exploration Production" (fig. 5 and 6), cell's quantity of complete model –$36 \times 77 \times 33$ [13].
- MATLAB Reservoir Simulation Toolbox. [17] Data of the "Geological Storage of CO2: Mathematical Modelling and Risk Analysis"(MatMoRA) project, cell's quantity of complete model – $100 \times 100 \times 11$ (fig. 7 and 8) and cell's quantity of second complete model – $100 \times 100 \times 21$ (fig. 9) [22].
- Test model of layer – Johansen, cell's quantity of complete model – $149 \times 189 \times 16$ (fig. 10)
- MATLAB Reservoir Simulation Toolbox. Data of the project "Sensitivity Analysis of the Impact of Geological Uncertainties on Production"(fig. 11), cell's quantity of complete model – $40 \times 120 \times 20$ [15].
- Sample models, cell's quantity of complete model – $33 \times 33 \times 11$ (fig. 12) and $67 \times 49 \times 10$ (fig. 11).

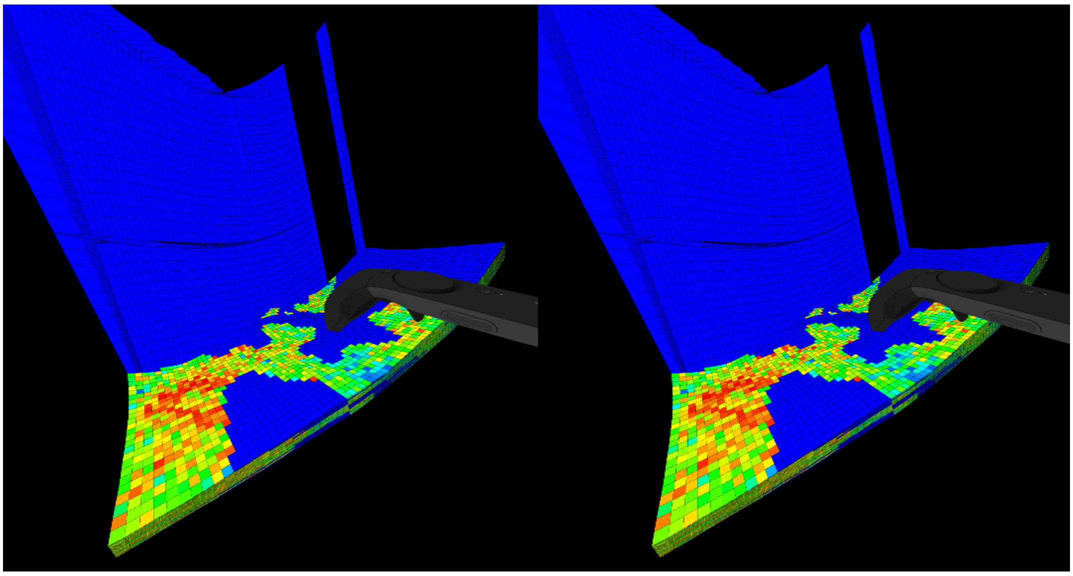

Figure 5: Test model of the field Kenbay, visualized in the virtual reality system

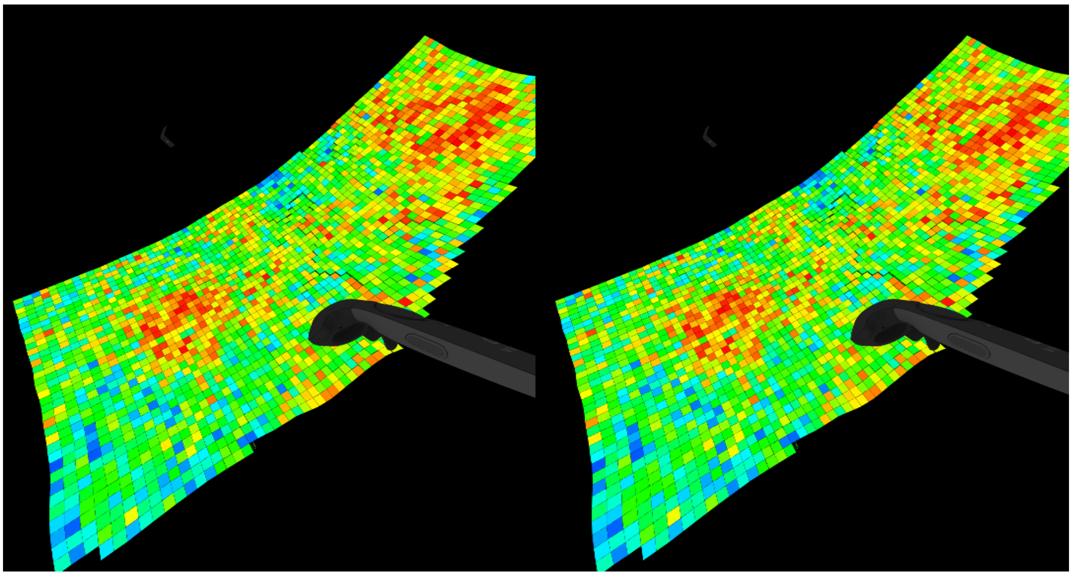

Figure 6: The edited model of the field Kenbay, visualized in the virtual reality system

In these figures, it is possible to watch an image projection from HMD to the monitor screen, where two frame buffers for the left and right eye with an angular displacement of the image are displayed. The figures also show a controller which carries out the control of test models of an oil and gas reservoir.

The developed program uses the graphic accelerator that helps to create more difficult objects, to add different matrixes for interactivity of application, to superimpose operation on GPU, at the same time reducing loading of CPU. However the developed program spends a certain amount of time for reading of large volume of data, with later processing of the

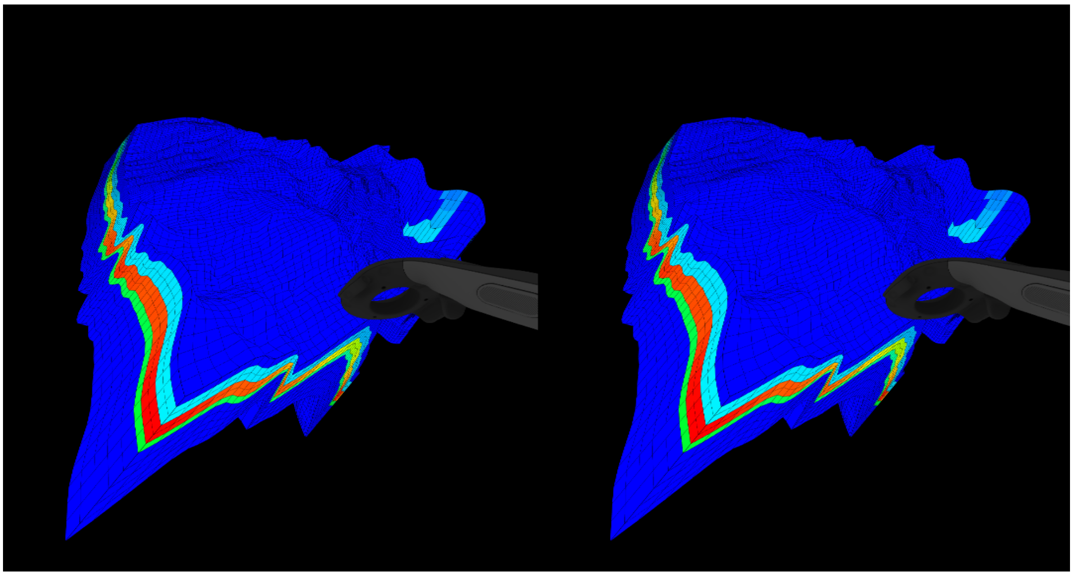

Figure 7: Test model of layer from MATLAB Reservoir Simulation Toolbox, visualized in the virtual reality system

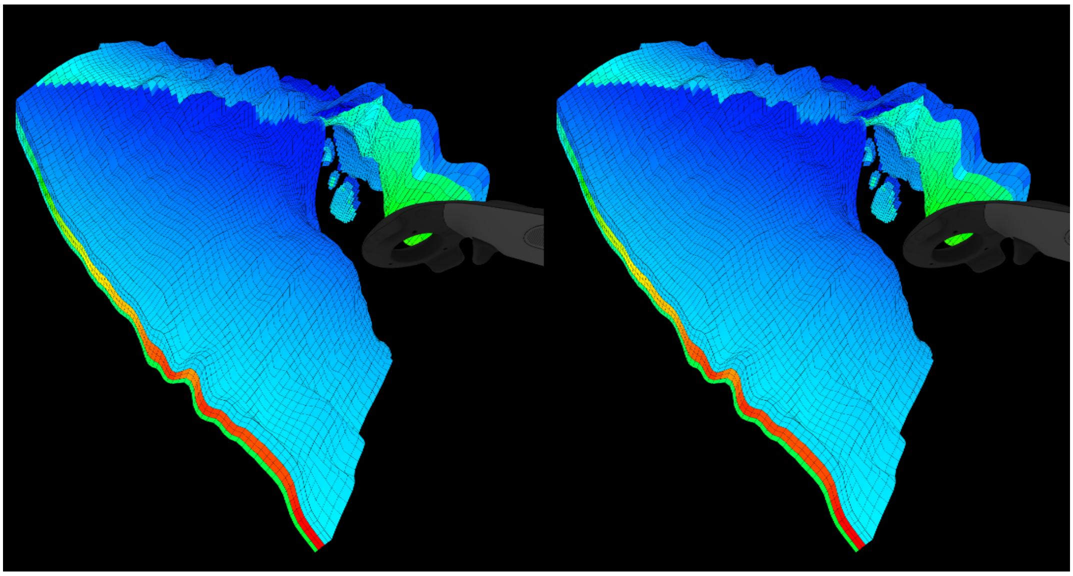

Figure 8: Test model of layer from MATLAB Reservoir Simulation Toolbox, visualized in the virtual reality system

acquired information for loading in the buffer, which means that this part of work is performed by the central processor.

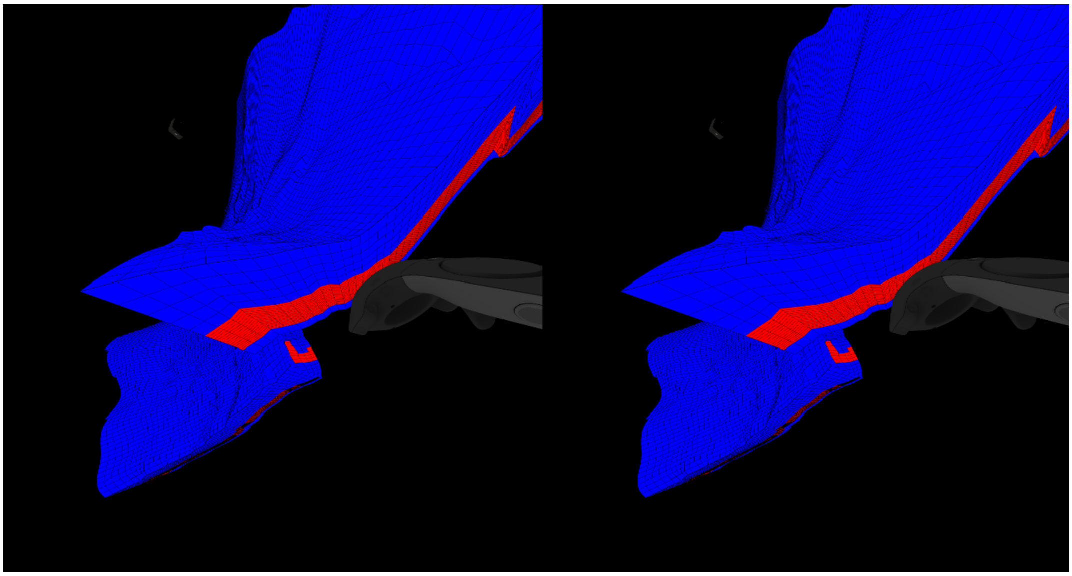

Figure 9: Test model of layer from MATLAB Reservoir Simulation Toolbox, visualized in the virtual reality system

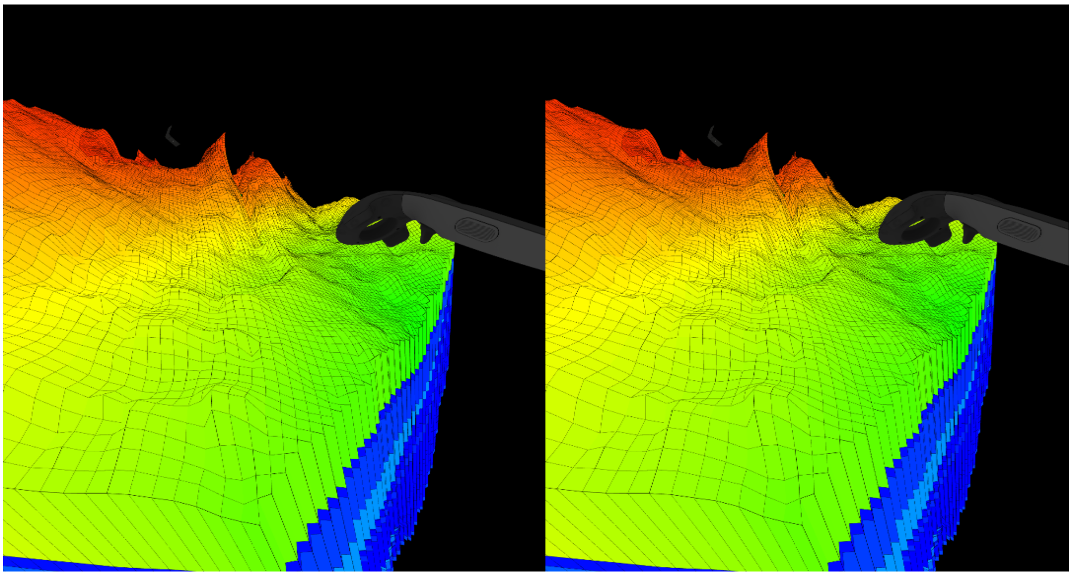

Figure 10: Test model of layer Johansen, visualized in the virtual reality system

5 Conclusion

The paper describes the main actions for developing an application, which represents three-dimensional grid model of oil and gas reservoir on VR headset, with necessary use of the appropriate equipment. The advantage of using virtual reality consists in improving of visual acceptability and dipping in the virtual environment with effect of presence. In VR headset display the quality of drawing of an object significantly differs from what can be watched on

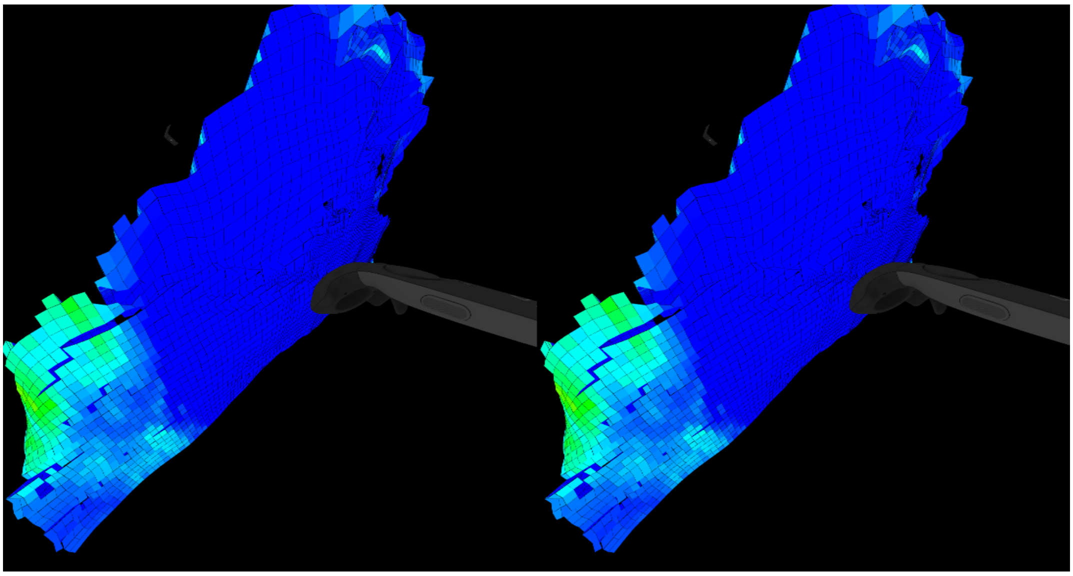

Figure 11: Test model of layer from MATLAB Reservoir Simulation Toolbox, visualized in the virtual reality system

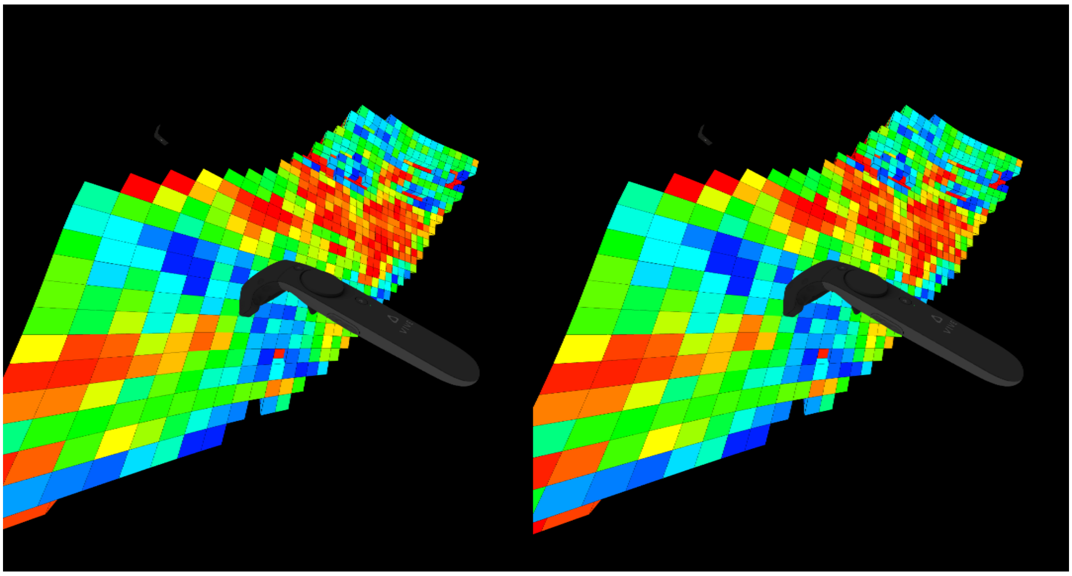

Figure 12: Test model of layer – Sample, visualized in the virtual reality system

the plane screen of the monitor. Also it is worth noting that in headset display it is easier to notice different errors, small details, which are difficult for noting out in a 2D format. This work also gives an opportunity to interact with the drawn model by means of controllers in real time that many times improves perception of the events. With use of OpenGL library there is also a possibility of interaction with model by means of a computer mouse. However the virtual reality and interactivity of operation by means of controllers has big advantage over primitive visualization with control of a mouse, since realistic images are created in the

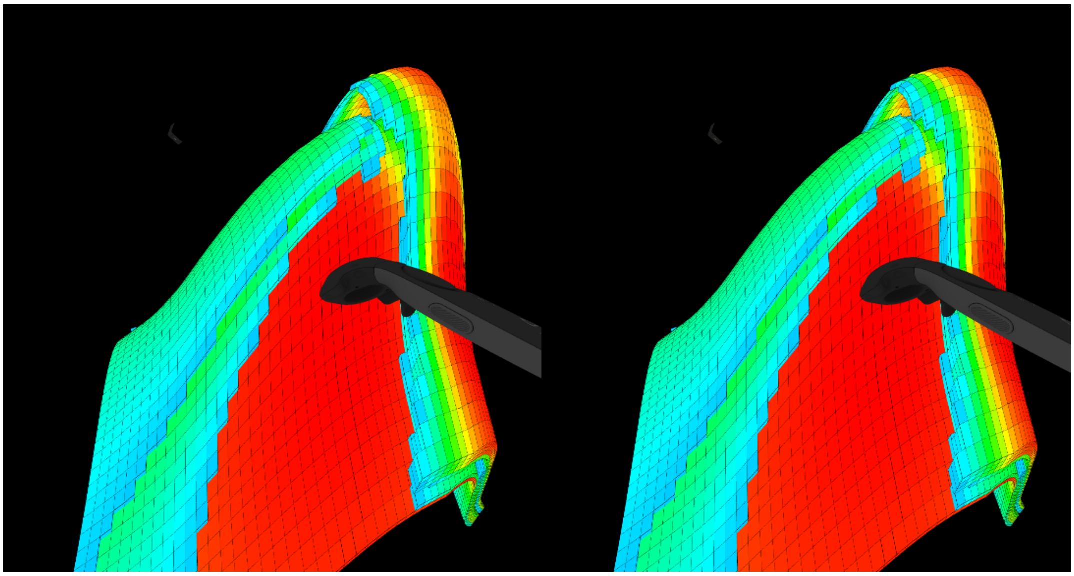

Figure 13: Test model of layer – Sample, visualized in the virtual reality system

virtual environment. They imitate physical presence for the user that leads to the effect of full presence.

6 Future work

In the future, it is planned to use the Vulkan standard, which was created to reduce the load on the central processor. It is also planned to add the user interface (UI), visualization in augmented reality system (AR) and in mixed reality system (MR).

7 Acknowledgements

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Educational Data and Learning Analytics in KazNU MOOCs Platform

Alimzhanov Ye.S., University of International Business,
Almaty, Kazakhstan, E-mail: aermek81@gmail.com
Mansurova M.Ye., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: mansurova01@mail.ru

The initial hype around massive open online courses (MOOCs) already subsided, but the number of new learners in MOOCs platforms is still growing. Due to low completion rates in the MOOCs compared to enrolled students it is important to establish and validate quality standards for these courses. Employing of educational data and learning analytics to improve lesson plans and course delivery become an innovative approach for teachers, curriculum developers and policy makers in education. Learning analytics of online courses can be also used for enhancement of classroom teaching by blending online and face-to-face learning models.
This work presents some observations about the behavior of students, obtained by analyzing the data generated during delivery of 13 MOOCs. Besides classification of learners by analysis their activity data, other interesting characteristics about platform learners like demographic, gender and level of education are described. The results indicate that the quality of interpersonal interaction within a course relates positively and significantly to student scores.
Key words: MOOCs, learning analytics, educational data, online learning, blended learning.

Әл-Фараби ат. ҚазҰУ-нің МООК платформасында білім беру мәліметтер мен ақпараттың сараптамасы

Әлімжанов Е.С., Халықаралық бизнес университеті, Алматы, E-mail: aermek81@gmail.com
Мансұрова М.Е., әл-Фараби ат. Қазақ ұлттық университеті, Алматы,
E-mail: mansurova01@mail.ru

Жаппай ашық онлайн курстар (ЖАОК) алғашқы дүрлікпе төмендеді, бірақ ЖАОК платформасындағы жаңа студенттер саны әлі де өсіп келеді. Тіркелген студенттер санымен салыстырғанда, ЖАОК-ты аяқтаушылар біршама аз болғандықтан, бұл курстардыө сапа стандарттарын құрастыру және бекіту маөызды қадам болып табылады. Оқу жоспары мен оқу курстарын жетілдіру үшін білім беру деректерін және ақпараттың талдауды қолдану оқу бағдарламаларын әзірлеушілерге және білім беру саясатын құрастырушыларға инновациялық тәсіл болып табылады. Онлайн-курстың деректерін талдауды қашықтықтан және дәстүрлі оқыту модельдерін араластыру арқылы оқу процесін жетілдіру үшін пайдаланылуы мүмкін.
Бұл мақалада 13 ЖАОК-та жинақталған деректерді талдау арқылы алынған курс қатысушыларының іс-әрекеті туралы кейбір зерттеулер келтірілген. Нәтижелер курстың ішіндегі өзара қарым-қатынастың сапасы студенттер үшін оң және маңызды екенін көрсетеді.
Түйін сөздер: ЖАОК, ақпараттың сараптамасы, білім беру мәліметтер, онлайн оқыту, арарлас оқыту.

Образовательные данные и аналитика обучения на платформе МООК КазНУ им. аль-Фараби

Алимжанов Е.С., Университет международного бизнеса, Алматы, E-mail: aermek81@gmail.com
Мансурова М.Е., Казахский национальный университет им. аль-Фараби, Алматы,
E-mail: mansurova01@mail.ru

Начальный ажиотаж вокруг массовых открытых онлайн курсов (МООК) пошел на спад, но число новых учащихся на платформах МООК все еще растет. Из-за низких показателей завершения МООК по сравнению с зарегистрированными студентами, важным этапом является установление и утверждение стандартов качества для этих курсов. Использование образовательных данных и аналитики обучения для улучшения планов уроков и предоставления курсов станет инновационным подходом для учителей, разработчиков учебных программ и политики в области образования. Аналитика обучения онлайн курсов может быть использована для улучшения образовательного процесса смешиванием дистанционных и традиционных моделей обучения.
В данной работе представлены некоторые наблюдения о поведении слушателей курсов, полученные путем анализа данных, накопленных при проведении 13 МООК. Результаты показывают, что качество межличностного взаимодействия в рамках курса имеет положительный и существенный характер для учащихся.
Ключевые слова: МООК, анализ обучения, образовательные данные, онлайн обучение, смешанное обучение.

1 Introduction

In 2012, Massive Open Online Courses (MOOCs) made a real sensation in the higher education sector, providing open access through the Internet to the best courses from the best professors and universities of the world [1]. For the last 6 years the number of MOOCs and open education platforms has continuously grow around the world. These MOOCs platforms are developing together with universities evolving into a new market of higher online education providing massive online specializations, credentials and academic degrees [3]. If we look at the numbers, now there have been released more than 7,000 online courses (Figure 1) from above 750 universities and institutions, which are located in more than 40 MOOCs resources where up to 60 million users are enrolled [2]. This numbers are given only according to the data of the Class Central MOOCs aggregator where many other online courses and providers are not taken into account.

The Learning Management System (LMS) allows to collect detailed information about the users' activities and interactions with course content during the learning in the online course. These data are actively used by researchers to improve the quality of educational resources and improve the content of online courses, as well as a deeper understanding of the learning process in online format and other practical purposes (see e.g., [4], [5]). In addition, the accumulated data is sufficiently large to facilitate the development of intelligent LMS and new methods of active learning in the future.

One of the negative indicators of MOOCs is a large dropout rate [6]. But in many cases they do not take into account the fact that learners participate in the MOOCs with different initial intention and motivation [7]. If the traditional university courses are mainly attended by full-time students whose main activity is studying, then MOOCs participants are mostly employed people with tertiary education [8]. According to statistics it is known that for in MOOCs about half enrolled students never engage with any of the content [9]. Most of the students do not reach the end of the course due to lack of time or lack of digital and learning skills for studying by online courses [10]. Therefore, the classification signed up for MOOCs students in their initial motivation will help determine the exact causes of failure and to understand how to improve the course to achieve their goals.

In this paper we try to describe some finding about our MOOCs learners and classify them by their activities. Also we try to answer to the following questions:

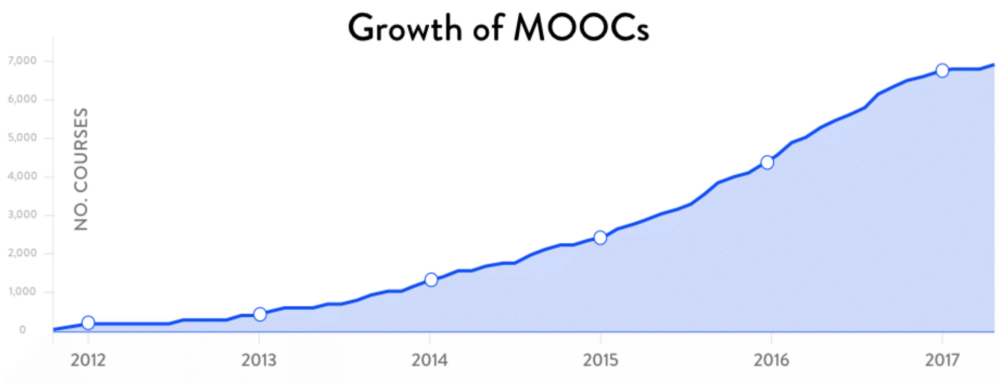

Figure 1: Growth of MOOCs number from 2012 according to Class Central data [3]

1. What is the motivation of each MOOCs learner?
2. How they interact with each other and teaching staff?
3. What they should do to successfully finish the course?

This findings and question answers can help to understand MOOCs developers and providers how improve course content, schedule and delivery methods, also policy makers and administration of universities can evaluate of MOOCs potential to include in academic process in appropriate blended learning model.

2 Literature Review

Several investigators (e.g. [11], [12]) expect that MOOCs can play an important role in future of global education system and even change it. The popularity of MOOCs has made a high volume of learner data available for analytic purposes. A number of scientists began to perform relative researches based on MOOCs data recently, which mainly focus on two aspects. The first is how to improve the MOOCs platform in personalization or to provide new features for both learners and instructors. For example, J. J. Williams and B. Williams [13] investigated how varying reminders and resources sent through emails to participants influence their use of course components like forums and their overall outcomes. C. Shi et al. [14] introduce VisMOOC, a visual analytic system to help analyze user learning behaviors by using video clickstream data from MOOC platforms. Kennedy et. al. [15] analyzed the relationship between a student's prior knowledge on end-of-MOOC performance.

The second aspect is to explore cognitive rules of learner by analyzing learning behavior and therefore to predict their following actions such as whether he will fall out the course. Predicting student performance in MOOCs is a popular and extensive topic. Kizilcec et al. [16] presented a simple, scalable, and informative classification method that identifies a small number of longitudinal engagement trajectories in MOOCs. Learner classification can be fulfilled by different criteria. Researchers from Stanford [17] divided learners into five categories by analyzing learning activities such as viewing a lecture and handing in an assignment for credit: Viewers, Solvers, All-rounders, Collectors, and Bystanders. Researchers from MIT [18] divided learners into four types based on whether or not they participated in the class forum or helped edit the class wiki pages: passive collaborator, wiki contributor, forum contributor, and fully collaborative.

Many researchers' works based on Person-Course Dataset AY2013 [19] which is provided by HarvardX-MITx (e.g. [20]) and others used CAROL Learner Data [21] by Stanford University (e.g. [22]). In this work we used learners data which collected during providing online courses in the Al-Farabi KazNU's own MOOCs platform [23].

3 Material and methods

3.1 MOOCs by Al-Farabi KazNU. Data Description

In 2014 al-Farabi KazNU became the first from Kazakhstani universities, which have joined the MOOCs movement and began work on producing own online courses. Initially, as the target audience was selected prospective students: graduates from secondary schools, students of vocational schools and colleges. Since the project was an initiative, funds for the development and delivery of courses was not provided. Despite the high teaching load and other professional duties of teaching staff we found and revealed among them enthusiasts and volunteers, which agreed to create courses in the new format.

In 2015 al-Farabi KazNU launched the first MOOCs for high school and undergraduate students. Since then we have developed, tested and implemented in the educational process of the University more than 35 courses in Kazakh, Russian and English. Currently on our website for open education registered more than 12 000 users and over 7 000 of them are actively studying the provided courses. Over the past years to KazNU MOOCs was enrolled about 7 200 learners: 6 624 from Kazakhstan, 432 from other countries of CIS, 42, 58 and 43 from EU, Asia and other countries respectively.

In the table 1 there are demographic characteristics of the platform learners. As you can see in he table most of learners are females, under the age of 25, with bachelor or associate degree. Average age of learners is equal to 25.2 and median age is 21.

Table 1: Demographic characteristic of learners (None means not provided)

Age between	**Male**	**Female**	**None**	**All**
under 25	1 768	3 103	9	4 880
25 and 35	315	735	1	1 051
36 and 50	173	650	0	823
over 50	92	231	0	323
None	37	78	7	122
Level of education				
Doctorate	108	285	0	393
Master's degree	314	922	3	1 239
Bachelor's degree	1 187	2 458	1	3 646
High school	669	864	4	1 537
Other	107	268	9	384
Total	**2 385**	**4 797**	**17**	**7 199**

Before the launch of the course, various marketing events were held to gather as much as possible the audience of learners. Most of the courses were conducted in the framework of programs for the refresher courses of teaching staff from universities and secondary schools. All certificates provided freely. That is why learners had very high motivation to get certificates and many courses have high completion rates than in usual MOOCs. In the table 2 the most popular courses are listed, where the number of successfully completed a course learners as well as external students from this number are indicated. Here external means MOOCs students from other institutions.

Table 2: Most popular online courses by Al-Farabi KazNU

Title of the course	**Language**	**Enrolled**	**Completed**	**External**
Management	English	699	508	212
Selected Issues of Inorganic Chemistry	Kazakh	450	169	90
Biophysics	Russian	186	32	32
Branding	Kazakh	405	206	109
Ethnography of the World Nations	Kazakh	326	177	41
Al-Farabi and Modernity	Kazakh	993	535	341
Constitutional Law of the RK	Russian	1028	650	318
Conflictology	Russian	129	68	45
Law Enforcement Bodies of the RK	Russian	370	107	46
Statistics	Kazakh	510	190	121
Probability Theory	Russian	553	125	25
Solving Physical Problems with prof. V. Kashkarov	Russian	725	68	12
Methods of Ethnological Research	Kazakh	454	276	109
Total	**1+6+6**	**6828**	**3111**	**1501**

The dataset collected from this courses (about 3 GB of JSON and CSV data) is used for analyzing users activity and classify them by their behavior. Initial row data is analyzed and reduced to 10% of original volume by dropping the insignificant attributes, personal data and the records with inconsistent and administrative information. Then we performed denormalization of the tables (*users*, *enrollments*, *certificates* and *tracking logs*) to get one universal table with the records where the most informative attributes collected. Below the attributes of cleaned dataset and their description are described: *user_id*: deidentificated id number of user; *course_id*: id of the courses; *viewed*: anyone who accessed the 'Courseware' tab; *explored*: anyone who accessed at least half of the chapters in the course content; *certified*: earned or not a certificate; *level of education*; *gender*; *year of birth*; *grade*: final grade of the course, ranged from 0 to 1; *start_time*: date of course registration; *last_time*: date of last interaction with course, blank if no interactions; *nevents*: number of the interactions with the course; *ndays_act*: number of unique days student interacted with course; *nvideo_plays*: number of play video events within the course; *nchapters*: number of chapters with which the learner interacted; *nforums*: number of posts to the Discussion Forum.

3.2 Methodology

Due to the great diversity of learners in age, education background, region, motivation and learning habits predicting of their successful course completing is a big challenge. Online survey is a good option to recognize learners' motivation, but most of them may not respond to an online survey. Therefore, learning activities may reflect a learner's motivation. The detailed records of learning activities in MOOCs platforms give us a chance to analyze a learner's motivation. Learners have different goals when following a MOOC. These goals are reflected in their behaviour patterns when following the course. Hill [24] has identified five categories of learners' behaviour in a MOOC:

- **No-shows:** register but never log in to the course whilst it is active.
- **Observers:** log in and may read content or browse discussions but do not take any form of assessment followed after videos.
- **Drop-ins:** perform some activity (watch videos, browse or participate in the discussion forum) for a select topic within the course but do not attempt to complete the entire course.
- **Passive participants:** view a course as content to consume. They may watch videos, take quizzes and/or read discussion forums but generally do not engage with the assignments.
- **Active participants:** fully intend to participate in the MOOC and take part in discussion forums, the majority of assignments and all quizzes.

A recent study by Wang and Baker [25] has shown that participants who expected to finish a MOOC were more likely to do so than participants who did not think they would complete the course. This motivation in the category of "active participants" is a good predictor for completing a MOOC. Although this finding is in line with the findings of other studies, the authors concluded that further research is needed to gain more insight into the motivations of MOOC participants and how these relate to MOOC design, in order to provide a learning experience worthwhile for a large community of learners.

Figure 2 displays the learners which have some activity entries, mark above 0 and those who obtained a certification. The average certification rate of 13 courses is 22,7%.

4 Results and Discussion

Figure 3 shows the average total activities and average video watchings of two group learners with mark above and equal to 0. The difference between these two groups is apparently huge especially in total events and activities. The average activities of learners with mark above 0 are three times more than learners with no mark at least. If a learner wants to earn certificate, he/she will spend more time on this course.

Based on the above analysis, learners can be divided into different categories according to their activities. Learners in category *Active participants* have highest activities while learners in category *Observers* have lowest ones. An activity index value was proposed to measure the engagement of a learner. According to above statistics, if a learner spend more time (days) in one course or with higher activities especially video playing events, he/she should obtain a higher grade value, while if a learner enrolled on too many courses, the engagement in one course will be less.

Understanding the reasons behind dropout rates in MOOCs and identifying areas in which these can be improved is an important goal for MOOC development. Many widely-quoted

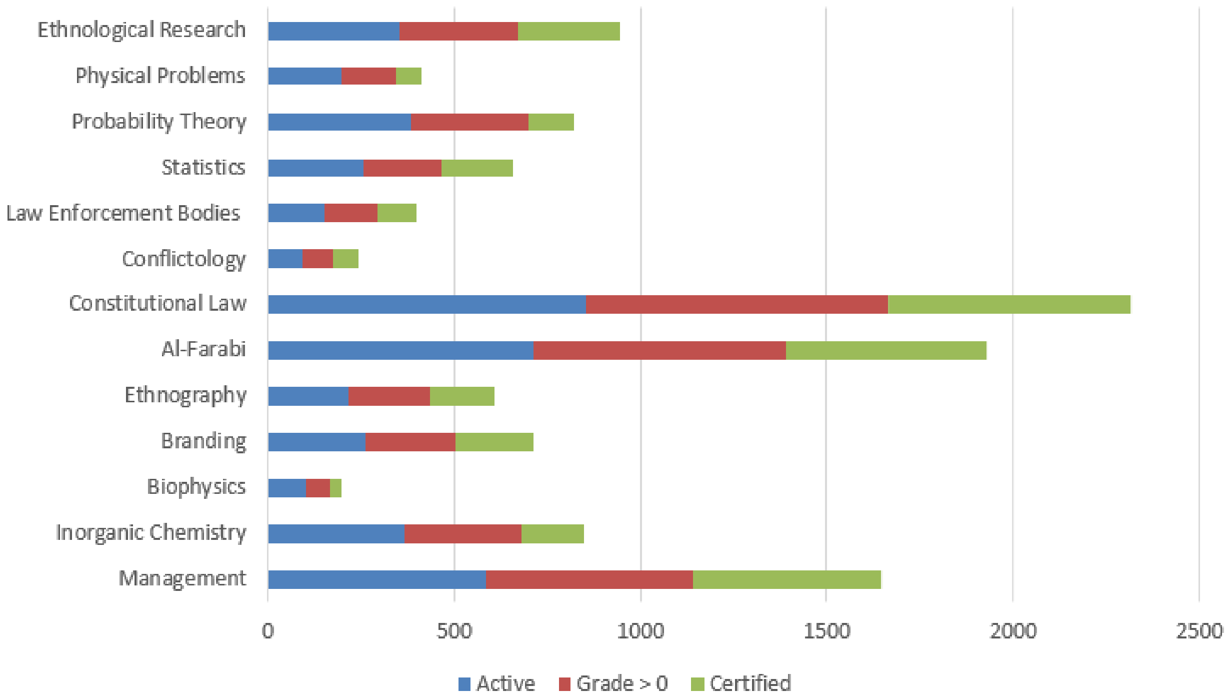

Figure 2: Course statistics on active, mark above 0 and certified learners

dropout rates are calculated from baseline numbers which include registrations by people who never engage with the course or who engage in their own way but without completing assessments. Despite this, it is clear that many of those who do wish to follow and complete a course are hindered by factors such as level of difficulty, timing and lack of digital and learning skills. These problems become even more acute when MOOCs are proposed as a replacement for traditional teaching (rather than just free, spare time activities) and particularly when they are suggested as the means to close gaps in education.

5 Conclusion

Employing of educational data and learning analytics to improve lesson plans and course delivery become an innovative approach for teachers, curriculum developers and policy makers in education. Learning analytics of online courses can be also used for enhancement of classroom teaching by blending online and face-to-face learning models. In the figure 4 weekly learners engagement in the course can be useful information for the teaching staff to apply motivating posts or emails to learners when learners become inert. Also learning analytics can help to identify flush of activity and reasons of occurrence which can help apply right conducting strategy during the course delivery.

In this paper, we first made an analysis about learning behaviors of learners in MOOCs and explored the differences and characteristic of learning behavior features between the learners with different grades. MOOCs learning is one kind of high-level behavior, learners may be influenced by many incentive factors of ultimate goal. The passing rate maybe not enough incentive for some learners. Survey research has bolstered the notion that effective

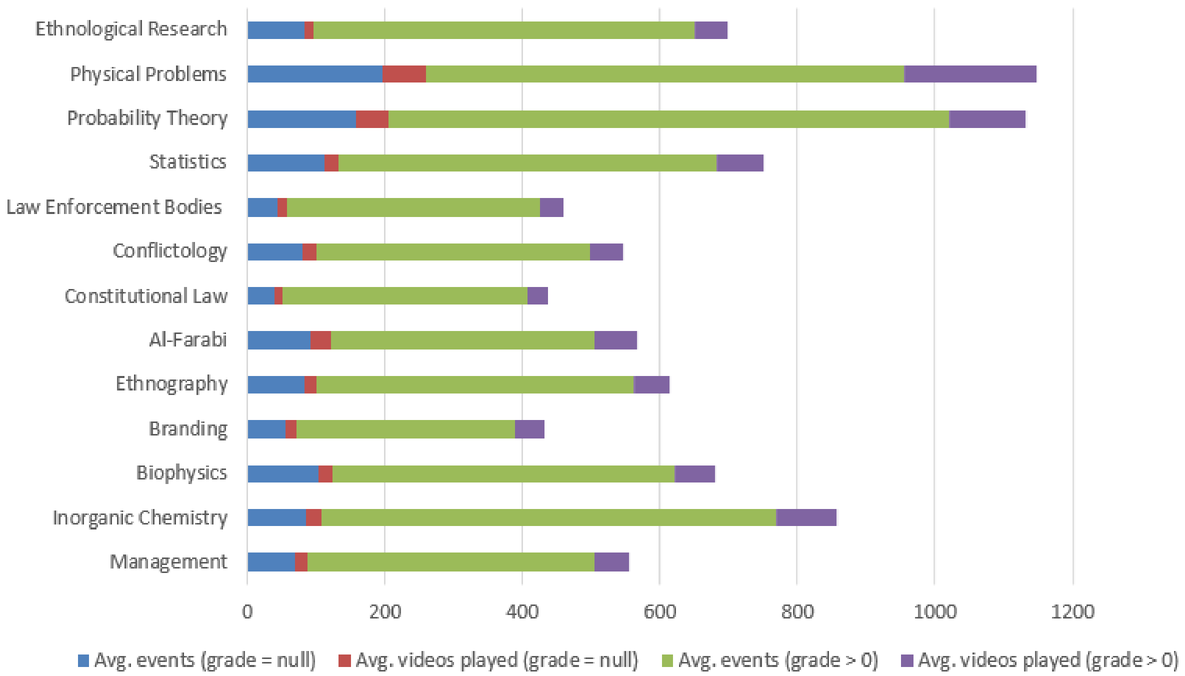

Figure 3: Average activities learners with grade above and equal to 0

learner–instructor and learner–learner interactions are critical to effective online learning and concluded that increased interpersonal interaction within the framework of the course, either with the instructor or with learner peers, positively affects student learning.

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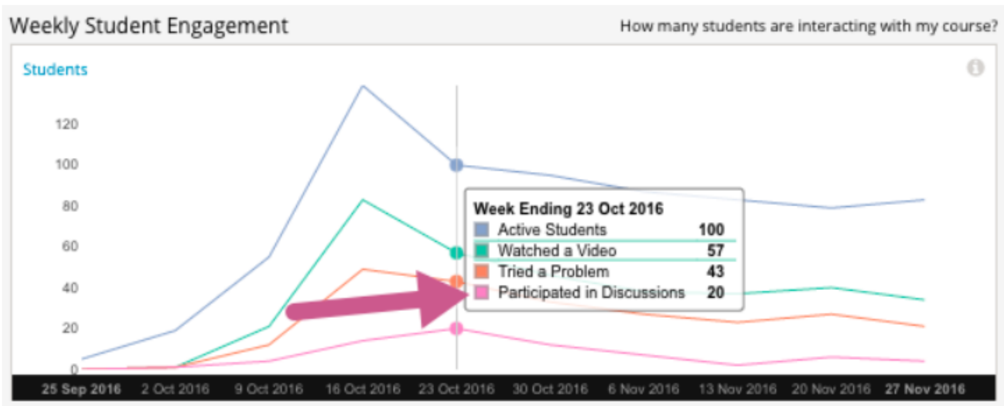

Figure 4: Weekly learners engagement chart of "Solving Physical Problems" online course

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Development of a hybrid parallel algorithm (MPI + OpenMP) for solving the Poisson equation

Kenzhebek Y.G., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: kenzhebekyerzhan@gmail.com
Baryssova S.B., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: sandugash.baryssova@gmail.com
Imankulov T.S., Al-Farabi Kazakh National University
Almaty, Kazakhstan, E-mail: imankulov_ts@mail.ru

This article presents the development of a hybrid parallel algorithm for solving the Dirichlet problem for the two-dimensional Poisson equation. MPI and OpenMP were chosen as the technology for parallelization. For the numerical sequential solution of the Poisson equation, an explicit "cross" scheme was used (the Jacobi iterative method). A parallel algorithm was implemented by the method of decomposition of regions, namely, one-dimensional decomposition. In the article in the form of tables and graphs shows the acceleration and efficiency of parallel algorithms using MPI and OpenMP technologies separately and were compared with the acceleration and efficiency of the MPI + OpenMP hybrid algorithm. Also, the choice of the hybrid program architecture is justified and the distribution of data between processes is explained. The results show the effectiveness of using a hybrid algorithm for solving such problems and show the acceleration of time by 1.5-2 times. The presented algorithm was tested on a cluster of the computing center of the Novosibirsk State University for a different number of points in the computational domain (from 64x64 to 1024x1024). The results of the presented work can be applied to the simulation of problems of hydrodynamics, ecology, aerodynamics, the spread of chemical reagents, the propagation of heat and other physical processes.
Keywords: high-performance computing, hybrid technologies, parallel computing, MPI, OpenMP.

MPI және OpenMP технологиялары негізінде Пуассон теңдеуін шешуге арналған гибрид параллельді алгоритм құру

Кенжебек Е.Ғ., Әл-Фараби атындағы Қазақ Ұлттық Университеті
Алматы қ., Қазақстан, E-mail: kenzhebekyerzhan@gmail.com
Барысова С.Б., Әл-Фараби атындағы Қазақ Ұлттық Университеті
Алматы қ., Қазақстан, E-mail: sandugash.baryssova@gmail.com
Иманкулов Т.С., Әл-Фараби атындағы Қазақ Ұлттық Университеті
Алматы қ., Қазақстан, E-mail: imankulov_ts@mail.ru

Бұл мақалада екі өлшемді Пуассон теңдеуі үшін Дирихле мәселесін шешуге арналған гибридті параллельді алгоритм ұсынылған. Параллельдеу технологиясы ретінде MPI және OpenMP таңдалды. Пуассон теңдеуінің сандық жүйелі шешімі үшін айқын «крест» схемасы қолданылды(Якоби итерациялық әдісі). Параллельді алгоритм облысты декомпозициялау әдісі бойынша жүзеге асырылды. Мақалада параллельді алгоритмдердің үдеуі және тиімділігі кестелер мен графиктер түрінде көрсетілген және гибридті алгоритмнің үдеуі және тиімділігімен салыстырулар жүргізілді. Сондай-ақ, гибридті бағдарлама архитектурасын таңдау себебі және процесаралық деректердің үлестірілуі түсіндіріледі. Алынған нәтижелер, гибридті алгоритмді осыған ұқсас есептерде қолдану тиімді екенін және уақыттың жеделдетілуі 1,5-2 есе артатынын көрсетеді. Бұл алгоритм Новосибирск Мемлекеттік Университетінің есептеуіш орталығының кластерінде есептеу облысының әртүрлі нүктелерінде (64x64-тен 1024x1024-ге дейін) сыналды. Жасалған жұмыстың нәтижелерін гидродинамиканың, экологияның, аэродинамиканың, химиялық реагенттердің таралуының, жылу мен басқа да физикалық үрдістердің таралуының мәселелерін модельдеуге қолдануға болады.
Түйін сөздер: жоғары өнімді есептеулер, гибрид технологиялар, параллельді есептеулер, MPI, OpenMP.

Разработка гибридного параллельного алгоритма (MPI+OpenMP) для решения уравнения Пуассона

Кенжебек Е.Г., Казахский национальный университет имени аль-Фараби
Алматы, Казахстан, E-mail: kenzhebekyerzhan.com
Барысова С.Б., Казахский национальный университет имени аль-Фараби
Алматы, Казахстан, E-mail: sandugash.baryssova@gmail.com
Иманкулов Т.С., Казахский национальный университет имени аль-Фараби
Алматы, Казахстан, E-mail: imankulov_ts@mail.ru

В данной статье представлена разработка гибридного параллельного алгоритма для решения задачи Дирихле для двумерного уравнения Пуассона. В качестве технологии для распараллеливания были выбраны MPI и OpenMP. Для численного последовательного решения уравнения Пуассона использовалась явная схема «крест» (итерационный метод Якоби). Параллельный алгоритм был реализован методом декомпозицией областей, а именно одномерная декомпозиция. В статье в виде таблиц и графиков показаны ускорения и эффективности параллельных алгоритмов при использовании технологий MPI и OpenMP по отдельности и были сравнены с ускорением и эффективностью гибридного алгоритма MPI + OpenMP. Так же, обоснован выбор архитектуры гибридной программы и объяснены распределения данных между процессами. Полученные результаты говорят об эффективности использования гибридного алгоритма для решения подобных задач и показывают ускорение времени в 1,5-2 раза. Представленный алгоритм протестирован на кластере вычислительного центра Новосибирского Государственного Университета для различного количества точек расчетной области (от 64x64 до 1024x1024). Результаты представленной работы можно применить для моделирования задач гидродинамики, экологии, аэродинамики, распространение химических реагентов, распространение тепла и других физических процессов.
Ключевые слова: высокопроизводительные вычисления, гибридные технологии, параллельные вычисления, MPI, OpenMP.

1 Introduction

Currently, parallel programming and high-performance computing systems are relevant in various fields of science and technology. High-performance computing uses parallel technologies, such as MPI, OpenMP and CUDA. The greatest productivity can be achieved by creating hybrids of the above technologies. The most high-performance, under a certain range of tasks, will be the merging of CUDA, MPI and OpenMP technologies into a single whole. Therefore, at present the development of hybrid parallel programs is very relevant.

The most complex of the parallel types are hybrid tasks. Particular interest to them is the trend towards the use of multi-core architectures and SMP-clusters for high-performance computing. One of the most effective programming approaches for such clusters is the hybrid, based on the combined use of MPI and OpenMP. The hybrid approach assumes that the algorithm is split into parallel processes, each of which is itself multi-threaded. Thus, there are two levels of parallelism: parallelism between MPI processes and parallelism within the MPI process at the thread level [1].

2 Literature review

There are many works devoted to the research of the MPI / OpenMP approach [2-4]. As practice shows [5-7], by consolidating MPI processes and reducing their number, a hybrid model can eliminate a number of MPI deficiencies, such as large overhead for message transmission and poor scalability with an increase in the number of processes [8]. However, the performance of a hybrid technology depends very much on the mode of its launch and execution, which determines the ratio of MPI processes and OpenMP threads on one computing node [9]. Chan and Yang [10] argue that MPI can be more favorable with the scalability of clusters. However, OpenMP can favor the speed of shared memory. In addition, the application performance can be affected by the type of problem that is being solved and its size. They show that the effect of MPI communication is the main weakness of this programming model. And finally, they conclude that OpenMP prevails over MPI especially with using a multi-core processor.

It is well known that the implementation of MPI for algorithms in which data is naturally distributed across processes demonstrates very high efficiency (almost linear scaling in time from the number of MPI processes). As it was shown, for example, in [11], in order to achieve comparable performance on one compute node in the case of OpenMP implementation, it is required to implement OpenMP using the concept on which MPI technology is based, but taking into account the presence of shared memory on the node.

Hybrid parallel programming enables to explore the best that is offered by distributed and shared architecture in HPC [12]. Hybrid programming models can match better the architecture characteristics of an SMP cluster, and that would replace message passing communication with synchronized thread-level memory access [13-15]. However, the hybrid programming model can not be regarded as the ideal for all codes [16, 17].

3 Materials and methods

3.1 Purpose of the work and formulation of the problem

The purpose of this work was the creation of a hybrid program that solves the two-dimensional Poisson equation using Jacobi's iterative method in the C ++ programming language using MPI and OpenMP technologies.

The two-dimensional Poisson equation of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f(x, y) \tag{1}$$

Where x, y are the coordinates; u (x, y) is the desired function; f (x, y) is a continuous function on a rectangular domain with Dirichlet boundary conditions.

$$f\left(x, y\right) = 2x\left(1 - x\right) + 2y\left(1 - y\right) \tag{2}$$

The Dirichlet boundary conditions for the problem under consideration are:

$$\begin{aligned} &u\left(0, y\right) = 0; \\ &u\left(1, y\right) = 0; \\ &u\left(x, 0\right) = 0; \\ &u\left(x, 1\right) = 0; \end{aligned} \tag{3}$$

3.2 Methods of solution

The most common approach for the numerical solution of differential equations is the method of finite differences. Following this method, the solution domain is represented as a discrete set of points [18]. For sampling internal grid points, a five-point pattern is used, thus using the Jacobi method to perform iterations, the equation takes the following form:

$$u_{i,\ j}^{n+1} = 0.\ 25\left(u_{i+1,\ j}^{n} + u_{i-1,\ j}^{n} + u_{i,\ j+1}^{n} + u_{i,\ j-1}^{n} + h^2 f_{ij}\right) \tag{4}$$

Here, $u_{i,\ j}^{n+1}$ is a new layer of Jacobi iterations, and $u_{i,\ j}^{n}$is the previous iteration layer. As shown in Figure 1, to calculate the value of each point of the new layer $u_{i,\ j}^{n+1}$, we need the values of four neighboring points of the previous layer $u_{i+1,\ j}^{n},\ u_{i-1,\ j}^{n},\ u_{i,\ j+1}^{n},\ u_{i,\ j-1}^{n}$.

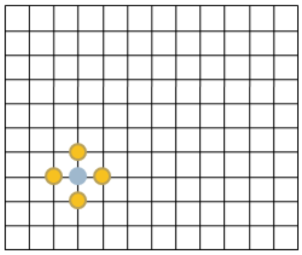

Figure 1: Jacobi method

3.3 Parallelizing a task in MPI

The first thing to solve when parallelizing such tasks is the way to share data between compute nodes. In the problem under consideration for solving the Poisson equation, a tape scheme was used to separate the data. With this division of data, the computing area can be broken down into several horizontal bands. For each process that performs processing of any band, the boundary lines of the previous and next bands were duplicated. The resulting enlarged bands are shown in Figure 2 with dashed frames. Calculations in each band are performed independently of each other and before each new iteration of Jacobi it is necessary to update the duplicated boundary lines.

The exchange of boundary lines between processes consists of two data transfers. First, each process passes its lower boundary to the next process and receives the upper boundary of the line of this process. In the second case, the transfer of boundary lines is performed in the opposite direction, that is, each process passes its upper boundary line to the previous process and receives the lower boundary line from that process. For this operation, combined reception and transmission of MPI_SendRecv messages was used.

3.4 Parallelizing a task in OpenMP

When organizing the problem in question using OpenMP (Open Multi-Processing) technology, the compiler directives were added to the sequential program code. Within this technology, these directives are used to allocate several parallel areas in which processing is performed

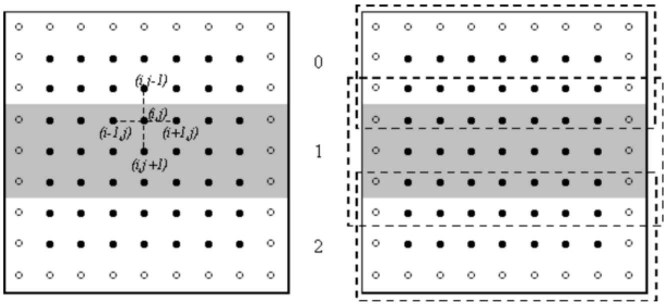

Figure 2: Data distribution

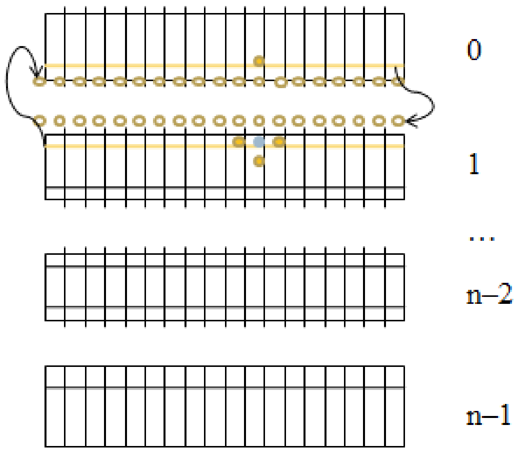

Figure 3: Exchange of boundary lines between processes

using threads. The processors used are multi-core, in order to optimally load all the kernels, there should be several parallel threads in the program. The number of threads that are specified in the program should not exceed the number of cores [19].

3.5 Hybrid method MPI + OpenMP

After creating parallel MPI and OpenMP algorithms, a hybrid method MPI + OpenMP was developed for parallel computation. When creating a hybrid program, a suitable architecture was considered to solve the problem under consideration and the advantages of each technology were taken into account.

MPI technology is used to parallelize a task between SMP nodes for processes, which allows using address spaces and processor computing resources. When performing calculations, each node does not take advantage of the shared memory between the cores, so OpenMP technology is used to parallelize the cores of each of these SMP nodes. MPI was run in a clustered configuration that uses the computational resources of several processors and runs several separate MPI processes on each used node [20].

The distribution of data between the processes was carried out using MPI, and parallel calculation of data within each of the processes was handled by OMP threads.

The architecture used is shown in Figure 4. This architecture has several modes for working with MPI processes and OpenMP threads. In a hybrid technology, the multiplication of MPI processes and threads specified in the program should correspond to the total number of cores used in the computer system. This is used to correctly load all the kernels. If you exceed a certain number of threads assigned to each MPI process, this can lead to a collision of threads between them. Thus, this will lead to an increase in the execution time of the work.

As shown in Figure 4, for example, if we have two nodes with eight cores on each, the multiplication of the processes and threads used should not exceed 16. This will ensure the balancing of work between processes and threads.

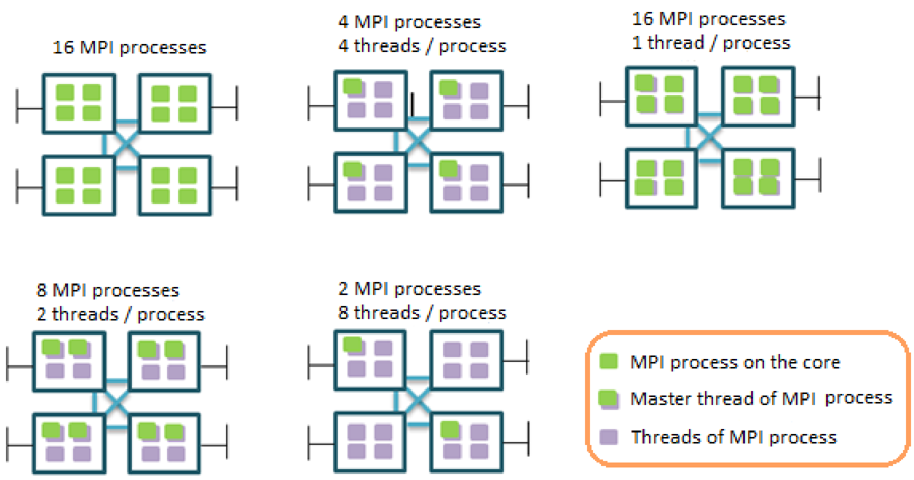

Figure 4: The architecture of the hybrid program MPI + OpenMP

4 Results and discussion

All parallel programs were tested on the Novosibirsk State University (NSU) cluster, which had 2 nodes available. Each node has two 4-core Intel (R) Xeon processor (R) CPU E5-2603 v2 1.80GHz. The tables show the averaged values of time based on several measurements.

Table 1. Time of parallel program execution using MPI technologies(p-process)

Grid size	The execution time of the sequential program, sec	The execution time of the MPI parallel program, sec			
		p = 2	p = 4	p = 8	p = 16
64x64	0,11	0,099	0,092	0,1	0,13
128x128	1,41	0,9	0,59	0,49	0,52
256x256	17,08	9,12	5,1	3,81	2,88
512x512	176,6	90,3	47,3	28,82	191,2
1024x1024	1549,3	781,74	335,7	191,2	135,2

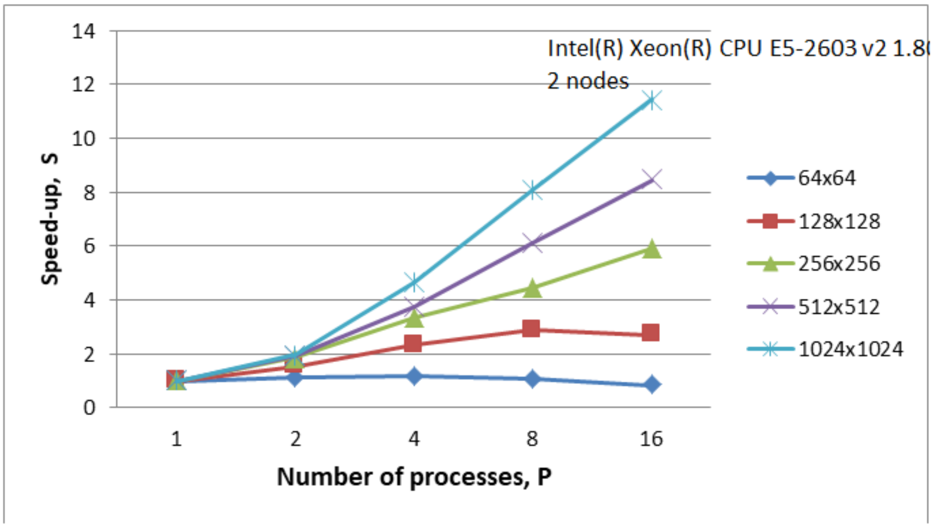

Figure 5: Speed-up of parallel version of the MPI program

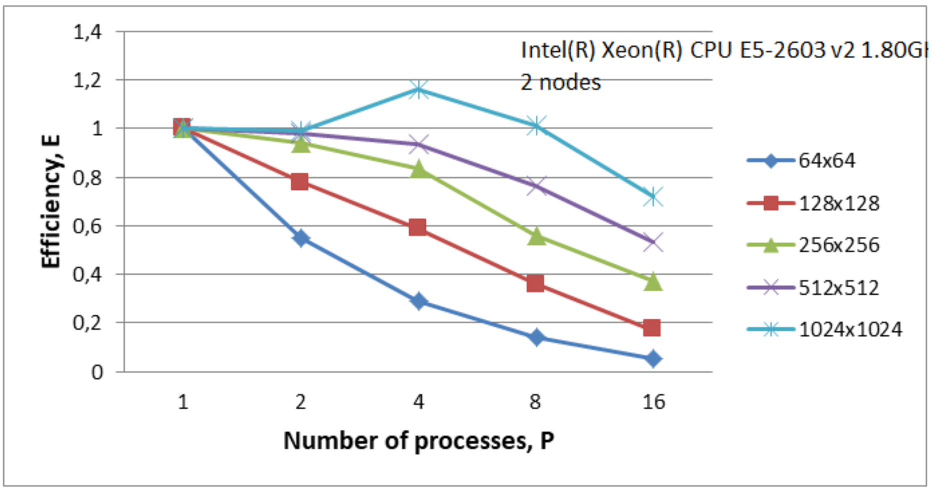

Figure 6: The efficiency of parallel version of the MPI program

Table 2. Time of execution of the parallel program using OpenMP technologies (thread)

Grid size	The execution time of the sequential program,sec(thread=1)	The execution time of the OpenMP parallel program, sec		
		thread=2	thread=4	thread=8
64x64	0,08	0,06	0,04	0,05
128x128	1,03	0,55	0,34	0,26
256x256	12,25	6,2	3,4	1,9
512x512	320,6	187,4	95,5	708,3
1024x1024	2317,3	1381,3	708,3	406,3

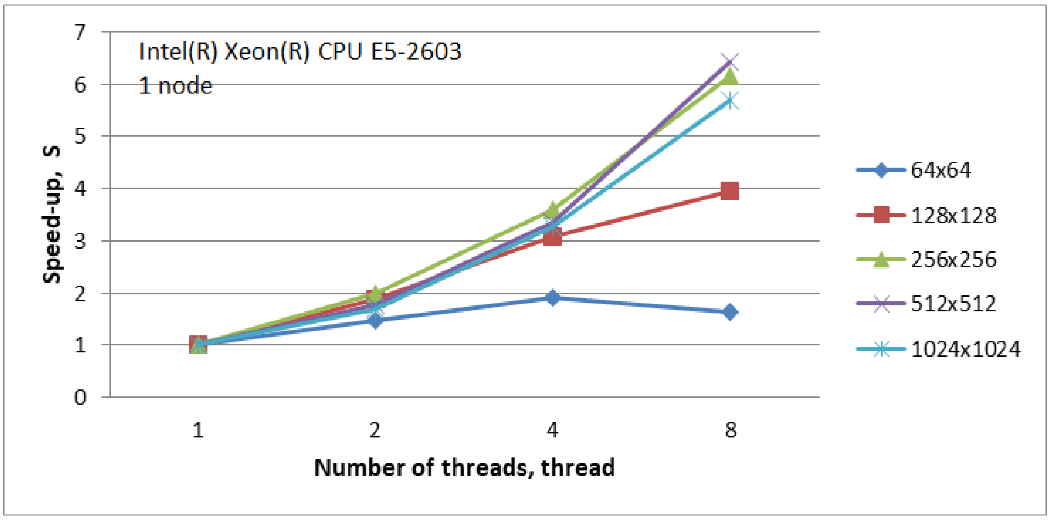

Figure 7: Speed-up of parallel version of the OpenMP program

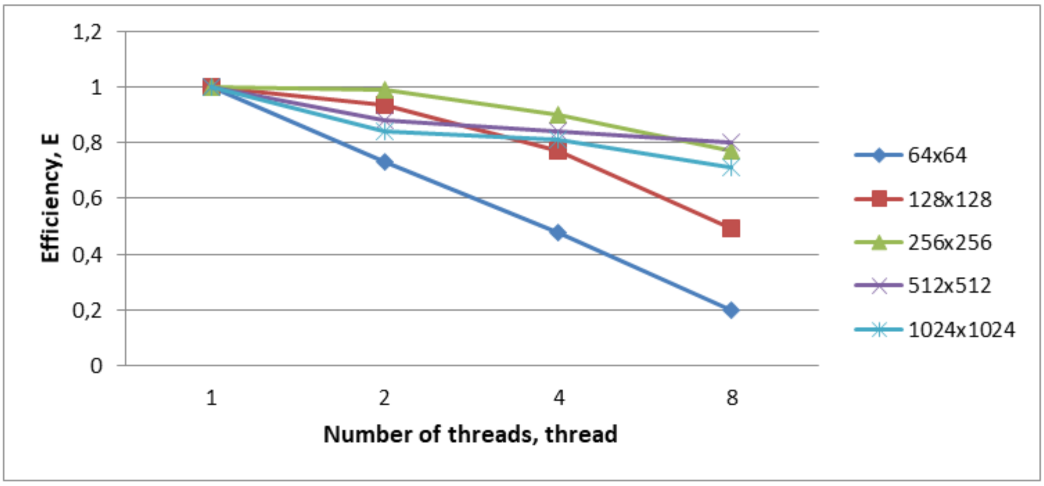

Figure 8: The efficiency of parallel version of the OpenMP program

Table 3. The execution time of the hybrid parallel program using MPI and OpenMP technologies (p-process, thread-thread)

Grid size	The execution time of the sequential program,sec (p=1)	The execution time of the hybrid parallel program MPI + OpenMP, sec			
		p=2, thread=8	p=4, thread=4	p=8, thread=2	p=16, thread=1
64x64	0,11	0,1	0,08	0,095	0,12
128x128	1,41	0,69	0,43	0,42	0,51
256x256	17,08	5,65	2,87	2,81	2,96
512x512	176,6	53,4	23,4	19,45	21,73
1024x1024	1549,3	406,8	150,6	127,1	132,17

Based on the obtained data, the average speed-up and efficiency of parallel programs MPI and MPI + OpenMP (Hybrid) were calculated for solving the Poisson equation. The results are shown in Figures 9 and 10.

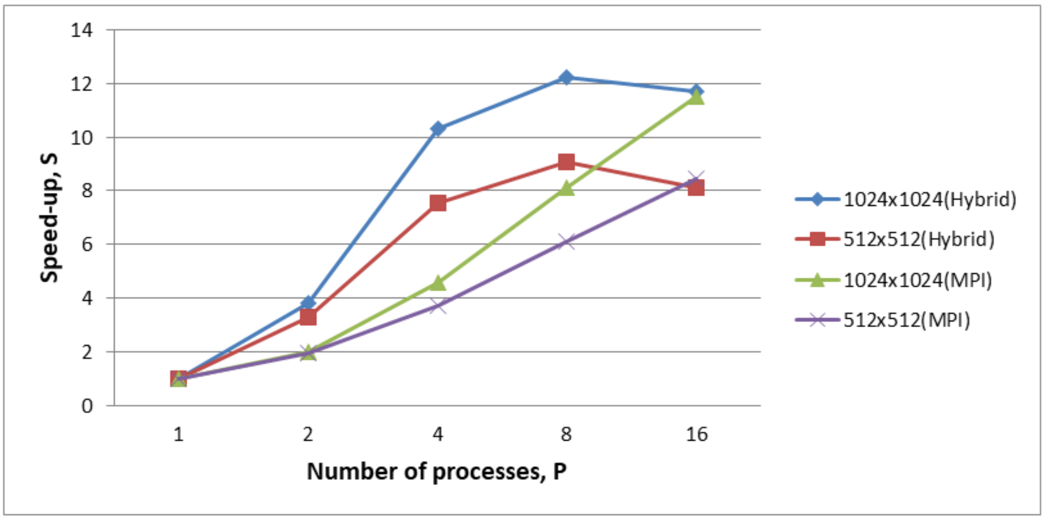

Figure 9: Speed-up for parallel versions of programs

On the efficiency figure of the parallel MPI program, it is noticeable that at the number of 1024x1024 points the efficiency at four processes becomes higher than one. The main reason for this is the total cache size available for the parallel program. With a large number of processors (or cores), one has access to more cache memory. At some point, most of the data fits into the cache memory, which greatly speeds up the calculation. Another way to take this into account is that the more processors are used, the less data that each gets until this part can fit inside the cache of a separate processor. For example, in our case, the cache memory of the Intel (R) Xeon processor (R) CPU E5-2603 has 10 MB of capacity. And in the Poisson problem under consideration there are 3 quantities of the double type. Therefore, when executing the program on one processor, the data did not fit into the cache memory. And

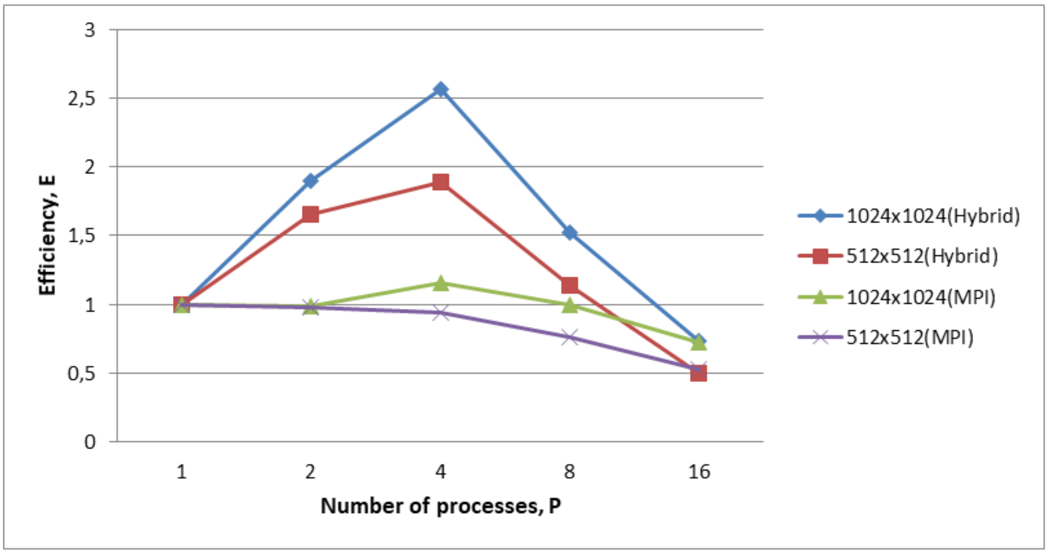

Figure 10: The efficiency of parallel versions of programs

when the program used all the processors in the computer system, each processor had a piece of data that was placed in the cache memory, thereby ensuring speed-up of the calculation.

The resulting superlinear acceleration using hybrid technology MPI + OpenMP is explained by the fact that MPI processes and threads use all the performance of the cores of the computer system. Because the threads assigned to each MPI process effectively use the computing resources of several computers. This hybrid program uses MPI technology to distribute data between processes, and OpenMP separates the iterations of the loop between threads of the program. This ensured the acceleration of work on each compute node.

5 Conclusion

This work was devoted to the development of a hybrid program using MPI and OpenMP technologies. Hybrid implementation of the program is more efficient when working with a large number of nodes and using multi-core processors, because this hybrid technology has the ability to use the cores of several computing nodes. The hybrid program MPI + OpenMP for solving the Poisson equation accelerated the performance of the work by 1.5-2 times in comparison with the MPI program.

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